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On the Sum of Strictly k-zero Matrices

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ABSTRACT

Let k be an integer such that $k \geq 2$. An n-by-n matrix A is said to be strictly k-zero if $A^k = 0$ and $A^m \neq 0$ for all positive integers m with $m < k$. Suppose A is an n-by-n matrix over a field with at least three elements. We show that, if A is a nonscalar matrix with zero trace, then (i) A is a sum of four strictly k-zero matrices for all $k \in \{2,..., n\}$; and (ii) A is a sum of three strictly k-zero matrices for some $k \in \{2,..., n\}$. We prove that, if A is a scalar matrix with zero trace, then A is a sum of five strictly k-zero matrices for all $k \in \{2,..., n\}$. We also determine the least positive integer m, such that every square complex matrix A with zero trace is a sum of m strictly k-zero matrices for all $k \in \{2,..., n\}$.

Keywords: Nilpotent matrix, trace, Jordan canonical form

INTRODUCTION

Let F be a field. We denote by M_n(F) the set of n-by-n matrices with entries from F. A matrix $A \in M_n$(F) is said to be nilpotent or k-zero if $A^k = 0_n$ for some positive integer k. If A is nilpotent, the least positive integer k, such that $A^k = 0_n$, is called the index of nilpotence of A. We say that A is strictly k-zero if A is nilpotent with index k. If A is (strictly) 2-zero, we say that A is (strictly) square zero.

The sum of nilpotents problem, which is the problem of expressing a square matrix as a sum of a finite number of nilpotent matrices, was first considered by J. Wang and P. Wu in 1991. In particular, they showed that a square matrix A over a complex Hilbert space is expressible as a sum of two strictly square zero matrices if and

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only if A is similar to $-A$ (Wang and Wu 1991). K. Takahashi (1999) gave necessary and sufficient conditions for a diagonalizable square complex matrix with two distinct nonzero eigenvalues to be expressible as a sum of three strictly square zero matrices. J.D. Botha (2012) generalized Wang and Wu's result over arbitrary fields. C.D. Pazzis (2017) proved that every trace zero matrix over an arbitrary field is a sum of four strictly square zero matrices. He also proved that every trace zero matrix over a field with characteristic two is a sum of three strictly square zero matrices.

The sum of nilpotents problem has different variations, depending on the conditions imposed on the nilpotent matrices, such as the index, number of summands, and underlying field. One can ask: for a fixed field F and a fixed index k, when is an n-by-n matrix over F a sum of m strictly k-zero matrices? For F = C (the field of complex numbers) and $k = 2$, this has been answered by Wang and Wu (1991) for $m = 2$, and was considered by Takahashi (2000) for $m = 3$. One can also consider the question: for a fixed index k and a fixed number m of summands, when is an n-by-n matrix a sum of m strictly k-zero matrices over an arbitrary field? For $k = 2$, this has been answered by J.D. Botha (2012) for $m = 2$, and was considered by C.D. Pazzis (2017) for $m = 3$. However, these results have only considered the case when the index is two. In this paper, we look into the sum of nilpotents problem for all indices and for fields with at least three elements.

Suppose F is a field with at least three elements. By examining all possible Jordan canonical forms of a nilpotent matrix, we show that every nilpotent $A \in M_n$(F) is expressible as a sum of two strictly k-zero matrices for all $k \in \{2,\ldots, n\}$ (see Theorem 12). We use this theorem together with a result of P. A. Fillmore (1969) to show that, if $A \in M_n$(F) is a nonscalar matrix, then A has zero trace if and only if A is expressible as a sum of two strictly k-zero matrices and two strictly l-zero matrices for any $k, l \in \{2,\ldots, n\}$ (see Theorem 13). Thus, every trace zero nonscalar matrix over a field with at least three elements is a sum of four strictly k-zero matrices for all $k \in \{2,\ldots, n\}$ (see Corollary 14). We also show that every trace zero nonscalar matrix is a sum of three strictly k-zero matrices for some $k \in \{2,\ldots, n\}$ (see Corollary 15). We prove that every trace zero scalar matrix over a field with at least three elements is a sum of five strictly k-zero matrices for all $k \in \{2,\ldots, n\}$ (see Corollary 16). Since zero trace is a necessary condition for a given square matrix to be expressible as a sum of nilpotent matrices, we prove that four is the minimum number m of summands, such that every $A \in M_n$(C) with $tr(A) = 0$ is a sum of m of strictly k-zero matrices for all $k \in \{2,\ldots, n\}$ (see Corollary 17). Furthermore, if $A \in M_n$(C) is not expressible as a sum of two strictly k-zero matrices for some $k \in \{2,\ldots, n\}$, we show that there exist a positive integer m and

$B \in M_m(\mathbb{C})$, such that $A \oplus B$ is expressible as a sum of two strictly l-zero matrices for all $l \in \{2,\ldots, n\}$ (see Theorem 18). We also show that, if A is a diagonalizable complex matrix with two distinct eigenvalues which is not expressible as a sum of three strictly square zero matrices, then there exist a positive integer t and a matrix $C \in M_t(\mathbb{C})$, such that $A \oplus C$ is a sum of three strictly square zero matrices (see Theorem 19).

PRELIMINARIES

Let F be a field. We denote the characteristic of F by $char$ (F). Given $A = [a_{ij}] \in M_n(\mathrm{F})$, we define the spectrum of A, denoted $\sigma(A)$, as the set of eigenvalues of A and the trace of A as $tr\,A = \sum_{i=1}^{n} a_{ii}$. For a positive integer k and $\lambda \in$ F, we define the k-by-k Jordan block corresponding to λ as $J_1(\lambda) = [\lambda]$ for $k = 1$, and

$$J_k(\lambda) := \begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & 0 & \lambda \end{bmatrix},$$

for $k > 1$. Note that $J_k(0)$ is strictly upper triangular, and hence, nilpotent. In particular, $J_k(0)$ is a strictly k-zero matrix.

Let $A \in M_n$(F) be strictly k-zero. Then, 0 is the only eigenvalue of A, and by the Jordan Canonical Form (JCF) Theorem, A is similar to

$$\bigoplus_{i=1}^{m} J_{n_i}(0),$$

for some positive integers n_i, such that $k = n_1 \geq n_2 \geq \ldots n_m$ and $\sum_{i=1}^{m} n_i = n$. Every matrix similar to A is also strictly k-zero. If $B \in M_m$(F) is strictly l-zero, then $A \oplus B \in M_{m+n}$(F) is strictly j-zero, where $j = max\,\{k,l\}$.

Let m and k be positive integers. We denote by $\mathrm{N}_k^{(m)}$(F) the set of all square matrices over F which are expressible as a sum of m strictly k-zero matrices. Note that $\mathrm{N}_k^{(1)}$(F) is the set of all strictly k-zero matrices over F, and $\mathrm{N}_1^{(1)}$(F) is the set of all zero square matrices over F. The following proposition gives some elementary properties of matrices in $\mathrm{N}_k^{(m)}$(F).

Proposition 1. Let $A \in \mathrm{N}_k^{(m)}(\mathrm{F})$.

(a) Then, the trace of A is zero.

(b) If $0 \neq \alpha \in \mathrm{F}$, then $\alpha A \in \mathrm{N}_k^{(m)}(\mathrm{F})$.

(c) If A is similar to C, then $C \in \mathrm{N}_k^{(m)}(\mathrm{F})$.

(d) If $r \leq k$ and $B \in \mathrm{N}_r^{(m)}(\mathrm{F})$, then $A \oplus B \in \mathrm{N}_k^{(m)}(\mathrm{F})$.

(e) If F has at least three elements, then $A \in \mathrm{N}_k^{(m+1)}(\mathrm{F})$. That is, $\mathrm{N}_k^{(m)}(\mathrm{F}) \subseteq \mathrm{N}_k^{(m+1)}(\mathrm{F})$ for all positive integers m and k.

Proof. Let $A \in \mathrm{N}_k^{(m)}(\mathrm{F})$. Note that (a) holds because every nilpotent matrix has zero trace, and the trace function is additive. Since nilpotence is preserved under multiplication by a nonzero scalar and under similarity, (b) and (c) hold. Since the direct sum of a strictly k-zero matrix and a strictly r-zero matrix is also strictly k-zero whenever $r \leq k$, (d) is true. If $B \in \mathrm{N}_k^{(1)}(\mathrm{F})$ and F has at least three elements, then there exists $a \in \mathrm{F} - \{0,1\}$ so that $B = aB + (1-a)B$ is a sum of two strictly k-zero matrices. Thus, (e) holds.

We state a consequence of Theorem 2 in P.A. Filmore (1969), which describes nonscalar matrices over arbitrary fields and with zero trace up to similarity.

Theorem 2. Let F be an arbitrary field. If $A \in M_n(\mathrm{F})$ is a nonscalar matrix with zero trace, then A is similar to a matrix with zero diagonal entries.

If $A \in M_n(\mathrm{F})$ is a nonscalar matrix with zero trace, then Theorem 2 guarantees that A is similar to a matrix B whose diagonal entries are zero. Now, B is a sum of an upper triangular matrix and a lower triangular matrix, each of which has diagonal entries zero. It follows that A is a sum of two nilpotent matrices. Conversely, because the trace of a nilpotent matrix is zero, if A is a sum of two nilpotent matrices, then the trace of A is zero. We summarize this in the following corollary.

Corollary 3. Let F be an arbitrary field and $A \in M_n(\mathrm{F})$ be a nonscalar matrix. Then $tr A = 0$ if and only if A is a sum of two nilpotent matrices.

Note that Corollary 3 is not true if we fix the index of nilpotence of the two summands. This is evident from Theorems 1 and 2 in J.D. Botha (2012). A polynomial $f \in \mathrm{F}[x]$ is said to be even-powered (odd-powered) if $f(x)=g(x^2)$ ($f(x)=xg(x^2)$) for some $g \in \mathrm{F}[x]$.

Theorem 4. Let $A \in M_n(\mathrm{F})$.

(a) If $char\,(\mathrm{F}) \neq 2$, then $A \in \mathrm{N}_2^{(2)}(\mathrm{F})$ if and only if A is similar to $-A$.

(b) If $char\,(\mathrm{F}) = 2$, then $A \in \mathrm{N}_2^{(2)}(\mathrm{F})$ if and only if all invariant factors of A are even-powered or odd-powered monic polynomials.

We include Proposition 1 in K. Takahashi (2000), which gives necessary and sufficient conditions for a diagonalizable complex matrix with two distinct eigenvalues to be expressible as a sum of three strictly square zero matrices.

Theorem 5. Let $A \in M_n(\mathrm{C})$ with $tr\,A = 0$ and assume that A is similar to

$$\bigoplus_{i=1}^{m} diag\ (\beta, \alpha) \oplus \alpha I_r ,$$

where $\alpha \neq \beta$, and m and r are positive integers. Then, $A \in \mathrm{N}_2^{(3)}(\mathrm{C})$ if and only if r is a divisor of $2m$.

We cite Theorems 1.1 and 1.3 in C.D. Pazzis (2017) concerning trace zero matrices over arbitrary fields, and over fields with characteristic two.

Theorem 6. Let F be a field and $A \in M_n(\mathrm{F})$.

(a) Then, $tr\,A = 0$ if and only if $A \in \mathrm{N}_2^{(4)}(\mathrm{F})$.

(b) If $char\,(\mathrm{F}) = 2$, then $tr\,A = 0$ if and only if $A \in \mathrm{N}_2^{(3)}(\mathrm{F})$.

By Proposition 1(a), if $A \in \mathrm{N}_m^{(k)}(\mathrm{F})$ for any positive integers m and k, then $tr\,A = 0$. Thus, it follows from Theorem 6 that, if $char\,(\mathrm{F}) = 2$, then $\mathrm{N}_2^{(3)}(\mathrm{F}) = \mathrm{N}_2^{(m)}(\mathrm{F})$ for all integers $m \geq 3$, and if $char\,(\mathrm{F}) \neq 2$, then $\mathrm{N}_2^{(3)}(\mathrm{F}) \subseteq \mathrm{N}_2^{(4)}(\mathrm{F}) = \mathrm{N}_2^{(m)}(\mathrm{F})$ for all integers $m \geq 4$.

Let $char\,(\mathrm{F}) = 2$. Let $n > 2$ and $A \in M_n(\mathrm{F})$ be the companion matrix associated with the monic polynomial $p(x) = x^n + x + 1$. Note that $p(x)$ is the only invariant factor of A, and $p(x)$ is neither odd-powered nor even-powered for all integers $n > 2$. Since the coefficient of x^{n-1} in $p(x)$ is zero, it follows that $tr\,A = 0$. By Theorems 4(b) and 6(b), it follows that $A \notin \mathrm{N}_2^{(2)}(\mathrm{F})$ and $A \in \mathrm{N}_2^{(3)}(\mathrm{F})$, respectively. Thus, we have that $\mathrm{N}_2^{(2)}(\mathrm{F}) \subset \mathrm{N}_2^{(3)}(\mathrm{F})$ whenever $char\,(\mathrm{F}) = 2$.

Let $char\,(\mathrm{F}) \neq 2$. Suppose $A \in M_n(\mathrm{F})$, such that $tr\,A = 0$ and A is not similar to $-A$. In Theorem 1.2 of C.D. Pazzis (2017), he proved that $B := A \oplus 0_n \in \mathrm{N}_2^{(3)}(\mathrm{F})$. However,

$B \notin N_2^{(2)}(F)$ by Theorem 4(a). Hence, it follows that $N_2^{(2)}(F) \subset N_2^{(3)}(F)$ whenever $char(F) \neq 2$. Therefore, $N_2^{(2)}(F) \subset N_2^{(3)}(F)$ for any field F.

MAIN RESULTS

Definition. Let $A \in M_n(F)$ and $B \in M_n(F)$. We define $A \boxtimes B$ by

$$A \boxtimes B = \begin{bmatrix} 0_{m,n} & A \\ B & 0_{n,m} \end{bmatrix} \in M_{m+n}(F).$$

Note that $A \boxtimes B$ has full rank if and only if A and B have full ranks. Hence, $A \boxtimes B$ is nonsingular if and only if A and B are nonsingular. If m and n are positive integers, such that $m, n \geq 2$, then

$$J_n(0) = I_{n-1} \boxtimes 0_1 \text{ and } J_n(0) \oplus J_m(0) = (I_{n-1} \oplus 0_1 \oplus I_{m-1}) \boxtimes 0_1.$$

Observe that, if A is an upper triangular matrix, then $A \boxtimes 0_1$ is strictly upper triangular. Moreover, if $A, C \in M_m(F)$ and $B, D \in M_n(F)$, then

$$(A \boxtimes B) + (C \boxtimes D) = (A+C) \boxtimes (B+D),$$

and, if $m = n$, then

$$(A \boxtimes B)\ (C \boxtimes D) = \begin{bmatrix} AD & 0 \\ 0 & BC \end{bmatrix}$$

Matrices over fields with at least three elements

Throughout this section, we assume that F is a field with at least three elements.

Lemma 7. Let m be a positive integer. Then, $J_m(0) \in N_k^{(2)}(F)$ for any integer k with $2 \leq k \leq m$.

Proof. Let m and k be integers, such that $2 \leq k \leq m$. We consider two cases: $m \geq 2k-1$ and $m < 2k-1$.

Case 1. Suppose $m \geq 2k-1$. Since $k \geq 2$, we have $2k-2 > 0$. By the Division Algorithm, there exist nonnegative integers n and r, such that $m = n(2k-2) + r$ and $0 \leq r < 2k-2$. Consider the matrix

$$A = \bigoplus_{i=1}^{n} (J_k(0) \oplus 0_{k-2}) \oplus X_r,$$

where

$$X_r = \begin{cases} J_r(0), & \text{if } r \leq k \\ J_k(0) \oplus 0_{r-k}, & \text{if } k < r < 2k-2 \end{cases}$$

and X_r is absent if $r = 0$. Let s be a nonnegative integer, such that

$$s = \begin{cases} r-(k-1), & \text{if } r \geq k-1 \\ r+(k-1), & \text{if } r < k-1. \end{cases}$$

Define

$$Y_S = \begin{cases} J_s(0), & \text{if } 1 \leq s \leq k \\ J_k(0) \oplus 0_{s-k}, & \text{if } s > k, \end{cases}$$

and consider the matrix

$$B = \begin{cases} 0_{k-1} \oplus \left[\bigoplus_{i=1}^{n} (J_k(0) \oplus 0_{k-2})\right] \oplus Y_s, & \text{if } r \geq k-1 \\ 0_{k-1} \oplus \left[\bigoplus_{i=1}^{n-1} (J_k(0) \oplus 0_{k-2})\right] \oplus Y_s, & \text{if } r < k-1, \end{cases}$$

where Y_s is absent if $s = 0$. Then, $A, B \in M_m(\mathrm{F})$ are strictly k-zero matrices, such that $J_m(0) = A + B$.

Case 2. Suppose $m < 2k-1$. Then, it follows that $2k-m-1 \geq 1$ and $0 \leq m-k < k-1$. Since F has at least three elements, there exists $a \in \mathrm{F} - \{0, 1\}$. Then, we have

$$J_m(0) = [(I_{m-k} \oplus aI_{2k-m-1} \oplus 0_{m-k}) \boxtimes 0_1] + [(0_{m-k} \oplus (1-a)I_{2k-m-1} \oplus I_{m-k}) \boxtimes 0_1].$$

One can check that the two summands are strictly -zero matrices.

Note that $J_2(0) \notin \mathrm{N}_2^{(2)}(\mathrm{Z}_2)$. Otherwise, there exists $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(\mathrm{Z}_2)$, such that A and $J_2(0) - A$ are strictly square zero matrices, which implies that $0 = \det(J_2(0)\ A) = ad-(b-1)c = \det A + c = 0 + c$. That is, $c = 0$. Since $A^2 = 0$, we get $a^2 + bc = 0$ and $d^2 + bc = 0$.

Hence $a = d = 0$. Since $A \neq 0_2$ and $A \in M_2(\mathrm{Z}_2)$, we have $A = J_2(0)$, which is a contradiction because $J_2(0) - A$ is strictly square zero. Thus, Lemma 7 fails to hold when $\mathrm{F} = \mathrm{Z}_2$ and $m = 2$. However, by Theorem 6, we have $J_2(0) \in \mathrm{N}_2^{(3)}(\mathrm{Z}_2)$. Moreover, by using Case 1 of Lemma 7, it follows that for any integer $n > 2$, $J_2(0) \in \mathrm{N}_2^{(2)}(\mathrm{Z}_2)$.

Lemma 8. Let $A \in M_n(\mathrm{F})$ with $A = \oplus_{i=1}^{m} J_{n_i}(0)$ and $n_1 \geq n_2 \geq \ldots \geq n_m \geq 2$. Then, $A \in \mathrm{N}_n(\mathrm{F})$.

Proof. Let $A \in M_n(\mathrm{F})$ with $A = \oplus_{i=1}^{m} J_{n_i}(0)$ and $n_1 \geq n_2 \geq \ldots \geq n_m \geq 2$. Suppose $a \in \mathrm{F} - \{0,1\}$. Then, we have

$$\begin{aligned} A = &\left(\bigoplus_{i=1}^{m-1} (aI_{n_i-1} \oplus [a]) \oplus aI_{n_m-1} \right) \boxtimes 0_1 \\ &+ \left(\bigoplus_{i=1}^{m-1} ((1-a)I_{n_i-1} \oplus [-a]) \oplus (1-a)I_{n_m-1} \right) \boxtimes 0_1. \end{aligned}$$

which is a sum of two strictly n-zero matrices.

In the previous lemma, we showed that, if $A \in M_n(\mathrm{F})$ is nilpotent without $J_1(0)$ blocks in its JCF, then A is a sum of two strictly n-zero matrices. We now show that the same A is a sum of two strictly k-zero matrices for $n - n_{j+1} < k < n$.

Lemma 9. Let $A \in M_n(\mathrm{F})$ with $A = \oplus_{i=1}^{j+1} J_{n_i}(0)$, where $n_1 \geq n_2 \geq \ldots \geq n_{j+1} \geq 2$. Let $l = \sum_{i=1}^{j} n_i$ and k be an integer, such that $l < k < n$. Then, $A \in \mathrm{N}_k^{(2)}(\mathrm{F})$.

Proof. Let $A \in M_n(\mathrm{F})$ with $A = \bigoplus_{i=1}^{j+1} J_{n_i}(0)$, where $n_1 \geq n_2 \geq \ldots \geq n_{j+1} \geq 2$. Let $l = \sum_{i=1} n_i$ and let k be an integer, such that $l < k < n$. There exists a unique positive integer k_1, such that $k = l + k_1$ and $0 < k_1 < n_{j+1}$. Note that

$$A = \bigoplus_{i=1}^{j+1} J_{n_i}(0) = \left[\bigoplus_{i=1}^{j} (I_{n_i-1} \oplus 0_1) \oplus I_{n_{j+1}-1} \right] \boxtimes 0_1.$$

Let $X = \bigoplus_{i=1}^{j} \left(I_{n_i-1} \oplus 0_1 \right) \oplus I_{n_{j+1}-1}$. Let $a \in \mathrm{F} - \{0,1\}$, and consider

$$Y = I_{n_{j+1}-k_1} \oplus aI_{n_1-(n_{j+1}-k_1+1)} \oplus [a] \oplus \left(\bigoplus_{i=2}^{j} \left(aI_{n_i-1} \oplus [a] \right) \right) \oplus aI_{k_1-1} \oplus 0_{n_{j+1}-k_1}$$

and

$$\begin{aligned} Z = &\ 0_{n_{j+1}-k_1} \oplus (1-a)I_{n_1-(n_{j+1}-k_1+1)} \oplus [-a] \\ &\oplus \left(\bigoplus_{i=2}^{j} \left((1-a)I_{n_i-1} \oplus [-a] \right) \right) \oplus (1-a)I_{k_1-1} \oplus I_{n_{j+1}-k_1}. \end{aligned}$$

Observe that $X, Y, Z \in M_{n-1}$(F) and that Y and Z have rank k–1. Since $X = Y + Z$ and $A = X \boxtimes 0_1$, we have $A = (Y \boxtimes 0_1) + (Z \boxtimes 0_1)$. Moreover, $Y \boxtimes 0_1$ and $Z \boxtimes 0_1$ are strictly k-zero.

We now apply Lemmata 7 – 9 to show that the n-by-n matrix described in Lemmata 8 and 9 is a sum of two strictly k-zero matrices for $2 \leq k < n$.

Lemma 10. Let $A \in M_n$ (F) with $A = \oplus_{i=1}^{m} J_{n_i}(0)$ and $n_1 \geq n_2 \geq \ldots \geq n_m \geq 2$. Then, $A \in \mathrm{N}_k^{(2)}$ (F) for any integer k with $2 \leq k < n$.

Proof. Let $A \in M_n$ (F) with $A = \oplus_{i=1}^{m} J_{n_i}(0)$ and $n_1 \geq n_2 \geq \ldots \geq n_m \geq 2$. Suppose k and n are integers, such that $2 \leq k < n$. We consider five cases.

Case 1. Suppose $k \leq n_m$. Then, $k \leq n_i$ for all $i = 1, \ldots, m$. By Lemma 7, we can write $J_{n_i}(0) = A_i + B_i$, where $A_i, B_i \in M_{n_i}$ (F) are strictly k-zero. Hence,

$$A = \bigoplus_{i=1}^{m} A_i + \bigoplus_{i=1}^{m} B_i$$

is a sum of two strictly k-zero matrices.

Case 2. Suppose $k = n_j$ for some $j \in \{1, \ldots, m-1\}$. Then, $n_i \leq k$ for all $i > j$ so that $J_{n_i}(0)$ is k-zero for all $i > j$, and hence, $\oplus_{i=j+1}^{m} J_{n_i}(0)$ is k-zero. For $\imath \leq j$, we have $n_i \geq k$. If $\imath \leq j$, then, by Lemma 7, we can write $J_{n_i}(0) = A_i + B_i$, where $A_i, B_i \in M_{n_i}$ (F) are strictly k-zero. Thus,

$$A = \left(\bigoplus_{i=1}^{j} A_i \oplus \bigoplus_{i=j+1}^{m} J_{n_i}(0) \right) + \left(\bigoplus_{i=1}^{j} B_i \oplus 0_{n_{j+1}+\ldots+n_m} \right)$$

is a sum of two strictly k-zero matrices.

Case 3. Suppose $n_m < k \leq n_1$ and $k \neq n_j$ for all $j \in \{1, \ldots, m-1\}$. Then, there exists a unique $j \in \{1, \ldots, m-1\}$, such that $n_j > k > n_{j+1}$. For $i > j$, we have $n_i < k$, so that $J_{n_i}(0)$ is k-zero for all $i > j$ and $\oplus_{i=j+1}^{m} J_{n_i}(0)$ is k-zero. For $\imath \leq j$, we have $n_i > k$. Thus, by Lemma 7, we can write $J_{n_i}(0) = A_i + B_i$, where $A_i, B_i \in M_{n_i}$ (F) are strictly k-zero. As in Case 2, we conclude that $\oplus_{i=1}^{m} J_{n_i}(0) \in \mathrm{N}_k^{(2)}$(F) .

Case 4. Suppose $k = \sum_{i=1}^{j} n_i$ for some $j \in \{2, \ldots, m\}$. By Lemma 8, we can write $\oplus_{i=1}^{m} J_{n_i}(0) = X + Y$, where $X, Y \in M_k$(F) are strictly k-zero. Note that $k > n_1 \geq n_i$, so that $J_{n_i}(0)$ is k-zero for all $i = 1, \ldots, m$. Thus, if $j < m$, then $\oplus_{i=j+1}^{m} J_{n_i}(0)$ is k-zero, and

$$\bigoplus_{i=1}^{m} J_{n_i}(0) = \left(X \oplus \bigoplus_{i=j+1}^{m} J_{n_i}(0) \right) + \left(Y \oplus 0_{n_{j+1}+\ldots+n_m} \right),$$

which is a sum of two strictly k-zero matrices. If $j = m$, then $k = n$ and $A \in \mathrm{N}_n^{(2)}$(F) by Lemma 8.

Case 5. Suppose $n_1 < k < n$ and $k \neq n_1+n_2+...+n_i$ for all $i \in \{2,..., m\}$. Then, there exist a unique positive integer k_1 and a unique $j \in \{2,..., m-1\}$, such that $k = \sum_{i=1}^{j} n_i + k_1$ and $0 < k_1 < n_{j+1}$. Let $l = \sum_{i=1}^{j} n_i$, so that $k = l + k_1$ and $l < k < \sum_{i=1}^{j+1} n_i$. If $j = m-1$, then we are done by Lemma 9. Since $n_1 < k$, it must be that $n_1 < k$ for all $i = 1, ...,$ m, which means that $J_{n_i}(0)$ is k-zero for all $i = 1, ..., m$. If $j < m-1$, then $\oplus_{i=j+2}^{m} J_{n_i}(0)$ is k-zero. By Lemma 9, we can write $\oplus_{i=1}^{j+1} J_{n_i}(0) = X+Y$, where $X, Y \in M_{l+n_{j+1}}(\mathrm{F})$ are strictly k-zero, and so

$$A = \left(X \oplus \bigoplus_{i=j+2}^{m} J_{n_i}(0) \right) + \left(Y \oplus 0_{n-l-n_{j+1}} \right),$$

which is a sum of two strictly k-zero matrices. Since we have exhausted all possible cases, we conclude that $A = \oplus_{i=1}^{m} J_{n_i}(0) \in \mathrm{N}_k^{(2)}(\mathrm{F})$ for any integer k with $2 \leq k < n$.

Thus far, the matrix A considered in Lemmata 8–10 did not include $J_1(0)$ blocks. The following shows that, when $J_1(0)$ blocks are included, then $A \oplus 0_r$ is a sum of two strictly k-zero matrices for $2 \leq k < n + r$.

Lemma 11. Let $A \in M_n(\mathrm{F})$ with $A = \oplus_{i=1}^{m} J_{n_i}(0)$ and $n_1 \geq n_2 \geq ... \geq n_m \geq 2$. If r is a positive integer, then $A \oplus 0_r \in \mathrm{N}_k^{(2)}(\mathrm{F})$ for any integer k with $2 \leq k < n + r$.

Proof. Let $A \in M_n(\mathrm{F})$ with $A = \oplus_{i=1}^{m} J_{n_i}(0)$ and $n_1 \geq n_2 \geq ... \geq n_m \geq 2$. Let r be a positive integer. Let k be an integer, such that $2 \leq k < n + r$. We consider two cases.

Case 1. Suppose $2 \leq k \leq n$. If we apply Lemmata 8 and 10 to A, we can write $A = X + Y$, where X and Y are strictly k-zero matrices. Thus, we have $A \oplus 0_r = (X \oplus 0_r) + (Y \oplus 0_r)$, which is a sum of two strictly k-zero matrices.

Case 2. Suppose $n < k \leq n + r$. Suppose $a \in \mathrm{F} - \{0,1\}$. Then,

$$\begin{aligned} A \oplus 0_r &= \left(\bigoplus_{i=1}^{m-1} \left(aI_{n_i-1} \oplus [a] \right) \oplus aI_{n_m-1} \oplus aI_{k-n} \oplus 0_{r+n-k} \right) \boxtimes 0_1 \\ &+ \left(\bigoplus_{i=1}^{m-1} \left((1-a)I_{n_i-1} \oplus [-a] \right) \oplus (1-a)I_{n_m-1} \oplus -aI_{k-n} \oplus 0_{r+n-k} \right) \boxtimes 0_1 \end{aligned}$$

is a sum of two strictly -zero matrices.

We summarize Lemmata 7-11 in the following theorem.

Theorem 12. Let F be a field with at least three elements. If $A \in M_n(\mathrm{F})$ is nilpotent, then $A \in \mathrm{N}_k^{(2)}(\mathrm{F})$ for all integers k, such that $2 \leq k < n$.

Proof. Let $A \in M_n(\mathrm{F})$ be nilpotent. Suppose k is an integer with $2 \le k \le n$. By the Jordan Canonical Form theorem, A is similar to $\oplus_{i=1}^{m} J_{n_i}(0)$, where $n_1 \ge n_2 \ge \ldots \ge n_m$. If $n_m \ge 2$, then, by Lemmata 8 and 10, we have $\oplus_{i=1}^{m} J_{n_i}(0) \in \mathrm{N}_k^{(2)}(\mathrm{F})$ for any integer k with $2 \le k \le n$. Otherwise, let $j \in \{1,\ldots,m\}$ be least, such that $n_j = 1$. Then, for all $i \ge j$, we have $n_i = 1$. That is, the Jordan Canonical Form of A is $\bigoplus_{i=1}^{j-1} J_{n_i}(0) \oplus 0_{m-j+1}$. By Lemma 11, we conclude that $\bigoplus_{i=1}^{j-1} J_{n_i}(0) \oplus 0_{m-j+1} \in \mathrm{N}_k^{(2)}(\mathrm{F})$ for any integer k with $2 \le k \le n$. By Proposition 1(c), $A \in \mathrm{N}_k^{(2)}(\mathrm{F})$ for any integer k with $2 \le k \le n$.

The following example shows that the converse of Theorem 12 does not hold. Let $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Then, $B = J_2(0)+J_2(0)^T$, but B is not nilpotent. As another example, take $A = B \oplus -B \in M_4(\mathrm{F})$. Suppose $a, b, x, y \in \mathrm{F}-\{0\}$. Observe that

$$A = \left[J_2(0) \oplus (-J_2(0))\right] + \left[J_2(0)^T \oplus (-J_2(0)^T)\right],$$

which is a sum of two strictly square zero matrices. Moreover,

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & -a & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

which is a sum of two strictly cube zero matrices, and

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -a & b \\ 0 & 0 & -x & -y \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & a & -b \\ 1 & 0 & x & y \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix},$$

which is a sum of two strictly four zero matrices. Therefore, $A \in \mathrm{N}_k^{(2)}(\mathrm{F})$ for $k = 2,3,4$, but A is not nilpotent, since A is nonsingular.

The remaining results in this section are applications of Corollary 3 and Theorem 12.

Theorem 13. Let F be a field with at least three elements and $A \in M_n(\mathrm{F})$ be a nonscalar matrix. Then, $trA = 0$ if and only if A is a sum of two strictly k-zero matrices and two strictly l-zero matrices for all integers k and l, such that $2 \le k, l \le n$.

Proof. Let F be a field with at least three elements and $A \in M_n(\mathrm{F})$ be a nonscalar matrix. Suppose $tr\,A = 0$, and k and l are integers, such that $2 \le k, l \le n$. By Corollary 3, we can write $A = X + Y$, where X and Y are nilpotent matrices. By Theorem 12, $X \in \mathrm{N}_k^{(2)}(\mathrm{F})$ and $Y \in \mathrm{N}_l^{(2)}(\mathrm{F})$ for any integers k, l, such that $2 \le k, l \le n$. Thus, A is a sum of two strictly k-zero matrices and two strictly l-zero matrices. The backward implication follows from Proposition 1(a).

If we take $k = l$ in Theorem 13, we get the following result.

Corollary 14. Let F be a field with at least three elements and $A \in M_n(\mathrm{F})$ be a nonscalar matrix. Then, $tr\,A = 0$ if and only if $A \in \mathrm{N}_k^{(4)}(\mathrm{F})$ for all integers k, such that $2 \le k \le n$. In particular, if $char(\mathrm{F}) = 0$ and $A \in M_n(\mathrm{F})$, then $tr\,A = 0$ if and only if $A \in \mathrm{N}_k^{(4)}(\mathrm{F})$ for all integers k, such that $2 \le k \le n$.

Corollary 15. Let F be a field with at least three elements and $A \in M_n(\mathrm{F})$ be a nonscalar matrix. Then, $tr\,A = 0$ if and only if $A \in \mathrm{N}_k^{(3)}(\mathrm{F})$ for some integer k, such that $2 \le k \le n$.

Proof. Let F be a field with at least three elements and $A \in M_n(\mathrm{F})$ be a nonscalar matrix. Suppose $tr\,A = 0$. By Corollary 3, we can write $A = X + Y$, where X and Y are nilpotent matrices. Since $A \neq 0_n$, at least one of X and Y is not zero. Suppose $X \neq 0_n$ with index of nilpotence k, where $2 \le k \le n$. By Theorem 12, $Y \in \mathrm{N}_l^{(2)}(\mathrm{F})$ for any integer l, such that $2 \le l \le n$. Choose $l = k$, so that $A \in \mathrm{N}_k^{(3)}(\mathrm{F})$. The backward implication follows from Proposition 1(a).

Corollary 16. Let F be a field with at least three elements. If $A = \lambda I_n$ with zero trace, then $A \in \mathrm{N}_k^{(5)}(\mathrm{F})$ for all integers k, such that $2 \le k \le n$.

Proof. Let F be a field with at least three elements. Suppose $A = \lambda I_n$ with $tr\,A = 0$. Let k be an integer, such that $2 \le k \le n$ and $N_k \in M_n(\mathrm{F})$ be a strictly k-zero matrix. Note that $A - N_k$ is a nonscalar matrix with zero trace. By Corollary 14, it follows that $A - N_k \in \mathrm{N}_k^{(4)}(\mathrm{F})$. Thus, $A \in \mathrm{N}_k^{(5)}(\mathrm{F})$ for all integers k, such that $2 \le k \le n$.

MATRICES OVER THE COMPLEX FIELD

By the discussion after Theorem 6, we know that $\mathrm{N}_2^{(2)}(\mathrm{C}) \subset \mathrm{N}_2^{(3)}(\mathrm{C})$. Let $0 \neq \alpha \in \mathrm{C}$. Suppose m and r are positive integers, such that r is not a divisor of $2m$. Let $\beta = -\frac{\alpha}{m}(m+r)$, and consider the matrix $B = \bigoplus_{i=1}^{m} diag(\beta, \alpha) \oplus \alpha I_r$ which has zero trace. By Corollary 14, $B \in \mathrm{N}_k^{(4)}(\mathrm{C})$ for all $k \in \{2,\ldots,2m+r\}$. But, $B \notin \mathrm{N}_2^{(3)}(\mathrm{C})$ by

Theorem 5. Thus, we have that $N_2^{(3)}(C) \subset N_2^{(4)}(C)$ by Proposition 1(e). Combining this with Corollary 14 gives us the following result.

Corollary 17. The least positive integer m, such that every $A \in M_n(C)$ with zero trace is a sum of m strictly k-zero matrices for all integers k, such that $2 \le k \le n$ is $m = 4$.

The following theorem deals with n-by-n matrices which are not expressible as a sum of two strictly k-zero matrices for some $2 \le k \le n$, but can be augmented so that it is a sum of two strictly l-zero matrices for $2 \le l \le n$.

Theorem 18. Let $A \in M_n(C)$. Suppose k is an integer, such that $2 \le k \le n$, and $A \notin N_k^{(2)}(C)$. Then, there exist a positive integer m and $B \in M_m(C)$, such that $A \oplus B \in N_l^{(2)}(C)$ for all $2 \le l \le n$.

Proof. Let $A \in M_n(C)$. Suppose k is an integer, such that $2 \le k \le n$ and $A \notin N_k^{(2)}(C)$. By the Jordan Canonical Form theorem, A is similar to $X \oplus Y$, where $X \in M_r(C)$ is nonsingular and $Y \in M_s(C)$ is nilpotent. By Theorem 12, it follows that A is not nilpotent, which in turn implies that $r > 0$. Let m be an integer, such that $m \ge r$. Then, $m > 0$ and $m + s \ge n$. Let $B = -X \oplus 0_m \in M_{r+m}(C)$. Then, $A \oplus B \in M_{n+r+m}(C)$ with $tr\,(A \oplus B) = 0$ and $A \oplus B$ is similar to $(X \oplus -X) \oplus (Y \oplus 0_m)$. By Theorem 4, there exist strictly square zero matrices $U, V \in M_{2r}(C)$, such that $X \oplus -X = U + V$. By Theorem 12, for any integer l, such that $2 \le l \le n$, there exist strictly l-zero matrices $M_l, N_l \in M_n(C)$, such that $Y \oplus 0_{n-s} = M_l + N_l$. Thus, we have

$$(X \oplus -X) \oplus (Y \oplus 0_m) = (U \oplus M_l \oplus 0_{m-n+s}) + (V \oplus N_l \oplus 0_{m-n+s})\,.$$

If we let $C = U \oplus M_l \oplus 0_{m-n+s}$ and $D = V \oplus N_l \oplus 0_{m-n+s}$, which are strictly l-zero matrices, then $(X \oplus -X) \oplus (Y \oplus 0_m) = C + D$. By Proposition 1(d), we conclude that $A \oplus B \in N_l^{(2)}$ for any integer l with $2 \le l \le n$.

The last theorem concerns diagonalizable n-by-n matrices, which are not expressible as a sum of three strictly square zero matrices, but can be augmented to form either a sum of two strictly square zero matrices or a sum of three strictly square zero matrices.

Theorem 19. Let $A \in M_n(C)$ with $tr\,A = 0$ and assume that A is similar to

$$\bigoplus_{i=1}^{m} diag\,(\beta, \alpha) \oplus \alpha I_r\,,$$

where $\alpha, \beta \in \mathbb{C}$, $\alpha \neq \beta$, and m and r are positive integers. Suppose $A \notin N_2^{(3)}(\mathbb{C})$.

(a) There is no diagonalizable matrix B with $tr\, B = 0$ and $\sigma(B) = \{\alpha, \beta\}$, such that $A \oplus B \in N_2^{(3)}(\mathbb{C}) - N_2^{(2)}(\mathbb{C})$.

(b) There exist a positive integer s and a matrix $B \in M_s(\mathbb{C})$, such that $A \oplus B \in N_2^{(2)}(\mathbb{C})$.

(c) There exist a positive integer t and a matrix $C \in M_t(\mathbb{C})$, such that $A \oplus C \in N_2^{(3)}(\mathbb{C}) - N_2^{(2)}(\mathbb{C})$.

Proof. Let $A \in M_n(\mathbb{C})$ with $tr\, A = 0$ and assume A is similar to

$$\oplus_{i=1}^{m} diag(\alpha, \beta) \oplus \alpha I_r$$

where $\alpha, \beta \in \mathbb{C}$, $\alpha \neq \beta$, and m and r are positive integers. Then, $\alpha \neq 0$, $\beta \neq 0$, and $\alpha + \beta \neq 0$. Suppose $A \notin N_2^{(3)}(\mathbb{C})$. Then, by Theorem 5, r is not a divisor of $2m$.

We prove (a). Suppose there is a diagonalizable matrix B with $tr\, B = 0$ and $\sigma(B) = \{\alpha, \beta\}$, such that $A \oplus B \in N_2^{(3)}(\mathbb{C}) - N_2^{(2)}(\mathbb{C})$. We divide the proof into three cases depending on the algebraic multiplicities of α and β in B.

Case 1. Suppose the algebraic multiplicity of β exceeds that of α in B. Then, there exist positive integers s and t, such that B is similar to

$$\oplus_{i=1}^{s} diag(\alpha, \beta) \oplus \beta I_t .$$

By Proposition 1(c), we can assume without loss of generality that

$$A \oplus B = \bigoplus_{i=1}^{m+s} diag\,(\alpha, \beta) \oplus \alpha I_r \oplus \beta I_t \in N_2^{(3)}(\mathbb{C}) - N_2^{(2)}(\mathbb{C}).$$

If $r<t$, then $A \oplus B$ is similar to $\bigoplus_{i=1}^{m+s+r} diag(\alpha, \beta) \oplus \beta I_{t-r}$, and $t - r$ divides $2(m+s+r)$ by Theorem 5, say $2(m+s+r) = k(t-r)$ for some positive integer k. Since $tr(A \oplus B)=0$, we have $(\alpha+\beta)(m+s+r) + \beta(t-r)=0$. Hence,

$$m + s + r = \left(-\frac{\beta}{\alpha + \beta}\right)(t - r) ,$$

and it follows that $k = -\dfrac{2\beta}{\alpha + \beta}$ is a positive integer. Consequently, we have $\beta = -\left(\dfrac{k}{k+2}\right)\alpha$, and $\alpha + \beta = \dfrac{2\alpha}{k+2}$. Since $tr\,(A) = 0$, we have $0 = (\alpha+\beta)m + \alpha r = \alpha\left(\dfrac{2m}{k+2} + r\right)$. Since $\alpha \neq 0$, we have $\dfrac{2m}{k+2} + r = 0$, which is a contradiction, since k, m, and r are positive integers.

If $r > t$, then the roles of α and β in the previous subcase are reversed, and we get $A \oplus B = \bigoplus_{i=1}^{m+s+t} diag(\alpha, \beta) \oplus \alpha I_{r-t}$. A similar argument as in the subcase $r < t$ leads to a contradiction.

If $t = r$, then it follows that $A \oplus B = \bigoplus_{i=1}^{m+s+r} diag(\alpha, \beta)$. Since $tr(A \oplus B) = (m+s+r)(\alpha+\beta) = 0$, it follows that $\alpha + \beta = 0$, which is a contradiction.

Case 2. Suppose the algebraic multiplicity of α exceeds that of β in B. Then, there exist positive integers s and t, such that B is similar to

$$\bigoplus_{i=1}^{s} diag\ (\alpha, \beta) \oplus \alpha I_r.$$

By Proposition 1(b), we can assume without loss of generality that

$$A \oplus B = \bigoplus_{i=1}^{m+s} diag(\alpha, \beta) \oplus \alpha I_{r+t} \in \mathrm{N}_2^{(3)}(\mathrm{C}) - \mathrm{N}_2^{(2)}(\mathrm{C}).$$

A similar argument in the previous case leads to a contradiction.

Case 3. Suppose the algebraic multiplicities of α and β are equal. Then, there exists a positive integer s, such that B is similar to

$$\bigoplus_{i=1}^{s} diag(\alpha, \beta).$$

By Proposition 1(c), we can assume without loss of generality that

$$A \oplus B = \bigoplus_{i=1}^{m+s} diag(\alpha, \beta) \oplus \alpha I_r \in \mathrm{N}_2^{(3)}(\mathrm{C}) - \mathrm{N}_2^{(2)}(\mathrm{C}).$$

A similar argument in Case 1 leads to a contradiction. Since we have exhausted all possible cases, we conclude that there is no diagonalizable matrix B with $tr\, B = 0$ and $\sigma(B) = \{\alpha, \beta\}$, such that $A \oplus B \in \mathrm{N}_2^{(3)}(\mathrm{C}) - \mathrm{N}_2^{(2)}(\mathrm{C})$.

We prove (b). Take $B = -A \in M_n(\mathrm{C})$. By Theorem 4, we conclude that $A \oplus B \in \mathrm{N}_2^{(2)}(\mathrm{C})$.

We prove (c). Let p be a positive integer and $B = \bigoplus_{i=1}^{p} diag(\alpha, -2\alpha) \oplus \alpha I_p$. Note that $trB = 0$ and p divides $2p$. By Theorem 5, $B \in \mathrm{N}_2^{(3)}(\mathrm{C})$. Let $C = -A \oplus B \in M_t(\mathrm{C})$, where $t = n + 3p$. Then, we have $A \oplus C = A \oplus -A \oplus B$. Since $\mathrm{N}_2^{(2)}(\mathrm{C}) \subseteq \mathrm{N}_2^{(3)}(\mathrm{C})$, it follows that $A \oplus -A \oplus \in \mathrm{N}_2^{(3)}(\mathrm{C})$. Since $A \oplus -A$ and B are both elements of $\mathrm{N}_2^{(3)}(\mathrm{C})$, we conclude that $A \oplus -A \oplus B = A \oplus C \in \mathrm{N}_2^{(3)}(\mathrm{C})$. By Theorem 4, it follows that $A \oplus C \notin \mathrm{N}_2^{(2)}(\mathrm{C})$.

Note that s and B in Theorem 19(b) are not unique, because if $N \in M_m(\mathbb{C})$ is a nilpotent matrix, then we can also take $B = -A \oplus N$. Since p is arbitrary in the proof of Theorem 19(c), we remark that t and B are not unique. Moreover, we can take any $B \in \mathrm{N}_2^{(3)}(\mathbb{C}) - \mathrm{N}_2^{(2)}(\mathbb{C})$ so that $A \oplus C \in \mathrm{N}_2^{(3)}(\mathbb{C}) - \mathrm{N}_2^{(2)}(\mathbb{C})$.

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REFERENCES

Botha JD. 2012. Sums of two square-zero matrices over an arbitrary field. Linear Algebra and its Applications. 436(3):516-524.

Fillmore PA. 1969. On similarity and the diagonal of a matrix. The American Mathematical Monthly. 76(2):167-169.

Pazzis CD. 2017. A note on sums of three square-zero matrices. Linear and Multilinear Algebra. 65(4);787-805.

Takahashi K. 2000. On sums of three square-zero matrices. Linear Algebra and its Applications. 306(1-3):45-57.

Wang JH, Wu PY. 1991. Sums of square-zero operators. Studia Mathematica. 99(2):115-127.

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