Derivation of Third Order MHD Equations

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ABSTRACT

As a continuation and improvement on previous works, we derive a third order MHD equation through a projection and perturbation formalism that we will apply to various MHD flows. It is shown that the model can be linked to the theory where structure and geometry of the particle plays a role in explaining turbulence.

Key words: projection techniques, turbulence, non-equilibrium phase transition

INTRODUCTION

The standard MHD equations can be solved analytically if applied to some of MHD flows with the theory in good agreement with experiments as long as the flow remained laminar. However, any agreement breaks down whenever the flow becomes turbulent. Consequently the standard MHD equations cannot be used for the description of turbulent flows. The reason may be found in the common assumption of hard-sphere structureless particles used in their derivation. Elastic collisions between the particles of the fluid do not affect the form of standard MHD equations.

Here we derive third order MHD equation containing two control parameters related to the internal structure and geometry of the particle. The control parameter related to the internal structure was previously found to be inversely proportional to the critical Reynolds number and thus dependent on energy gap between ground and excited states of the particles if one adopts the hypothesis of quantum origin of turbulence. There are dissipative effects due to the excitations of internal degrees of freedom of the particle leading to inelastic collisions. In case no excitations occur the collisions remain elastic.

PRELIMINARIES

Following the projection formalism of Zwanzig, Dresden, and Muriel and the perturbation procedure outlined in the kinetic equations correct to the appropriate order k=0,1,2 are derived from the Liouville equation for N-particle distribution function

$$i\frac{\partial f^{(N)}}{\partial t} = Lf^{(N)} \qquad (1)$$

where the Liouville operator L may be written as

$$L = L_o + \lambda L_i + L_e, \text{ with } L_0 = -i\sum \frac{\vec{p}_i}{m}\vec{\nabla}_{r_i},$$

$$L_i = \frac{1}{2} i \sum_{j\neq j'} \vec{\nabla}_{r_j} V_{jj'} (\vec{\nabla}_{p_j} - \nabla_{p_{j'}}) \text{ and } L_e = -i\sum_i F_i \vec{\nabla}_{p_i}.$$

Define a projector, $P = \frac{1}{\Omega^{N-1}}\int\ldots\int d\vec{r}_2\ldots d\vec{r}_N$

where Ω is the volume of the system. The complementary projector is $1-P$. Applying both projectors to the Liouville equation, it may be reduced to an exact equivalent of the first equation in the BBGKY hierarchy of equations for the one particle distribution function

$$\frac{\partial f}{\partial t} = -i(L_o + L_e)f - i\lambda PL_i G(1-P)f^{(N)}(0)$$
$$-\lambda^2 PL_i \int_0 G(t-s)L_i f(s)ds \qquad (2)$$

We then take a simple approach. Formally expand the distribution function $f=\sum_{n=0}^{\infty}\lambda^k f^{(k)}$ in orders of l and the propagator $G = e^{i(t-s)L}$ in Taylor series, and then substitute into Eq. 2. Then we pick the terms to the appropriate order *k*.

THIRD ORDER PERTURBATION THEORY

Zero order (k=0). Zero-order kinetic equation for the one-particle, (electron r ion) distribution function $f_{\mp}^{(0)}$ is the Boltzmann equation with BBGKY-like elastic collision term in the presence of total electromagnetic field f $\frac{\partial f_{\mp}^{(0)}}{\partial t} + \frac{\vec{p}}{m}\vec{\nabla} f_{\mp}^{(0)} + \vec{F}\vec{\nabla}_p f_{\mp}^{(0)} = (\frac{\partial f_{\mp}}{\partial t})_{coll}$

with $\vec{F} = \mp e(\vec{E} + \frac{\vec{p}\times\vec{B}}{mc})$ as the Lorentz force acting on a charged particle, which can be reduced to the standard MHD equation.

First order (k=1). We obtain MHD equation with one correction term, $\partial_j < U_i U_j >_t$, known as the Reynolds MHD equation. Corresponding Fourier transformed form is used in most theories of turbulence. The turbulence is considered merely as a flow phenomenon and the particle structure is here unimportant or unnecessary. The first order theory does not tell us anything about the transition from the laminar to turbulent regime.

Second order (k=2). The second order kinetic equation can be reduced to an MHD equation for mean velocity of the fluid $\vec{U}$ using a renormalization due to McComb.

$$\rho\left(\frac{\partial \vec{U}}{\partial t} + (\vec{U}\vec{\nabla})\vec{U}\right) + \vec{\nabla}p - \rho\nu\vec{\nabla}^2\vec{U} - \frac{\vec{j}\times\vec{B}}{c} =$$

$$-\frac{1}{m}\int_0^t b(t-s)\vec{\nabla}\rho\, ds - \frac{1}{m}\int_0^t b(t-s)^2\,\vec{\nabla}^2\vec{U}ds \quad (4)$$

where p is the mean pressure, ρ is the density, ν is the kinematic viscosity, the parameter $b = {}_{(\Omega)}\int(\vec{\nabla}V)^2\, d\Omega$ with Ω as the volume and V is the collision interaction potential. We have b=0 for elastic collisions. This suggests b≥0 is a measure of the strength of inelastic interactions and is related to the internal structure of the molecule. The quantum kinetic model of turbulence serves as a basis in relating to other internal structure parameters. The parameter b becomes significant in the turbulent regime and an application such as the Hartmann flow is used to illustrate the effect of b. Numerical simulation has shown intersecting velocity profiles indicative of non-equilibrium phase transition from laminar to turbulent flow.

Third order (k=3). To obtain the third order kinetic equation we pick simply terms containing λ^3 in Eq.2 using the appropriate expansions for one-particle distribution function and propagator. The third order equation may be written as $\frac{\partial f_{\mp}^{(3)}}{\partial t} = -i(L_0 + L_e)f_{\mp}^{(3)} - PL_i\int_0^t\{\frac{1}{6}i(t-s)^3[(L_0^2L_i^2 + L_0L_iL_0L_i + L_iL_0^2L_i)f_{\mp}^{(0)}(s) - L_0^3L_if_{\mp}^{(1)}(s)\}ds$ + terms with zero contribution if multiplied by one component of momentum and integrated over momentum space.

Using explicit forms of the Liouville operators we may rewrite Eq. 5 as

$$\frac{\partial f_{\mp}^{(3)}}{\partial t} + \frac{\vec{p}}{m}\vec{\nabla}f_{\mp}^{(3)} + \frac{\vec{F}}{m}\vec{\nabla}_p f_{\mp}^{(3)}$$

$$(\frac{\partial f_{\mp}^{(3)}}{\partial t})_{coll} - \frac{1}{m}\int_0^t b(t-s)[\vec{\nabla}_p\vec{\nabla} + \vec{p}\vec{\nabla}^2{}_p\vec{\nabla}]f_{\mp}^{(1)}(s)ds$$

$$-\frac{1}{2m}\vec{\nabla}^2\int_0^t b(t-s)^2[2\vec{p}\vec{\nabla}_p + \vec{p}^2\vec{\nabla}^2{}_p]f_{\mp}^{(1)}(s)ds$$

$$-\frac{2}{3m^2}\vec{\nabla}^3\int_0^t b(t-s)^3[3\vec{p}^2\vec{\nabla}_p + \vec{p}^3\vec{\nabla}^2{}_p]f_{\mp}^{(1)}(s)ds$$

$$-\frac{2}{m^2}\vec{\nabla}^2\int_0^t \vec{c}(t-s)^3[2\vec{p}\vec{\nabla}^2{}_p + \vec{p}^2\vec{\nabla}^3{}_p]f_{\mp}^{(0)}(s)ds \qquad (5)$$

where the new vector parameter $\vec{c} = \int\limits_{(\Omega)} (\vec{\nabla} V)^3 \, d\Omega$ is related to the geometry of the particle and is interpreted as a measure of the asymmetry of the particle. It vanishes for perfectly spherical particle. Multiplying Eq.6 by a component of the momentum and integrating over momentum space, the third order MHD equation may be written as

$$\rho[\frac{\partial \vec{U}}{\partial t} + (\vec{U}\vec{\nabla})\vec{U}] + \vec{\nabla}p - \nu\rho\vec{\nabla}^2\vec{U} - \frac{\vec{j}\times\vec{B}}{c} =$$

$$-\frac{1}{m}\int_0^t b[(t-s)\vec{\nabla}\rho + (t-s)^2\vec{\nabla}^2\vec{U}]ds$$

$$-\frac{4}{m^2}\int_0^t (t-s)^3[5\vec{c}\vec{\nabla}^2\rho + b\dot{\nabla}^3\vec{U}^2]ds \tag{6}$$

using renormalization of the density, pressure and velocity to their true values due to McComb. Eq.7 is the MHD equation with proposed correction terms in integral form correct up to the third order. It contains the second order equation and there are new correction terms with a vector parameter $\vec{c}$, a measure of the asymmetry of the particle and scalar b. It is interesting to observe that the geometry of the particle does not affect the flow of an incompressible fluid; however there is a new term with control parameter $b \propto R^{-1}_{\text{critical}}$ affecting the flow. The relation of the assymetry parameter $\vec{c}$, to the critical Reynolds number is not yet known. If coupled with Maxwell equations it may be reduced after somewhat lengthy but elementary procedure to a fifth order non-linear differential equation for Hartmann or Couette flow using a configuration where $\vec{U} = (U,0,0), \vec{B} = (0,0,B_0)$,

$$\frac{\rho}{\nu}\frac{\partial^5 U}{\partial t^5} + \frac{\sigma B_0^2}{\nu c^2}\frac{\partial^4 U}{\partial t^4} - \frac{\partial^6 U}{\partial t^4 z^2} - \frac{2b}{m\nu}\frac{\partial^3 U}{\partial t \partial z^2}$$

$$-\frac{4!2b}{m^2}\left(3\frac{\partial U}{\partial z}\frac{\partial^2 U}{\partial z^2} + U\frac{\partial^3 U}{\partial z^3}\right) = 0 \tag{7}$$

where s is the conductivity. We expect a solution containing at most five incommensurable frequencies describing chaotic behaviour of the system. This is improvement over second order for the numeric solutions as well as in the time validity.

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