Static Behaviors of Confined Time-Arrival Operators

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ABSTRACT

We show that the quantization of the classical Time-of-Arrival (TOA) for arbitrary position X still leads to a class of self-adjoint TOA-operator for a confined particle. The spectrum of the TOA-operator is studied for different cases.

INTRODUCTION

When does a given particle prepared in some initial quantum state arrive at a given spatial point?

In standard quantum formalism, this raises the time-of-arrival (TOA) at the level of quantum observable where the TOA distribution is supposedly derivable from the spectral resolution of a self-adjoint TOA-operator canonically conjugated to the driving Hamiltonian. Recently, Galapon (2000) has shown that objections in constructing such TOA operators, due to Pauli's Theorem, do not hold within the single Hilbert space formulation of quantum mechanics. Also, researchers have evidently not been discouraged from seeking an expression for the TOA distribution within a consistent theoretical framework (Muga & Leavens, 2000).

We construct the TOA-operator for arbitrary detector position as a generalization of the operator constructed for the detector position at $X = 0$ (Galapon, 2002), which has shown that a class of self-adjoint and canonical TOA operator can be constructed for a spatially confined particle in the interval $[-l,+l]$.

By considering the symmetry properties of the constructed TOA operator, theoretical predictions for the probability distributions were obtained and compared with numerical results.

CONFINED TOA AND TOA OPERATORS

The TOA at X of a classical particle with position q, momentum p, and mass μ is given by

$$t = -\frac{\mu}{p}(q - X) \qquad (1)$$

Symmetrizing the classical expression (Eq. (1)) for the TOA at X gives (Muga & Leavens, 2000)

$$T = -\frac{\mu}{2}\left[(q - X)P^{-1} + P^{-1}(q - X)\right] \qquad (2)$$

in which T, q, and P are the operator versions of t, q, and p, respectively.

We attach the Hilbert space $H = L^2[-l,l]$. The position operator is unique and is given by the bounded multiplicative operator, q, whose domain is the entire Hilbert space. We rename the momentum operator P

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by $P_\gamma = -i\hbar\frac{\partial}{\partial q}$ with domain $D(P_\gamma) = \{\phi \in H : \phi' \in H,$ $\phi'(-l) = \exp(-2i\gamma)\phi'(l)\}$. The Hamiltonian operator is $H_\gamma = -\frac{\hbar^2}{2\mu}\partial_q^2\phi$ whose domain is $D(H_\gamma) = \{\phi \in D(P_\gamma) : \phi''(q) \in H, \phi'(-l) = \exp(-2i\gamma)\phi'(l)\}$.

We consider $\gamma \in (-\pi, \pi)$ to cover the entire symmetry of the classical TOA in the quantum domain. Different values of γ correspond to different physics. We also rename T by T_γ, such that Eq. (2) becomes

$$T_\gamma = -\frac{\mu}{2}\left[(q - X)P_\gamma^{-1} + P_\gamma^{-1}(q - X)\right] \tag{3}$$

The momentum and the Hamiltonian operators commute and have a common set of eigenvectors,

$$\phi_\gamma(q) = \frac{1}{\sqrt{2l}}\exp\left(i(\gamma + n\pi)\frac{q}{l}\right), n = 0, \pm 1, \pm 2, \ldots,$$

and both have pure point spectra.

Non-periodic boundary condition

Since q appears in first power and X is just a parameter in Eq. (3), T_γ is an operator if the inverse of the momentum operator P_γ^{-1} exists. For this non-periodic case, zero is not an eigenvalue of P_γ, thus, the inverse of P_γ exists. P_γ is unbounded and self-adjoint, thus P_γ^{-1} is bounded, everywhere defined (by extension) and self-adjoint. Then it follows that for every $\gamma \in (-\pi, \pi)$, T_γ is bounded, everywhere defined, and is a symmetric operator. Thus, T_γ is self-adjoint.

In coordinate representation, Eq. (3) assumes the form of a Fredholm integral operator (Galapon, 2000)

$$T_\gamma\phi(q) = \int_{-l}^{l} T_\gamma(q, q')\phi(q')dq' \tag{4}$$

with the non-periodic kernel

$$T_\gamma(q, q') = -\frac{\mu}{4\hbar\sin\gamma}(q + q' - 2x) \left(\exp(i\gamma)H(q - q') + \exp(-i\gamma)H(q' - q)\right) \tag{5}$$

in which $H(q,q')$ is the Heaviside function. T_γ is canonically conjugate to H_γ in the canonical domain

$$D_c^\gamma = \left\{\phi(q) \in D(H_\gamma) : \int_{-l}^{l}\phi(q')dq' = 0, \phi^k(-l) = \phi^k(l) = 0, k = 0, 1\right\}. \tag{6}$$

Periodic boundary condition

For the periodic case, zero is an eigenvalue of the momentum operator, thus, the inverse of P_γ doesn't exist. But TOA is a valid question only if the particle is in motion, otherwise it goes nowhere. We then expect that the non-periodic kernel Eq. (5) has a finite part corresponding to the non-vanishing momentum components in the limit as $\gamma \to 0$. The finite part is extracted by removing the divergent contribution of the vanishing momentum eigenvalue (Galapon, 2002), such that the kernel becomes

$$T_0(q, q') = \frac{-i\mu}{4\hbar}(q - q' - 2x)\left[\operatorname{sgn}(q' - q) - \frac{1}{l}(q' - q)\right] \tag{7}$$

and the canonical domain for the Hamiltonian and TOA-operator, T_0

$$D_c^0 = \left\{\phi(q) \in D(H_0) : \int_{-l}^{l} q'\phi(q')dq' = 0, \phi^k(-l) = \phi^k(l) = 0, k = 0, 1\right\} \tag{8}$$

Both kernels corresponding to the non-periodic and the periodic boundary conditions are symmetric and bounded, reaffirming the self-adjointness of the TOA operators. Also, they are compact and the canonical domains are closed, such that the pair (H_γ,T_γ) forms a canonical pair on this closed subspace of the Hilbert space (Galapon, 2002).

CONFINED TIME-OF-ARRIVAL (TOA) SYMMETRIES

By symmetry consideration, we derive some properties of the confined TOA-operator and infer relationships among the eigenfunctions. We particularly consider the behavior of the TOA-operator on the actions of the parity operator, Π and the time reversal operator, Θ.

The actions of Π and Θ are $\Pi\varphi(q,t)=\varphi(-q,t)$ and $\Theta\varphi(q,t)=(q,-t)$, respectively, where $\varphi(q,t)=\varphi(q,0)$ are vectors of the Hilbert space (we particularly considered the initial state, $\varphi(q,0)$).

Non-periodic $(\gamma \neq \pi/2)$ case

The symmetries of the non-periodic TOA-operators follow directly from the invariance of their kernels under the following operations

$$T_{[\gamma,X]}(q,q',X)=-T^{*}_{[\gamma,-X]}(-q,-q',-X) \tag{9}$$

$$T_{[\pi-\gamma,X]}(q,q',X)=-T^{*}_{[\pi-\gamma,-X]}(q,q',X) \tag{10}$$

$$T_{[-\gamma,X]}(q,q',X)=-T^{*}_{[-\gamma,X]}(q,q',X) \tag{11}$$

We denote $T_{[\gamma,X]}$ as the TOA-operator at X for the case γ with the kernel in Eq. (4) as $T_{[\gamma,X]}(q,q',X)$ and $\varphi_{[\gamma,X]}$ as the corresponding eigenfunctions. For every $\varphi \in H$, it can be shown that $T_{[\gamma,-X]}\Theta\Pi\varphi=-\Theta\Pi T_{[\gamma,X]}\varphi$, and using Eq. (9), the probability density relations in coordinate and momentum representations are

$$\left|\varphi_{[\gamma,-X]}(q)\right|^{2}=\left|\varphi_{[\gamma,X]}(-q)\right|^{2} \tag{12}$$

$$\left|\varphi_{[\gamma,-X]}(k)\right|^{2}=\left|\varphi_{[\gamma,X]}(k)\right|^{2} \tag{13}$$

We also find $T_{[-\gamma,X]}\Theta\varphi=-\Theta T_{[\gamma,X]}\varphi$ and using Eq. (11) leads to

$$\left|\varphi_{[-\gamma,X]}(q)\right|^{2}=\left|\varphi_{[\gamma,X]}(q)\right|^{2} \tag{14}$$

$$\left|\varphi_{[-\gamma,X]}(k)\right|^{2}=\left|\varphi_{[\gamma,X]}(-k)\right|^{2} \tag{15}$$

Also, it can be shown that, $T_{[\pi-\gamma,X]}\Theta\varphi=-\Theta T_{[\pi-\gamma,X]}\varphi$ with the following probability distributions from Eq. (10)

$$\left|\varphi_{[\pi-\gamma,X]}(q)\right|^{2}=\left|\varphi_{[\gamma,X]}(q)\right|^{2} \tag{16}$$

$$\left|\varphi_{[\pi-\gamma,X]}(k)\right|^{2}=\left|\varphi_{[\gamma,X]}(-k)\right|^{2} \tag{17}$$

If we let $T_{[\gamma,X]}\varphi=\tau_{[\gamma,X]}\varphi$, where $\tau_{[\gamma,X]}$ is the eigenvalue of the TOA-operator, we found that

$$T_{[\pi-\gamma,X]}\varphi(q)=-\tau_{[\gamma,X]}\varphi(q) \tag{18}$$

$$T_{[-\gamma,X]}\varphi(q)=-\tau_{[\gamma,X]}\varphi(q) \tag{19}$$

$$T_{[\gamma,-X]}\varphi(q)=-\tau_{[\gamma,X]}\varphi(q) \tag{20}$$

Periodic (γ = 0 and γ = π/2) case

We note the following symmetries of the periodic kernel

$$T_{[0,x]}(q,q',X)=-T^{*}_{[0,X]}(q,q',X) \tag{21}$$

$$T_{[0,x]}(q,q',X)=-T^{*}_{[0,-X]}(-q,-q',-X) \tag{22}$$

For every $\varphi \in H$, we note $T_{[0,X]}\Theta\varphi=-\Theta T_{[0,X]}\varphi$. It can be shown that $T_{[0,X]}$ has positive and negative eigenvalues of equal magnitudes, with corresponding eigenfunctions, φ^{+} and φ^{-}. Using Eq. (21), we get the following probability densities

$$\left|\varphi^{-}(q)\right|^{2}=\left|\varphi^{+}(q)\right|^{2} \tag{23}$$

$$\left|\varphi^{-}(k)\right|^{2}=\left|\varphi^{+}(k)\right|^{2} \tag{24}$$

For opposite X, we have $T_{[0,-X]}\Theta\Pi\varphi=-\Theta\Pi T_{[0,X]}\varphi$ and by using Eq. (22), we find

$$\left|\varphi_{[0,-X]}(q)\right|^{2}=\left|\varphi_{[0,X]}(-q)\right|^{2} \tag{25}$$

$$\left|\varphi_{[0,-X]}(k)\right|^{2}=\left|\varphi_{[0,X]}(k)\right|^{2} \tag{26}$$

Spectrum of the time-of-arrival (TOA) operator

The solution to Eq. (4) reduces to an eigenvalue problem

$$\int_{-l}^{l}T(q,q')\phi_{\tau}^{\gamma}(q)\,dq'=\tau_{\gamma}\phi_{\tau}^{\gamma}(q) \tag{27}$$

which is solved using the Nystrom Method for second order homogenous Fredholm integral equation (Delves & Mohamed, 1985). We produce eigenfunctions and eigenvalues for the TOA-operator, which were not done in current literatures (Muga & Leavens, 2000), although we will not emphasize on the numerical values of the simulation, but more on the behaviors of eigenfunctions and the spectrum.

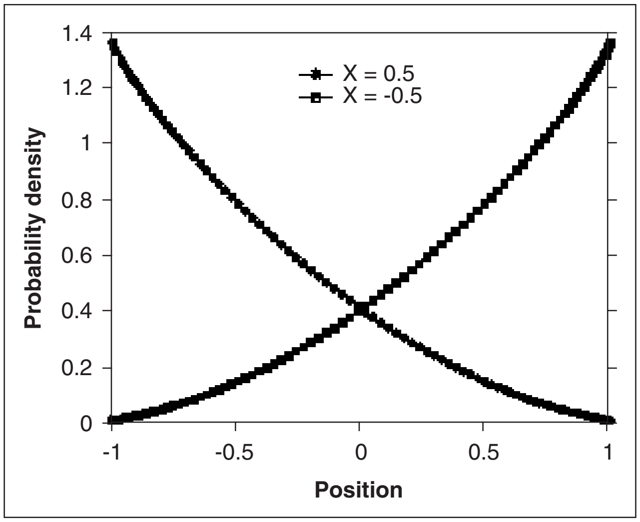

Fig. 1. Position probability distribution for detector positions $X = 0.5$ and $X = -0.5$; with $\gamma = 3$ and eigenvalues $\tau_{[3,0.5]} = -4.3944$ and $\tau_{[3,-0.5]} = 4.3944$. The distributions are mirror images of each other.

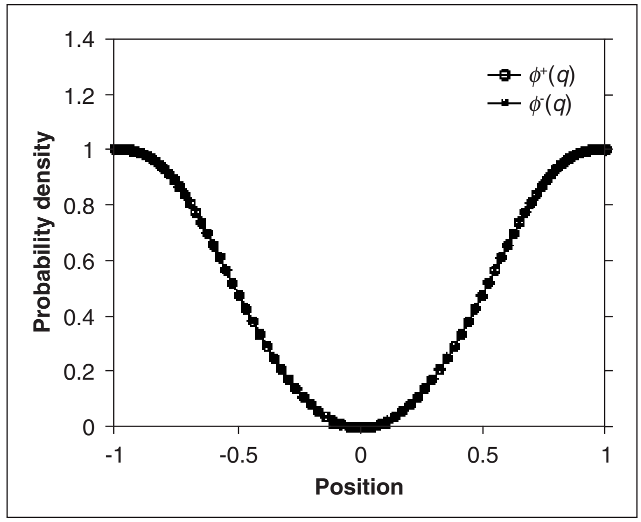

Fig. 3. Probability density of the position corresponding to the eigenfunctions of the negative (ϕ^-) and positive (ϕ^+) eigenvalue at $X = 0$. The probability densities overlap and their eigenvalues differ in signs only.

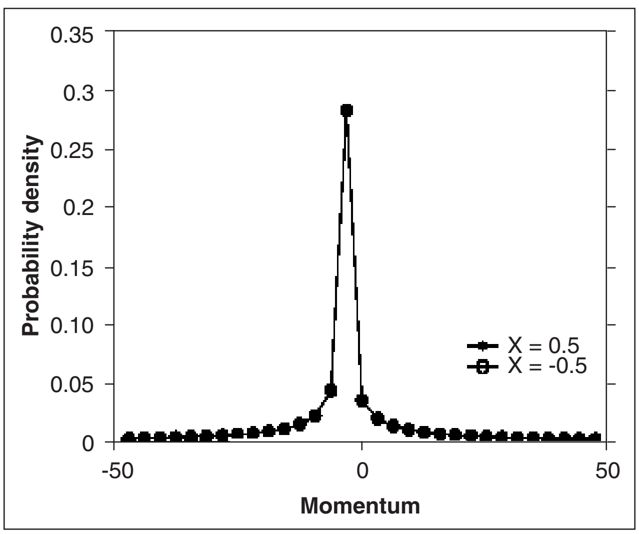

Fig. 2. Momentum probability distribution for detector positions $X = 0.5$ and $X = -0.5$, with $\gamma = 3$. The x-axis is the momentum value $n\pi$, where $n = 0, +1, ...$ The distributions overlap.

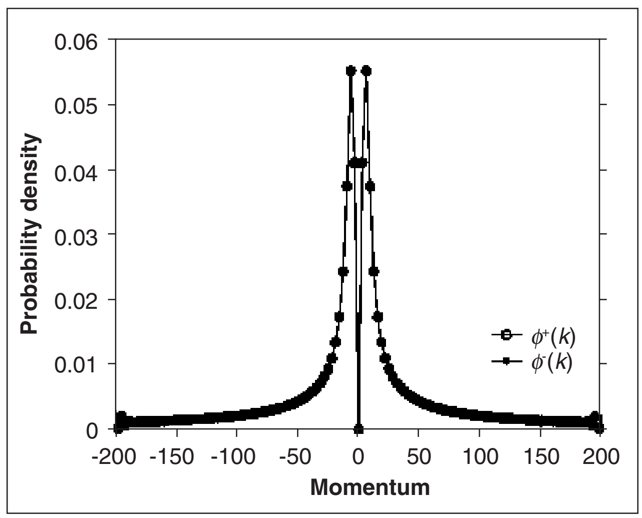

Fig. 4. The momentum probability density for periodic boundary condition at $X = 0$. The probabilities corresponding to the negative and positive eigenvalues overlap.

For the non-periodic case, we are able to verify Eqs. (12) to (17). In particular, in Fig. 1, relations of the position probability conforms with Eq. (12) and the eigenvalue of operators $T_{[\gamma,X]}$ and $T_{[\gamma,-X]}$, which are opposite in sign but are of equal magnitude are consistent with Eq. (20). The corresponding momentum probability for Fig. 1 is shown in Fig. 2. This is consistent with Eq. (13).

For the periodic boundary case, we particularly focused on the detector at $X = 0$. But we have also confirmed Eqs. (25) and (26). This operator was constructed in Galapon (2000). On Figs. 3 and 4, the probability densities corresponding to the negative and positive eigenvalue are again overlapping for both the momentum and position. These are consistent with Eqs. (23) and (24).

CONCLUSION

We have shown that the quantization of the classical TOA at arbitrary *X* for a spatially confined particle allows a construction of a quantum mechanical counterpart, a TOA-operator, which is canonically conjugate to the free Hamiltonian. The eigenvalues are supposed to be the outcome of a TOA measurement. From symmetry considerations, we derived some properties of the TOA for different cases. But a better insight on the TOA operator properties is to be obtained by evolution of the given stationary states.

REFERENCES

Delves, L.M. & J.L. Mohamed, 1985. Computational methods for integral equations. Cambridge University Press.

Galapon, E.A., 2000. Canonical pairs, spatially-confined motion, and the quantum time-of-arrival problem. quant-ph/ 0001062.

Galapon, E.A., 2002. Pauli's theorem and quantum canonical pairs: The consistency of a bounded self-adjoint time operator canonically conjugate to a Hamiltonian with non-empty point spectrum. *Proc. Roc. Soc*. 487: 451.

Muga, J.G. & C.R. Leavens, 2000. Arrival time in quantum mechanics. *Phys. Rep*. 338: 353-438.

Press, W.H., et al., 1992. Numerical recipes in Fortran77: the art of scientific computing. Cambridge University Press.