A study on $\hat{g}^{**}s - R_0$ and $\hat{g}^{**}s - R_1$ spaces In Topological spaces

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Abstract

In this paper, we introduces the concept of R_0-space, R_1-space, door space, submaximal using $\hat{g}^{**}s$-closed set and investigate its properties. We have also study their relationship with some other higher separation axioms. We have also, introduced a new definition $S\hat{g}^*$-space by using semi-closed and $\hat{g}^*$-closed sets and study its relationship with other closed sets using $\hat{g}^{**}s$-closed set.

Keywords. $\hat{g}^{**}s - R_0$, $\hat{g}^{**}s - R_1$ space, $\hat{g}^{**}s$ -door spaces, $\hat{g}^{**}s$ –submaximal, $S\hat{g}^*$-space.

2010 AMS subject classification: 54D15, 54G05‡

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‡ Received on July 22, 2022. Accepted on October 15, 2022. Published on January 30, 2023. doi: 10.23755/rm.v45i0.989. ISSN: 1592-7415. eISSN: 2282-8214.

1. Introduction

Topology is a major area of mathematics concerned with spatial properties that are preserved under continuous deformations of objects. It emerged through the development of concepts from geometry and set theory, such as space, dimension, and transformation. Ideas that are now classified as topology were expressed as early as 1736. By the middle of the 20th century, topology had become an important area of study within mathematics. The word topology is used both for the mathematical discipline and for a family of sets with certain properties that are used to define a topological space, a basic object of topology. Topology includes many subfields namely point-set topology, algebraic topology and geometric topology.

In the literature of General Topology, the concept of semi open sets was introduced by Levine in 1963 [11] and g-closed sets in 1970 [12]. M.K.R.S. Veera Kumar defined $\hat{g}$-closed sets in 2001 [10] and $\hat{g}^*$-closed sets in 1996 [9]. In the year 1995, J. Dontchev [2, 3], defined on door spaces and submaximal spaces.

The notion of R_0 topological space is introduced by N. A. Shanin [13] in 1943. Later, A. S. Davis [13] rediscovered it and studied some properties of this weak separation axiom. In 2010, S. Balasubramanian defined separation axioms on generalized sets [14]. Several topologists further investigated properties of R_0topological spaces and many interesting results have been obtained in various contexts. In the same paper, A. S. Davis also introduced the notion of R_1 topological space which are independent of both T_0 and T_1 but strictly weaker thanT_2. M. G. Murdeshwar and S. A. Naimpally [8] studied some of the fundamental properties of the class of R_1 topological spaces. In 1963, N. Levine [11] offered a new notion to the field of general topology by introducing semi-open sets. He defined this notion by utilizing the known notion of closure of an open set, i.e., a subset of a topological space is semi-open if it is contained in the closure of its interior. Since the advent of this notion, several new notions are defined in terms of semi-open sets of which two are semi-R_0 and semi-R_1 introduced by S. N. Maheshwari and R. Prasad [15] and C. Dorsett [1], respectively. These two notions are defined as natural generalizations of the separation axioms R_0 and R_1 by replacing the closure operator with the semi closure operator and openness with semi-openness. Since then, this notion received wide usage in general topology.

In this paper, we continue the study of the above mentioned classes of topological spaces satisfying these axioms by introducing two more notions in terms of $\hat{g}^{**}s$-closed sets [7] called $\hat{g}^{**}s$-$R_0$and $\hat{g}^{**}s$-R_1. We have also applied the definition in [7] to door space and submaximal space.In this paper we introduced S$\hat{g}^*$-space using semi-closed and $\hat{g}^*$-closed sets.

2. Preliminaries

Throughout this paper (X, τ) represent the non-empty topological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset A of a space (X, τ), $cl(A)$ and $int(A)$ denote the closure and interior of A respectively.

Definition. 2.1

i) A space (X, τ) is R_0 [13] if for each open set U of X, x ∈ U implies cl({x}) ⊆ U.
ii) A topological space (X, τ) is R_1 [13] if for x, y ∈ X such that cl({x}) ≠ cl({y}), there are disjoint open sets U and V such that cl({x}) ⊆ U and cl({y}) ⊆ V.
iii) A topological space (X, τ) is a door space [3] if every subset of X is either open or closed in X.
iv) A topological space (X, τ) is a submaximal space [4] if every dense subset of X is open in X.

3. $\hat{g}^{**}s - R_0$ And $\hat{g}^{**}s - R_1$ spaces

Definition. 3.1. A space (X, τ) is said to be $\hat{g}^{**}s - R_0$ if for any open set $U, x \in U$, then $\hat{g}^{**}scl\{x\} \subseteq U$.

Definition. 3.2. A space (X, τ) is said to be $\hat{g}^{**}s - R_1$ if for x and y in X, with $\hat{g}^{**}scl(\{x\}) \neq \hat{g}^{**}scl(\{y\})$, there exist two disjoint $\hat{g}^{**}$s-open sets U and V such that $\hat{g}^{**}scl(\{x\}) \subseteq U$ and $\hat{g}^{**}scl(\{y\}) \subseteq V$

Theorem. 3.3. Every $\hat{g}^{**}s - R_0$-space is $\hat{g}^{**}s - T_0$-space.

Proof. Let (X, τ) be a $\hat{g}^{**}s - R_0$-space. Let x and y be two distinct points of X. By hypothesis, for any open set U and $x \in U \Rightarrow \hat{g}^{**}scl(\{x\}) \subseteq U$. Also, every open set is $\hat{g}^{**}$s-open, which implies U is $\hat{g}^{**}$s-open and $x \in U$. Therefore, for any distinct points x and y, there exist a $\hat{g}^{**}$s-open set U containing x and not y. Hence, (X, τ) is a $\hat{g}^{**}s - T_0$-space.

Theorem. 3.4. Every $\hat{g}^{**}s - R_0$-space is $\hat{g}^{**}s - T_2$-space.

Proof. Let (X, τ) be a $\hat{g}^{**}s - R_0$-space. Let x and y be two distinct points of X. By hypothesis, for any open set U and $x \in U \Rightarrow \hat{g}^{**}scl\{x\} \subseteq U$. If $\hat{g}^{**}scl(\{x\}) \nsubseteq U$, there exist another open set V not containing x, so $y \in V$. Also, $U \cap V = \emptyset$. Therefore, for any distinct points x and y, there exist two disjoint $\hat{g}^{**}$s-open sets U and V. Hence, (X, τ) is a $\hat{g}^{**}s - T_2$-space.

Theorem. 3.5. If (X, τ) is $\hat{g}^{**}s - R_0$, then for any closed set U and $x \notin U$, there exist an $\hat{g}^{**}$s-open set G such that $U \subseteq G$ and $x \notin G$.

Proof. Suppose (X, τ) is $\hat{g}^{**}s - R_0$. Let U be any closed set and $x \notin U \Rightarrow X \backslash U$ is open and $x \in X \backslash U$. By assumption, $\hat{g}^{**}scl(\{x\}) \subseteq X \backslash U \Rightarrow U \subseteq X \backslash \hat{g}^{**}scl(\{x\})$. Put $G =$

$X\backslash\hat{g}^{**}scl\{x\}$.Since $\hat{g}^{**}scl\{x\}$ is $\hat{g}^{**}$s-open. Also, $U \subseteq G$ and$x \notin G$. Therefore, for any closed set U and $x \notin U$, there exist an $\hat{g}^{**}$s-open set G such that $U \subseteq G$ and$x \notin G$.
Conversely, suppose for any closed set U and $x \notin U$, there exist an $\hat{g}^{**}$s-open set G such that $U \subseteq G$ and $x \notin G$

Theorem. 3.6. Every $\hat{g}^{**}s - R_0$-space is $\hat{g}^{**}$s-regular.
Proof. Let (X, τ) be a $\hat{g}^{**}s - R_0$-space. Let F be a closed set and$x \in X - F$. By theorem: 3.3, there exist an $\hat{g}^{**}$s-open set G such that $U \subseteq G$ and$x \notin G$. Put$H = \{x\}$. Since every singleton set is open, so H is open. Also, we know that, every open set is $\hat{g}^{**}$s-open. Also, $G \cap H = G \cap \{x\} = \emptyset$. Thus, G and H are disjoint $\hat{g}^{**}$s-open sets containing x and H respectively. Therefore, (X, τ) is $\hat{g}^{**}$s-regular.

Theorem. 3.7. A topological space (X, τ) is $\hat{g}^{**}s - R_0$ iff for any points x and y in X, $x \neq y$ implies$\hat{g}^{**}scl(\{x\}) \cap \hat{g}^{**}scl(\{y\}) = \emptyset$.
Proof. Let X be $\hat{g}^{**}s - R_0$ and $x \neq y$ in $X \Rightarrow \{x\}$ be an open set and$y \notin \{x\}$. Since,$x \in \{x\}$, we have $\hat{g}^{**}scl(\{x\}) \subseteq \{x\}$. Thus$\hat{g}^{**}scl(\{x\}) = \{x\}$. Now,$\hat{g}^{**}scl(\{x\}) \cap \hat{g}^{**}scl(\{y\}) = \{x\} \cap \{y\} = \emptyset$. Conversely, Suppose for any points x and y in$X$, $x \neq y \Rightarrow \hat{g}^{**}scl(\{x\}) \cap \hat{g}^{**}scl(y) = \emptyset$. Let V be an open set and $x \in V$ and let $y \in \hat{g}^{**}scl(\{x\})$ (1). Suppose$y \notin V$. By assumption, $\hat{g}^{**}scl(\{x\}) \cap \hat{g}^{**}scl(\{y\}) = \emptyset \Rightarrow y \notin \hat{g}^{**}scl(\{x\})$this is a contradiction to (1). Therefore, $y \in V$ and$\hat{g}^{**}scl(\{x\}) \subseteq V$. Thus (X, τ) is$\hat{g}^{**}s - R_0$.

Corollary. 3.8. A topological space (X, τ) is $\hat{g}^{**}s - R_0$ iff for any points x and y in X, $x \neq y \Rightarrow cl(\{x\}) \cap cl(\{y\}) = \emptyset$.
Proof. Directly follows from theorem.3.5

Theorem. 3.9. If (X, τ) is$\hat{g}^{**}s - R_0$, then it is$\hat{g}^{**}s - T_1$.
Proof. Let (X, τ) be a $\hat{g}^{**}s - R_0$-space and $x \in X \Rightarrow \{x\}$ is open. By hypothesis, $\hat{g}^{**}scl(\{x\}) \subseteq \{x\} \Rightarrow \hat{g}^{**}scl(\{x\}) = \{x\} \Rightarrow \{x\}$ is $\hat{g}^{**}$s-closed $\Rightarrow$every singleton set is $\hat{g}^{**}$s-closed$\Rightarrow$ (X, τ) is$\hat{g}^{**}s - T_1$.

Theorem. 3.10. Every $\hat{g}^{**}s - R_0$-space is $\hat{g}^{**}s - R_1$-space.
Proof. Let (X, τ) is$\hat{g}^{**}s - R_0$. Let$x, y \in X$, with$\hat{g}^{**}scl(\{x\}) \neq \hat{g}^{**}scl(\{y\})$. By above theorem, $\hat{g}^{**}scl(\{x\}) = \{x\}$ and $\hat{g}^{**}scl(\{y\}) = \{y\} \Rightarrow \{x\}$ and $\{y\}$ are $\hat{g}^{**}$s-open sets and $\{x\} \cap \{y\} = \emptyset \Rightarrow (X, \tau)$ is $\hat{g}^{**}s - R_1$-space.

Theorem. 3.11. For any $\hat{g}^{**}$s-closed set H, $\hat{g}^{**}scl(\{x\}) \cap H = \emptyset$, for every$x \in X\backslash H$, then (X, τ) is $\hat{g}^{**}s - R_0$-space.
Proof. Assume that for any $\hat{g}^{**}$s-closed set H,$\hat{g}^{**}scl(\{x\}) \cap = \emptyset$, for every $x \in X\backslash H$. Let G be any open set and $x \in G$. Then $x \in G \Rightarrow x \in X(X\backslash G)$ and $X\backslash G$ is closed. Therefore, by assumption, $\hat{g}^{**}scl(\{x\}) \cap (X\backslash G) = \emptyset \Rightarrow \hat{g}^{**}scl(\{x\}) \subseteq G$. Hence proved.

Corollary. 3.12. For any $\hat{g}^{**}$s-closed set H,$cl(\{x\}) \cap H = \emptyset$, for every $x \in X\backslash H$, then (X,τ) is $\hat{g}^{**}s - R_0$-space.
Proof. Assume that for any $\hat{g}^{**}$s-closed set H,$cl(\{x\}) \cap H = \emptyset$, for every $x \in X\backslash H$. Let V be any open set in X and $x \in V$. Then, $x \in V = X\backslash(X\backslash V)$ and $X\backslash V$ is closed. Since every closed set is closed. Since, every closed set is $\hat{g}^{**}$s-closed which implies that $X\backslash V$ is $\hat{g}^{**}$s-closed. By assumption, $cl(\{x\}) \cap (X\backslash V) = \emptyset \Rightarrow cl(\{x\}) \subseteq V \Rightarrow \hat{g}^{**}\text{scl}(\{x\}) \subseteq cl(\{x\}) \subseteq V \Rightarrow \hat{g}^{**}\text{scl}(\{x\}) \subseteq V \Rightarrow (X,\tau)$ is $\hat{g}^{**}s - R_0$-space.

Theorem. 3.13. If a topological space, (X,τ) is $\hat{g}^{**}s - R_1$ and $\hat{g}^{**}s - T_0$-space.
Proof. Since every $\hat{g}^{**}s - T_1$ is $\hat{g}^{**}s - T_0$. Then, the result is obvious.

Theorem. 3.14. If (X,τ) is $\hat{g}^{**}s - R_0$ and $\hat{g}^{**}s - T_0$-space, then it is also a $\hat{g}^{**}s - T_1$-space.
Proof. Let $x \neq y$ be any two points of X. Since (X,τ) is$\hat{g}^{**}s - R_0$, there exist an open set U such that $x \in U$ and$\hat{g}^{**}\text{scl}(\{x\}) \subseteq U \Rightarrow x \notin X\backslash U$. Since, $y \notin U$, there exist another $\hat{g}^{**}$s-open set $X\backslash U = V(say)$ containing y but not x. Therefore, for any two distinct points x and y, there exist two distinct $\hat{g}^{**}$s-open sets U and V. Hence, (X,τ) is $\hat{g}^{**}s - T_1$-space.

Theorem. 3.15. If a topological space (X,τ) is$\hat{g}^{**}s - R_1$, then either $cl(\{x\}) = X$, for each $x \in X$ or $cl(\{x\}) \neq X$, for each$x \in X$.
Proof. Assume that (X,τ) is$\hat{g}^{**}s - R_1$. If $cl(\{x\}) = X$, for each$x \in X$, then the theorem is obvious. If not, then there exist $y \in X$ such that$cl(\{y\}) \neq X$. Suppose not, there exist $z \in X$ such that$cl(\{z\}) = X$. Now,$cl(\{y\}) \neq X = cl(\{z\})$. Since (X,τ) is$\hat{g}^{**}s - R_1$, there exist disjoint $\hat{g}^{**}$s-open sets U and V containing $cl(\{y\})$ and $cl(\{z\})$ respectively. Since $cl(\{z\}) = X$, we have $V = X \Rightarrow U \cap V = U \neq \emptyset$ which a contradiction is to$U \cap V = \emptyset$. Hence proved.

4. $\hat{g}^{**}$s-door space

Definition. 4.1. A topological space (X,τ) is called a $\hat{g}^{**}$s-door space if every subset is either $\hat{g}^{**}$s-open or $\hat{g}^{**}$s-closed.

Definition. 4.2. A topological space (X,τ) is called $\hat{g}^{**}$s-submaximal if every dense subset of X is $\hat{g}^{**}$s-open.

Definition. 4.3. A topological space (X,τ) is called $\hat{g}^{**}$s-extremally disconnected space in which the $\hat{g}^{**}$sclosure of every $\hat{g}^{**}$s-open subset is $\hat{g}^{**}$s-open.

Definition. 4.4. A topological space is called $S\hat{g}^*$-space if the intersection of a semi-closed set with $\hat{g}^*$-closed set is $\hat{g}^*$-closed.

Definition. 4.5. A topological space (X, τ) is called $\hat{g}^{**}$s-hyperconnected if every non-empty $\hat{g}^{**}$s-open subset of X is $\hat{g}^{**}$s-dense.

Theorem. 4.6. Every subspace of $\hat{g}^{**}$s-door space is a $\hat{g}^{**}$s-door space.
Proof. Let S be a subspace of X and $A \subseteq S$ is a subset of X. Since X is $\hat{g}^{**}$s-door space, A is either $\hat{g}^{**}$s-open or $\hat{g}^{**}$s-closed in X. Hence A is either $\hat{g}^{**}$s-open or $\hat{g}^{**}$s-closed in S. Therefore, S is $\hat{g}^{**}$s-door space.

Theorem. 4.7. In a $\hat{g}^{**}$s-hyperconnected space, every $\hat{g}^{**}$s-submaximal space is a $\hat{g}^{**}$s-door space.
Proof. Let $A \subset X$ and if A is dense in X, so A is $\hat{g}^{**}$s-open. (1)
If A is not dense, there exist a non-empty open set$B \subset A^c$. Since every open set is $\hat{g}^{**}$s-open, there exist a non-empty $\hat{g}^{**}$s-open set$B \subset A^c$. Since X is $\hat{g}^{**}$s-hyperconnected, B is dense and A^c is also dense. Again by definition of $\hat{g}^{**}$s-submaximal, A^c is $\hat{g}^{**}$s-open which implies A is $\hat{g}^{**}$s-closed. (2)
From (1) and (2), A is either $\hat{g}^{**}$s-open or $\hat{g}^{**}$s-closed. Therefore, (X, τ) is a $\hat{g}^{**}$s-door space.

Theorem. 4.8. In(X, τ), every door space is $\hat{g}^{**}$s-door space.
Proof. Let A be a door space. Every subset of X is either open or closed. Since every closed set is $\hat{g}^{**}$s-closed and every open set is $\hat{g}^{**}$s-open, every subset of X is either $\hat{g}^{**}$s-open or $\hat{g}^{**}$s-closed. Therefore, A is a $\hat{g}^{**}$s-door space.

Remark. 4.9. Converse of above theorem is not true.

Example. 4.10. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}\}$, $\tau^c = \{\emptyset, X, \{b, c\}\}$,
$\hat{g}^{**}so(X, \tau) = \{\emptyset, X, \{a\}, \{a, b\}, \{a, c\}\}$ and $\hat{g}^{**}sc(X, \tau) = \{\emptyset, X, \{b\}, \{c\}, \{b, c\}\}$.
Hence (X, τ) is a $\hat{g}^{**}$s-door space but it is not a door space.

Theorem. 4.11. For a subset A of a $S\hat{g}^*$-space (X, τ) the following are equivalent:
i) A is $\hat{g}^{**}$s-closed
ii) $\hat{g}^*cl(\{x\}) \cap A \neq \emptyset$, for each $x \in scl(A)$
iii) $scl(A) - A$ Contains no non-empty $\hat{g}^*$-closed set.
Proof.i) $\Rightarrow$ ii) Let$x \in scl(A)$. Suppose$\hat{g}^*cl(\{x\}) \cap A = \emptyset$. Then$A \subseteq X - \hat{g}^*cl(\{x\})$. Since A is $\hat{g}^{**}$s-closed, $scl(A) \subseteq X - \hat{g}^*cl(\{x\})$ which is a contradiction to$x \in scl(A)$. Therefore, $\hat{g}^*cl(\{x\}) \cap A \neq \emptyset$

ii) $\Rightarrow$ iii) Let F be a $\hat{g}$-closed set such that $F \subseteq scl(A) - A$. Suppose $x \in F$. Then $\hat{g}^*cl(\{x\}) \subseteq F$. Therefore $\emptyset \neq \hat{g}^*cl(\{x\}) \cap A \subseteq F \cap A \subseteq [scl(A) - A] \cap A = \emptyset$ which is a contradiction.
iii) $\Rightarrow$ i) Let $A \subseteq G$ and G be $\hat{g}^*$-open in X, Suppose scl (A) not a subset of G. Then $scl(A) \cap (X - G)$ is non-empty $\hat{g}^*$-closed subset of $scl(A) - A$ which is a contradiction. Therefore, $scl(A) \subseteq G$ and hence A is $\hat{g}^{**}$s-closed.

Theorem. 4.12. If B is a cl open subset of (X, τ), then B is a $\hat{g}^{**}$s-closed set.
Proof. Since B is cl open, B is both open and closed. Let U be a $\hat{g}^*$-open set in X and $B \subseteq U$. Since B is clopen, we have $int(cl(B)) = B \Rightarrow int(cl(B)) \subseteq B$ and $B \subseteq int(cl(A))$, (ie) $int(cl(B)) \subseteq B \Rightarrow B$ is semi-closed and $scl(B) - B$. Therefore, $scl(B) \subseteq U$. Hence B is $\hat{g}^{**}$s-closed in X.

Theorem. 4.13. In (X, τ), every $\hat{g}^{**}$s-dense is dense.
Proof. Let $A \subseteq X$ be $\hat{g}^{**}$s-dense in $(X, \tau) \Rightarrow \hat{g}^{**}scl(A) = X$. Since, $X = \hat{g}^{**}scl(A) \subseteq cl(A) \Rightarrow X \subseteq cl(A)$. Always, $cl(A) \subseteq X \Rightarrow cl(A) = X \Rightarrow A \subseteq X$ be dense in (X, τ).

Remark. 4.14. Converse of above theorem is not true.

Example. 4.15. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a, b\}\}$, $\tau^c = \{\emptyset, X, \{c\}\}$, $\hat{g}^{**}sc(X, \tau) = \{\emptyset, X, \{c\}, \{b, c\}, \{a, c\}\}$ and $\hat{g}^{**}so(X, \tau) = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$. Let $A = \{a\}$ is a dense subset of X but it is not $\hat{g}^{**}$s-dense.

Remark. 4.16. Every $\hat{g}^{**}s$-door space need not be a $\hat{g}^{**}s$-submaximal.

Example. 4.17. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a, b\}\}$, $\hat{g}^{**}sc(X, \tau) = \{\emptyset, X, \{c\}, \{b, c\}, \{a, c\}\}$ and $\hat{g}^{**}so(X, \tau) = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$. Here (X, τ) is a $\hat{g}^{**}s$-door space but it is not $\hat{g}^{**}s$-submaximal.

Remark. 4.18. Every $\hat{g}^{**}s$-submaximal space need not be $\hat{g}^{**}s$-door space.

Example. 4.19. Let $X = \{a, b, c, d\}, \tau = \{\emptyset, X, \{a\}, \{d\}, \{a, d\}, \{a, b, c\}\}$,

$$sc(X, \tau) = \{\emptyset, X, \{b\}, \{c\}, \{d\}, \{b, c\}, \{c, d\}, \{b, d\}, \{a, b, c\}, \{b, c, d\}\},$$

$$\hat{g}^*c(X, \tau) = \{\emptyset, X, \{b\}, \{c\}, \{d\}, \{a, b\}, \{b, c\}, \{a, c\}, \{b, d\}, \{c, d\}, \{a, b, c\}, \{b, c, d\}, \{a, c, d\}, \{a, b, d\}\},$$

$$\hat{g}^{**}sc\{X, \tau\} = \{\emptyset, X, \{b\}, \{c\}, \{d\}, \{b, c\}, \{c, d\}, \{b, d\}, \{a, b, c\}, \{b, c, d\}\},$$

$$\hat{g}^{**}so(X, \tau) = \{\emptyset, X, \{a\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{a, b, c\}, \{a, c, d\}, \{a, b, d\}\}.$$

Remark. 4.20. Every $S\hat{g}^*$-space need not be a door space.

Example. 4.21. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{b, c\}\}, \tau^c = \{\emptyset, X, \{a\}, \{b, c\}\}$, $sc(X, \tau) = \{\emptyset, X, \{a\}, \{b, c\}\}$ *and* $\hat{g}^*c(X, \tau) = \{\emptyset, X, \{a\}, \{b, c\}\}$. Here $sc \cap \hat{g}^*c = \hat{g}^*c$ but it is not a door space.

Theorem. 4.22. In $S\hat{g}^*$-space

i) $(sg)^*$-Closed set coincide with $\hat{g}^{**}s$-closed set.
ii) gs^{**}-Closed set coincide with $\hat{g}^{**}s$-closed set.
iii) $\hat{\eta}^*$-Closed set coincide with $\hat{g}^{**}s$-closed set.
iv) αg-Closed set coincide with $\hat{g}^{**}s$-closed set.
v) ps-Closed set coincide with $\hat{g}^{**}s$-closed set.
vi) gsp-Closed set coincide with $\hat{g}^{**}s$-closed set.
vii) sp-Closed set coincide with $\hat{g}^{**}s$-closed set.
viii) sg-Closed set coincide with $\hat{g}^{**}s$-closed set.
ix) $g\alpha$-Closed set coincide with $\hat{g}^{**}s$-closed set.
gs-Closed set coincide with $\hat{g}^{**}s$-closed set.

Theorem. 4.23. In $S\hat{g}^*$-space, semi-closed and $\hat{g}$-closed are independent.

Theorem. 4.24. The property of being $\hat{g}^{**}s$-door space is a topological property.
Proof. Let (X, τ) be a $\hat{g}^{**}s$-door space and let $f: X \to Y$ be a homeomorphism. Let$A \subseteq Y$, consider $f^{-1}(A) \subseteq X$, since X is a $\hat{g}^{**}s$-door space, then $f^{-1}(A)$ is either $\hat{g}^{**}s$-open or $\hat{g}^{**}s$-closed in X. Now,$f\left(f^{-1}(A)\right) = A$. Then A is either $\hat{g}^{**}s$-open or $\hat{g}^{**}s$-closed in Y. Therefore, Y is a $\hat{g}^{**}s$-door space.

Theorem. 4.25. The property of being a $\hat{g}^{**}s$-door space is an expensive property.
Proof. Suppose (X, τ) is a $\hat{g}^{**}s$-door space. Let $\tau \subset \tau_1$ and$A \subseteq X$. Since, (X, τ) is a $\hat{g}^{**}s$-door space, then A is either $\hat{g}^{**}s - \tau$-open or $\hat{g}^{**}s - \tau$-closed. Since$\tau_1 \subset \tau$, A is either $\hat{g}^{**}s - \tau_1$-open or $\hat{g}^{**}s - \tau_1$-closed. Then (X, τ_1) is a $\hat{g}^{**}s$-door space.

Theorem. 4.26. Let (X, τ) be a $\hat{g}^{**}s$-door space and $Y \subseteq X$ be a clopen subset of Y, then (Y, τ_Y) is also a $\hat{g}^{**}s$-door space.
Proof. Let $A \subseteq Y$ be a subset of Y. Now,$A \subseteq X$. By hypothesis, (X, τ) is a $\hat{g}^{**}s$-door space which implies that A is either $\hat{g}^{**}s$-open or $\hat{g}^{**}s$-closed in X. Since Y is both open and closed. Then A is either $\hat{g}^{**}s$-open or $\hat{g}^{**}s$-closed in Y. Therefore, Y is also a $\hat{g}^{**}s$-door space.

Proposition. 4.27. Let A and B be $\hat{g}^{**}$s-closed subsets of X such that $cl(A) = scl(A)$ and$cl(B) = scl(B)$. Thus $A \cup B$ is $\hat{g}^{**}$s-closed.

Proof. Let U be $\hat{g}^*$-open such that $A \cup B \subseteq U \Rightarrow A \subseteq U$ and $B \subseteq U \Rightarrow scl(A) \subseteq U$ and $scl(B) \subseteq U \Rightarrow cl(A) \subseteq U$ and $cl(B) \subseteq U \Rightarrow cl(A \cup B) = cl(A) \cup cl(B) \subseteq U \Rightarrow cl(A \cup B) \subseteq U \Rightarrow scl(A \cup B) \subseteq cl(A \cup B) \subseteq U$. Therefore, $A \cup B$ is $\hat{g}^{**}$s-closed.

Proposition. 4.28. If B is a regular open subset of a topological space (X, τ), then B is $\hat{g}^{**}$s-closed.
Proof. Let U be $\hat{g}^*$-open set such that $B \subseteq U$. Since B is regular open, $int(cl(B)) = B \Rightarrow int(cl(B)) \subseteq B \subseteq U \Rightarrow B$ is semi $-$ closed $\Rightarrow scl(B) = B \subseteq U \Rightarrow scl(B) \subseteq U$. Therefore, B is $\hat{g}^{**}$s-closed.

Theorem. 4.29. If A and B are subsets of X such that $A \subseteq B$ and A is g^*-closed, then B is $\hat{g}^{**}s$-closed.
Proof. Assume $B \subseteq U$ and U is $\hat{g}^*$-open. Therefore, $A \subseteq B \Rightarrow A \subseteq U$. Also, every $\hat{g}^*$-open set is g-open which implies U is g-open in X. Therefore, $A \subseteq U$ and U is g-open $\Rightarrow cl(A) \subseteq U \Rightarrow cl(B) \subseteq U \Rightarrow scl(B) \subseteq cl(B) \subseteq U \Rightarrow scl(B) \subseteq U$. Therefore B is $\hat{g}^{**}$s-closed.

Theorem. 4.30. If A and B are subsets of X such that $A \subseteq B$ and A is g^*s-closed, then B is $\hat{g}^{**}$s-closed.
Proof. Assume that $B \subseteq U$ and U is $\hat{g}^*$-open. Since $A \subseteq B \Rightarrow A \subseteq U$. Also, every $\hat{g}^*$-open set is gs-open which implies U is gs-open in X. Therefore, $A \subseteq U$ and U is gs-open $\Rightarrow scl(A) \subseteq U \Rightarrow scl(A) \subseteq scl(B) \subseteq U \Rightarrow scl(B) \subseteq U$. Hence proved.

5. Conclusion

We have studied the concept of R_0-space, R_1-space, door space, and submaximal space via $\hat{g}^{**}$s-closed set and examples are provided to state the converse doesn't implies. It shows that every $\hat{g}^{**}s - R_0$-space is $\hat{g}^{**}s - T_2$-space, $\hat{g}^{**}$s-regular and $\hat{g}^{**}s - R_1$-space. In addition we have shown that every $\hat{g}^{**}s$-door space need not be a $\hat{g}^{**}s$-submaximal and suitable examples are given for it. A new concept of $S\hat{g}^*$-space is also introduced. Further, we have compared $\hat{g}^{**}s$-closed set with other existing closed sets and we have also investigated the characterization of these spaces.

References

[1] C. Dorsett, Semi-T_1, Semi-R_1 and Semi-R_0 topological spaces, Ann. Soc. Sci. Bruxelles 92(1978), 143-150.

[2] Dontchev J. on submaximal spaces, Tamkang j math. 1995; 26:253-260.

[3] Dontchev J. on door spaces, Indian j pure Appl math. 1995; 26:873-881.

[4] I. Arockia Rani, Studies on generalizations of generalized closed sets and maps in topological spaces, Ph. D Thesis, Bharathiyar University, Coimbatore, (1997).

[5] J.N. Sharma. (1977). “General topology" Krishna Prakashan, Meerut (U.P).

[6] K. Anitha, On g*s-closed sets in Topological spaces, Int. J. Contemp. Math., 6 (19) (2011), 917-929.

[7] M. Anto and S. Andrin Shahila, $\hat{g}^{**}s$ -closed sets in topological spaces, IJMTT-ICIMCEH2020, ISSN: 2349 – 5758, 1-7.

[8] M. G. Murdeshwar and S. A. Naimpally, R1-topological spaces, Canad. Math. Bull. 9(1966), 521-523.

[9] M.K.R.S. Veerakumar, Between closed sets and g*-closed sets, Mem. Fac. Sci. Kochi. Univ. Ser. A, Math., 17 (1996), 33-42.

[10] M.K.R.S. Veera Kumar, $\hat{g}$-closed sets and G L C functions, Indian J. Math., Indian J. Math., 43 (2) (2001), 231-247

[11] N. Levine, Semi-open sets and semi-continuity in topological spaces, 70 (1963), 36- 41.

[12] N. Levine, generalized closed sets in topology, Rend. Circ. Mat. Palermo 19 (2) (1970), 89-96.

[13] N. A. Shanin. (1943), On separation in topological spaces, Dokl. Akad. Nauk SSSR, 38, 110-113.

[14] S. Balasubramanian, Generalized separation axioms, Scientia Magna, 6 (4) (2010), 1-14.

[15] S. N. Maheshwari and R. Prasad, On R_0 spaces, Portug. Math. 34(1975), 213-217.