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N$\hat{g}$*s-Continuous functions in Nano Topological Spaces

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Abstract

The aim of this paper is to introduces N$\hat{g}$*s-continuous function in nano topological spaces and we also study the relation between N$\hat{g}$*s-irresolute functions and N$\hat{g}$*s-continuous functions in different closed sets.

Keywords: $\hat{g}$*s-closed set, $\hat{g}$*s-continuous functions, N$\hat{g}$*s-irresolute.

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1. Introduction

Topology which is a branch of mathematics, is formally defined as the study of qualitative properties of certain objects that are invariant under a certain kind of transformation, especially those properties that are invariant under a certain kind of equivalence. The term topology was introduced by a German Mathematician called Johnn Benedict Listing in 1847. The modern topology largely based on the idea of set theory was developed by George Cantor in the later part of 19th century. Since its inception the topic has been growing in different level and in various fields.

Nano topology is one of the latest feathers in topology that applies to real life situations. Lellis Thivagar was the main brain behind developing the concept of nano topology. It is constructed in terms of lower and upper approximations and boundary region of a subset of a universe. The term "Nano" can be ascribed to any unit of measure.

The concept of continuity plays a very major role in general topology and they are now the research topics of many topologists worldwide. Indeed, a significant theme in general topology concerns the variously modified forms of continuity, separation axioms etc., by utilizing generalized open sets. N. Levine [7] introduced the concept of generalized closed sets in 1970. The concept of $\hat{g}$*s –closed sets was introduced by M. Anto [12]. In 2013, M. Lellis Thivagar [6] has introduced nano topological space with respect to a subset X of a universe U, which is defined in terms of lower and upper approximation of X. He has also defined nano-closed sets, nano-interior and nano-closure of a set. He has also introduced, among other, some certain weak form of nano open sets such as nano α-open sets, nano semi open sets and nano pre-open sets. The aim of this paper is to introduce a new class of sets on nano topological spaces called N$\hat{g}$*s –closed sets. Further, we investigate and discuss the relation of this new sets with existing ones.

2.Preliminaries

Definition 2.1 Let U be a non-empty finite set of objects called the universe and R be an equivalence relation on U named as the in-discernibility relation. Then U is divided into disjoint equivalence class, Elements belonging to the same equivalence class are said to be in discernible with one another. The pair (U, R) is said to be the approximation space. Let X ⊆ U.

- The lower approximation of X with respect to R is the set of all object which can be for certain classifies as X with respect to R and it is denoted by L_R(X). That is L_R(X) = $\bigcup_{x\in U}$\{R(x): R(x) ⊆ X\}, where R(X) denotes the equivalence classes determined by X ∈ U.
- The upper approximation of X with respect to R is the of all objects, which can be for certain classified as X with respect to R and it is denoted by U_R(X). That is, U_R(X) = $\bigcup_{x\in U}$\{R(x): R(x) ∩ $X \neq \emptyset$\}.

• The boundary of the region of X with respect to R is the set of all objects, which can be classified neither as X nor as not X with respect to R and it is denoted by B_R(X) = U_R(X) _ L_R(X).

Definition 2.2 If (UR) is an approximation space and X, Y ⊆ U, then

1. $L_R(X) \subseteq X \subseteq U_R(X)$
2. $L_R(\emptyset) = U_R(\emptyset) = \emptyset$ and $L_R(U) = U_R(U) = U$
3. $U_R(X \cup Y) = U_R(X) \cup U_R(Y)$
4. $U_R(X \cap Y) \subseteq U_R(X) \cap U_R(Y)$
5. $U_R(XY) \supseteq U_R(U_R(Y)$
6. $U_R(X \cap Y) = U_R(X\ U_R(Y)$
7. $L_R(X) \subseteq L_R(Y)$ and $U_R(X) \subseteq U_R(Y)$ whenever X ⊆U
8. $U_R((X^c)) = [L_R[(X)]^c$ and $L_R[(X^c)] = [U_R(X)]^c$
9. $U_R(U_R((X)) = L_R(U_R((X)) = U_R(X)$
10. $L_R(L_R(X)) = U_R((L_R(X)) = L_R(X)$

Definition2.3 Let U be the universe, R be an equivalence relation on U and $\tau_R(X)$ = {U, ∅, L_R(X), U_R(X), B_R(X)} where X ⊆ U. τ_R(X)satisfies the following axioms:

1. U and $\emptyset \in \tau_R(X)$
2. The union of elements of any sub collection of τ_R(X) is in τ_R(X).
3. The intersection of the elements of any finite sub collection of τ_R(X) is in τ_R(X). That is, τ_R(X) forms a topology on U is called the nano topology on U with respect to X. We call (U, τ_R(X)) is called the nano topological space.

Definition2.4 If (U, τ_R(X)) is a nano topological space with respect to X where X ⊆ U and if A ⊆ U, then the nano interior of A is defined as the union of all nano open subsets of A and it is de noted by Nint(A). That is, Nint(A) is the largest nano open subset of A. The nano closure of A is defined as the intersection of all nano closed sets containing A and its denoted by Ncl(A). Ncl(A) is the smallest nano closed set containing A.

REMARK 2.5 If τ_R(X) is the nano topology on U with respect to X, then the set B= {U, L_R(X), U_R(X), B_R(X)} is the basis for τ_R(X).

Definition 2.6 A function f: (U, τ_R(X)) → (V, σ_R(Y)) is called

• Nano-continuous [2] if f^{-1}(A) is nano-closed in (X, τ) for every nano-closed set A in (Y, σ).
• Nano α-continuous [7] if f^{-1}(A) is nano α-closed in (X, τ) for every nano-closed set A in (Y, σ).
• Nano semi-continuous [4] if f^{-1}(A) is nano semi closed in (X, τ) for every nano-closed set A in (Y, σ).
• Nano regular-continuous if f^{-1}(A) is nano regular closed in (X, τ) for every nano-closed set A in (Y, σ).

Definition 2.7 If (U, τ_R(X)) is a nano topological space if A ⊆ U., then A is said to be
- N$g^{\#}\alpha$-closed if Nαcl(A) ⊆ V whenever A ⊆ V and V is Nĝ-open in (U, τ_R(X)
- Ng^*-closed if Ncl(A) ⊆ V whenever A ⊆ V and V is Ng-open in (U, τ_R(X) • Ng^*s-closed if Nscl(A) ⊆ V whenever A ⊆ V and V is Ng-open in (U, τ_R(X)).
- Nsĝ-closed if Ncl(A) ⊆ U whenever A ⊆ U and U is Nsg-open in (U, τ_R(X)).
- Nĝα-closed if Nαcl(A) ⊆ U whenever A ⊆ U and U is Nĝ-open in (U, τ_R(X)) • Nαg^*s-closed if N$s$cl(A) ⊆ U whenever A ⊆ U and U is Nαg-open in (U, τ_R(X)). • N$g\alpha g$ - closed if Ncl(A) ⊆ U whenever A ⊆ U and U is Nαg-open in (U, τ_R(X))

Definition 2.8 If (U, τ_R(X)) is a nano topological space if A ⊆ U., then A is said to be
- N$g^{\#}\alpha$-continuous [9] if f^{-1}(A) is N$g^{\#}\alpha$-closed in (U, τ_R(X)) for every nano-closed set A in (V, σ_R(Y))
- Ng^*-continuous [12] if f^{-1}(A) is Ng^*-closed in (U, τ_R(X)) for every nano-closed set A in (V, σ_R(Y))
- Ng^*s-continuous [12] if f^{-1}(A) is Ng^*s-closed in (U, τ_R(X)) for every nano-closed set A in (V, σ_R(Y))
- Nĝ-continuous [6] if f^{-1}(A) is Nĝ-closed in (U, τ_R(X)) for every nano-closed set A in (V, σ_R(Y))
- Nsĝ-continuous [11] if f^{-1}(A) is Nsĝ-closed in (U, τ_R(X)) for every nano-closed set A in (V, σ_R(Y))
- Nĝα-continuous [10] if f^{-1}(A) is Nĝα-closed in (U, τ_R(X)) for every nano-closed set A in (V, σ_R(Y))
- Nαg^*s-continuous [8] if f^{-1}(A) is Nαg^*s-closed in (U, τ_R(X)) for every nano-closed set A in (V, σ_R(Y))
- N$g\alpha$g-continuous [5] if f^{-1}(A) is N$g\alpha$g-closed in (U, τ_R(X)) for every nano-closed set A in (V, σ_R(Y))

Definition 2.9[1] A subset A of a nano topological space (U, τ_R(X)) is called N$\hat{g}$*s - closed set if Nscl(A) ⊆ U whenever A ⊆ U and U is Nĝ-open in (U, τ_R(X)).

3. N$\hat{g}$*s-Continuous Functions

In this section we define N$\hat{g}$*s-Continuous functions and discuss some of their properties.

Definition 3.1 Let (U, τ_R(X)) and (V, σ_R(Y)) be two nano topological spaces. Then a mapping f: (U, τ_R(X)) → (V, σ_R(Y)) is called N$\hat{g}$*s-continuous if f^{-1}(S) is N$\hat{g}$*s-closed in (U, τ_R(X)) for every nano-closed set S in (V, σ_R(Y)).

Definition 3.2 A function f: (U, τ_R(X)) → (V, σ_R(Y)) is N$\hat{g}$*s-irresolute if f^{-1}(S) is N$\hat{g}$*s closed in (U, τ_R(X)) for each N$\hat{g}$*s -closed set S in (V, σ_R(Y)).

Example 3.3. Let (U, τ_R(X)) and (V, σ_R(Y)) be two nano topological spaces. Let U = {a, b, c, d} with U/R = {{b, c}, {a}, {d}} and X = {a}. Then the nano topology τ_R(X)

={∅,U,{a}}. Then N$\hat{g}$*s-C(U, τ_R(X)) = {∅, U,{b},{c},{a, b},{a, c}, {b, c}, {b, d}, {c, d}, {a, b, c}, {a, b, d}, {a, c, d }, {b, c, d}}. Let V = {a, b, c, d} with V/R = {{a}, {c}, {b, d}} and Y = {a, b, c}. σ_R(Y) = {∅, V, {a, c}, {b, d}}. (σ_R(Y)) = {∅, V, {a, c}, {b, d}}. N$\hat{g}$*s-C (V, σ_R(Y)) = {∅, U, {b}, {c}, {a, b}, {a, c}, {b, c}, {b, d}, {c, d}, {a, b, c}, {a, b, d}, {a, c, d}, {b, c, d}}. Define f: U → V as f (a) = b, f (b) = a, f(c) = d, f (d) = c. We have f^{-1} (a, c) = {b, d}, f^{-1} (b, d) = {a, c}. Thus, the inverse image of every N-closed set in V is N$\hat{g}$*s -closed U. Therefore, f is N$\hat{g}$*s –continuous.

4. Main Results

Proposition 4.1 Let (U, τ_R(X)) and (V, σ_R(Y)) be two nano topological spaces. A function f: (U, τ_R(X)) → (V, σ_R(Y)) is Nano-continuous then f is N$\hat{g}$*s-continuous.
Proof: Let A be any nano closed set in (V, σ_R(Y)). Since f is Nano continuous, f^{-1}(A) is nano closed in (U, τ_R(X)). Since every nano closed set is N$\hat{g}$*s-closed. Therefore f^{-1}(A) is N$\hat{g}$*s-closed in (U, τ_R(X)). Hence f is N$\hat{g}$*s-continuous.

Remark 4.2 The converse of the above theorem need not be true, as proved by the following example.

Example 4.3 Let (U, τ_R(X)) be a nano topological space where U = {a, b, c, d} with U/R = {{b}, {c}, {a, d}} and X = {a, b}. τ_R(X) = {∅, U, {b}, {a, d}, {a, b, d}}. $(\tau_R(X))^c$ = {∅, U, {a, c, d}, {c}, {b, c}}. Nĝ*s C (U, τ_R(X)) = {∅, U, {b}, {c},{a, c}, {a, d}, {b, c}, {c, d}, {a, b, c}, {a, c, d},{b, c, d}}. NSC (U, τ_R(X)) = {∅, U, {b}, {c}, {b, c}, {a, d}, {a, c, d}}.
NαC (U, τ_R(X)) = {∅, U, {b}, {c}, {b, c}, {a, d}, {a, c, d}}. Let (V, σ_R(Y)) be a nano topological space where V = {a, b, c, d} with V/ R = {{b}, {d}, {a, c}} and Y = {a, d}. Let σ_R(Y) = {∅, V, {d}, {a, c}, {a, c, d}}. $(\sigma_R(Y))^c$ = {∅, V, {a, b, c}, {b}, {b, d}}. Define f(a) = d, f(b) = a, f(c) = b, f(d) = c. Let f^{-1}(b) = {c}, f^{-1}(b, d) = {a, c}, f^{-1}(a, b, c) = {b, c, d}. Here {b, c, d} is Nĝ*s-closed but not N-closed.

Proposition 4.4 Let (U, τ_R(X)) and (V, σ_R(Y)) be two nano topological spaces. A function f: (U, τ_R(X)) → (V, σ_R(Y)) is Nα-continuous then f is N$\hat{g}$*s-continuous.
Proof Let A be any nano closed set in (V, σ_R(Y)). Since f is α-continuous, f^{-1}(A) is nano α-closed in (U, τ_R(X)). Since every nano α-closed set is N$\hat{g}$*s-closed. Therefore f^{-1}(A) is N$\hat{g}$*s-closed in (U, τ_R(X)). Hence f is N$\hat{g}$*s-continuous.

Remark 4.5 The converse of the above theorem need not be true, as proved by the following example.
From the example 3.6, the sub set A= {b, c, d} is not nano α-closed set in (U, τ_R(X)). Hence f is not nano α-continuous.

Proposition 4.6 Let (U, τ_R(X)) and (V, σ_R(Y)) be two nano topological spaces. A function f: (U, τ_R(X)) → (V, σ_R(Y)) is Nsemi-continuous then f is N$\hat{g}$*s-continuous.

Proof Let A be any nano closed set in (V, σ_R(Y)). Since f is semi-continuous, f^{-1}(A) is nano semi-closed in (U, τ_R(X)). Since every nano semi-closed set is N$\hat{g}$*s-closed. Therefore f^{-1}(A) is N$\hat{g}$*s-closed in (U, τ_R(X)). Hence f is N$\hat{g}$*s-continuous.

Remark 4.7 The converse of the above theorem need not be true, as proved by the following example.
From the example 3.6, the sub set A= {b, c, d} is not nano semi-closed set in (U, τ_R(X)). Hence f is not nano semi-continuous.

Proposition 4.8 Let (U, τ_R(X)) and (V, σ_R(Y)) be two nano topological spaces. A function f: (U, τ_R(X)) → (V, σ_R(Y)) is N$g^{\#}\alpha$-continuous then f is N$\hat{g}$*s-continuous.
Proof Let A be any nano closed set in (V, σ_R(Y)). Since f is N$g^{\#}\alpha$-continuous, f^{-1}(A) is N$g^{\#}\alpha$-closed in (U, τ_R(X)). Since every N$g^{\#}\alpha$-closed set is N$\hat{g}$*s-closed. Therefore f^{-1}(A) is N$\hat{g}$*s-closed in (U, τ_R(X)). Hence f is N$\hat{g}$*s-continuous.

Remark 4.9 The converse of the above theorem need not be true, as proved by the following example.

Example 4.10 Let (U, τ_R(X)) be a nano topological space where U = {a, b, c, d} with U/R = {{a}, {c}, {b, d}} and X = {a}. τ_R(X) = {∅, U, {a}}. Nĝ*sC (U, τ_R(X)) = {∅, U, {b},{c}, {d}, {a, b}, {a, d}, {a, c}, {b, c}, {b, d}, {c, d}, {a, b, d}, {a, b, c,},{a, c, d}}. N$g^{\#}\alpha$C (U, τ_R(X)) = {∅, U, {b}, {c}, {d}, {b, c}, {b, d}, {c, d}, {b, c, d}}. Let (V, σ_R(Y)) be a nano topological space where V = {a, b, c, d} with V/ R = {{a}, {c}, {b, d}} and Y = {a, b}. σ_R(Y) = {∅, V, {a}, {b, d}, {a, b, d}, (σ_R(Y))= {∅, V, {c}, {a, c}, {b, c, d}}. Define f(a) = b, f(b) = a, f(c) = c, f(d) = d. Let f^{-1}(c) = {c}. f^{-1}(a, c) ={b, c}, f^{-1}(b, c, d) ={a, c, d}. Here {a, c, d} is Nĝ*s closed but not N$g^{\#}\alpha$-closed.

Proposition 4.11 Let (U, τ_R(X)) and (V, σ_R(Y)) be two nano topological spaces. A function f: (U, τ_R(X)) → (V, σ_R(Y)) is N$\hat{g}\alpha$-continuous then f is N$\hat{g}$*s-continuous.
Proof Let A be any nano closed set in (V, σ_R(Y)). Since f is N$\hat{g}\alpha$-continuous, f^{-1}(A) is N$\hat{g}\alpha$-closed in (U, τ_R(X)). Since every N$\hat{g}\alpha$-closed set is N$\hat{g}$*s-closed. Therefore f^{-1}(A) is N$\hat{g}$*s-closed in (U, τ_R(X)). Hence f is N$\hat{g}$*s-continuous.

Remark 4.12 The converse of the above theorem need not be true, as proved by the following example.

Example 4.13 Let (U, τ_R(X)) be a nano topological space where U = {a, b, c, d} with U/R = {{a}, {b}, {c, d}} and X = {b, d}. τ_R(X) = {∅, U, {b}, {c, d}, {b, c, d}}. Nĝ*sC (U, τ_R(X)) = {∅, U, {a}, {b}, {a, b}, {a, d}, {a, c}, {c, d}, {a, b, d}, {a, b, c,}, {a, c, d}}. N$\hat{g}\alpha$C (U, τ_R(X)) = {∅, U, {a}, {a, b}, {a, c}, {a, d}, {c, d}, {a, b, c}, {a, b, d}, {a, c, d}}. Let (V, σ_R(Y)) be a nano topological space where V = {a, b, c, d} with V/ R = {{a}, {c}, {b, d}} and Y = {a, b, d}. σ_R(Y) = {∅, V, {a, b, d}}, (σ_R(Y)) = {∅, V, {c}}.

Define f(a) = b, f(b) = c, f(c) = d, f(d) = a. Let f^{-1}(c) = {b}. Here {b} is Nĝ*s-closed but not N$\hat{g}\alpha$-closed.

Proposition 4.14 Let (U, τ_R(X)) and (V, σ_R(Y)) be two nano topological spaces. A function f: (U, τ_R(X)) → (V, σ_R(Y)) is Ng*-continuous then f is Nĝ*s -continuous.
Proof Let A be any nano closed set in (V, σ_R(Y)). Since f is Ng*-continuous, f^{-1}(A) is Ng*-closed in (U, τ_R(X)). Since every Ng*-closed set is N$\hat{g}$*s-closed. Therefore f^{-1}(A) is N$\hat{g}$*s-closed in (U, τ_R(X)). Hence f is N$\hat{g}$*s-continuous.

Remark 4.15 The converse of the above theorem need not be true, as proved by the following example.

Example 4.16 Let (U, τ_R(X)) be a nano topological space where U = {a, b, c, d} with U/R = {{a}, {c}, {b, d}} and X = {b, c}. τ_R(X) = {∅, U, {c}, {b, d}, {b, c, d}}. $(\tau_R(X))^c$ = {∅, U, {a, b, d}, {a}, {a, c}}. Nĝ*sC (U, τ_R(X)) = {∅, U, {a}, {c}, {a, b}, {a, c}, {a, d}, {b, d}, {a, b, d}, {a, b, c,}, {a, c, d}}. Ng*C (U, τ_R(X)) = {∅, U, {a}, {a, b}, {a, c}, {a, d}, {b, d}, {a, b, c}, {a, b, d}, {a, c, d}}. Let (V, σ_R(Y)) be a nano topological space where V = {a, b, c, d} with V/ R = {{b}, {c}, {a, d}} and Y = {a, c}. σ_R(Y)= {∅, V, {c}, {a, d}, {a, c, d}}. $(\sigma_R(Y))^c$= {∅, V, {a, b, d}, {b}, {b, c}}. Define f(a) = c, f(b) = d, f(c) = b, f(d) = a. Let f^{-1}(a, b, d) = {a, b, c}, f^{-1}(b, c) = {a, c}, f^{-1}(b) = {c}. Here {c} is Nĝ*s-closed but not Ng*-closed.

Proposition 4.17 Let (U, τ_R(X)) and (V, σ_R(Y)) be two nano topological spaces. A function f: (U, τ_R(X)) → (V, σ_R(Y)) is Ng*s-continuous then f is Nĝ*s -continuous.
Proof Let A be any nano closed set in (V, σ_R(Y)). Since f is Ng*s-continuous, f^{-1}(A) is Ng*s-closed in (U, τ_R(X)). Since every Ng*s-closed set is N$\hat{g}$*s-closed. Therefore f^{-1}(A) is N$\hat{g}$*s-closed in (U, τ_R(X)). Hence f is N$\hat{g}$*s-continuous.

Remark 4.18 The converse of the above theorem need not be true, as proved by the following example.

Example 4.19
Let (U, τ_R(X)) be a nano topological space where U = {a, b, c, d} with U/R = {{a}, {b}, {c, d}} and X = {a, c}. τ_R(X) = {∅, U, {a}, {c, d}, {a, c, d}}. $(\tau_R(X))^c$ = {∅, U, {b, c, d}, {b}, {a, b}}. Nĝ*sC (U, τ_R(X)) = {∅, U, {a}, {b}, {a, b}, {b, c}, {b, d}, {c, d}, {a, b, d}, {a, b, c,}, {b, c, d}}. Ng*sC (U, τ_R(X)) = {∅, U, {a}, {a, b}, {b, c}, {b, d}, {c, d}, {a, b, c}, {a, b, d}, {b, c, d}}. Let (V, σ_R(Y)) be a nano topological space where V = {a, b, c, d} with V/ R = {{a}, {c}, {b, d}} and Y = {a, b}. σ_R(Y) = {∅, V, {a}, {b, d}, {a, b, d}}. $(\sigma_R(Y))^c$= {∅, V, {b, c, d}, {c}, {a, c}}. Define f(a) = d, f(b) = c, f(c) = b, f(d) = a. Let f^{-1}(b, c, d) = {a, b, c}, f^{-1}(a, c) = {b, d}, f^{-1}(c) = {b}. Here {b} is Nĝ*s-closed but not Ng*s-closed.

Proposition 4.20 Let (U, τ_R(X)) and (V, $\sigma_{R'}$(Y)) be two nano topological spaces. A function f: (U, τ_R(X)) → (V, σ_R(Y)) is Nsĝ-continuous then f is Nĝ*s -continuous.
Proof Let A be any nano closed set in (V, $\sigma_{R'}$(Y)). Since f is Nsĝ-continuous, f^{-1}(A) is Nsĝ -closed in (U, τ_R(X)). Since every Nsĝ-closed set is N$\hat{g}$*s-closed. Therefore f^{-1}(A) is N$\hat{g}$*s closed in (U, τ_R(X)). Hence f is N$\hat{g}$*s-continuous.

Remark 4.21 The converse of the above theorem need not be true, as proved by the following example.
Let (U, τ_R(X)) be a nano topological space where U = {a, b, c, d} with U/R = {{b}, {c}, {a, d}} and X = {a, b}. (U) = {∅, U, {b}, {a, d}, {a, b, d}}. $(\tau_R(X))^c$ = {∅, U, {a, c, d}, {c}, {b, c}}. Nĝ*sC (U, τ_R(X)) = {∅, U, {b}, {c}, {a, c}, {a, d}, {b, c}, {c, d}, {a, c, d}, {a, b, c,}, {b, c, d}}.
NsĝC (U, τ_R(X)) = {∅, U, {c}, {b, c}, {a, c, d}}. Let (V, σ_R(Y)) be a nano topological space where V = {a, b, c, d} with V/ R = {{a}, {b}, {c, d}} and Y = {b, c}. σ_R(Y) = {∅, V, {b}, {c, d}, {b, c, d}}. $(\sigma_R(Y))^c$= {∅, V, {a, c, d}, {a}, {a, b}}. Define f(a) = b, f(b) = a, f(c) = d, f(d) = c. Let f^{-1}(a, c, d) = {b, c, d}, f^{-1}(a, b) = {a, b}, f^{-1}(a) = {b}. Here {b} is Nsĝ-closed but not Ng*s-closed.

Proposition 4.22 Let (U, τ_R(X)) and (V, σ_R(Y)) be two nano topological spaces. A function f: (U, τ_R(X)) → (V, σ_R(Y)) is Ngαg-continuous then f is Nĝ*s -continuous.
Proof Let A be any nano closed set in (V, σ_R(Y)). Since f is Ngαg-continuous, f^{-1}(A) is Ngαg-closed in (U, τ_R(X)). Since every Ngαg-closed set is N$\hat{g}$*s-closed. Therefore f^{-1}(A) is N$\hat{g}$*s-closed in (U, τ_R(X)). Hence f is N$\hat{g}$*s-continuous.

Remark 4.23 The converse of the above theorem need not be true, as proved by the following example.

Example 4.24 Let (U, τ_R(X)) be a nano topological space where U = {a, b, c, d} with U/R = {{b}, {c}, {a, d}} and X = {a, b}. τ_R(X) = {∅, U, {b}, {a, d}, {a, b, d}}. $(\tau_R(X))^c$ = {∅, U, {a, c, d}, {c}, {b, c}}. Nĝ*sC (U, τ_R(X)) = {∅, U, {b}, {c}, {a, c}, {a, d}, {b, c}, {c, d}, {a, c, d}, {a, b, c,},{b, c, d}}. NsĝC (U, τ_R(X)) = {∅, U, {c}, {b, c}, {a, c, d}}. Let (V, σ_R(Y)) be a nano topological space where V = {a, b, c, d} with V/ R' = {{a}, {b}, {c, d}} and Y = {b, c}. σ_R(Y) = {∅, V, {b}, {c, d}, {b, c, d}}. $(\sigma_R(Y))^c$= {∅, V, {a, c, d}, {a}, {a, b}}. Define f(a) = b, f(b) = a, f(c) = d, f(d) = c. Let f^{-1}(a, c, d) = {b, c, d}, f^{-1}(a, b) = {a, b}, f^{-1}(a) = {b}. Here {b} is Nsĝ-closed but not Ng*s-closed.

Proposition 4.25 Let (U, τ_R(X)) and (V, σ_R(Y)) be two nano topological spaces. A function f: (U, τ_R(X)) → (V, σ_R(Y)) is Nαg^*s-continuous then f is Nĝ*s -continuous.
Proof Let A be any nano closed set in (V, σ_R(Y)). Since f is Nαg^*s-continuous, f^{-1}(A) is Nαg^*s-closed in (U, τ_R(X)). Since every Nαg^*s-closed set is N$\hat{g}$*s-closed. Therefore f^{-1}(A) is N$\hat{g}$*s-closed in (U, τ_R(X)). Hence f is N$\hat{g}$*s-continuous.

Remark 4.26 The converse of the above theorem need not be true, as proved by the following example.

Example 4.27 Let (U, τ_R(X)) be a nano topological space where U = {a, b, c, d} with U/R = {{b}, {c}, {a, d}} and X = {b, c}. τ_R(X) = {∅, U, {b, c}}. $(\tau_R(X))^c$ = {∅, U, {a, d}} Nĝ*sC(U, τ_R(X)) = {∅, U, {a},{d},{a, b}, {a, d}, {b, c}, {b, d}, {c, d}, {a, b, d}, {a, c, d}, {a, b, c,},{b, c, d}}.
N$\alpha g^* s$C (U, τ_R(X)) = {∅, U, {a}, {d}, {a, d}}. Let (V, $\sigma_{R'}$(Y)) be a nano topological space where V = {a, b, c, d} with V/ R' = {{b}, {c}, {a, d}} and Y = {a, c}. σ_R(Y) = {∅, V, {c}, {a, d}, {a, c, d}}. $(\sigma_R(Y))^c$= {∅, V, {a, b, d}, {b}, {b, c}}. Define f(a) = b, f(b) = c, f(c) = d, f(d) = a. Let f^{-1}(a, b, d) = {a, c, d}, f^{-1}(b, c) = {a, b}, f^{-1}(b) = {a}. Here {a, b} is N$\hat{g}$*s-closed but not Ng*s-closed.

Proposition 4.28
Composition of two Nĝ*s-continuous function need not be Nĝ*s-continuous.
Let (U, τ_R(X)) be a nano topological space where U = {a, b, c, d} with U/R = {{b}, {c}, {a, d}} and X = {a, b}. τ_R(X) = {∅, U, {b}, {a, b, d}, {a, d}}. $(\tau_R(X))^c$ = {∅, U, {a, c, d}, {c}, {b, c}} Nĝ*sC(U, τ_R(X)) = {∅, U, {b},{c},{a, c}, {a, d}, {b, c}, {c, d}, {a, b, c}, {a, c, d}, {b, c, d}}.
Define f(a) = d, f(b) = a, f(c) = b, f(d) = c. Let f^{-1}(a, b, c) = {b, c, d}, f^{-1}(b, d) = {a, c}, f^{-1}(b) = {c}. Here {a, b} is N$\hat{g}$*s-but not Ng*s-closed.
Let (V, σ_R(Y)) be a nano topological space Let V = {a, b, c, d} with V/ R = {{b}, {d}, {a, c}} and Y = {a, d}. σ_R(Y) = {∅, V, {d}, {a, c}, {a, c, d}}. $(\sigma_R(Y))^c$= {∅, V, {a, b, c}, {b}, {b, d}}. Nĝ*sC(U, τ_R(X)) = {∅, U, {b},{d},{a, b}, {a, c}, {b, c}, {b, d}, {a, b, c}, {a, b, d}, {b, c, d}}.
Let (W, (Z)) be a nano topological space Let W = {a, b, c, d} with W/ R = {{a}, {c}, {b, d}} and Z = {b, c}. (Z) = {∅, W, {c}, {b, d}, {b, c, d}}. $((Z))^c$ = {∅, V, {a, b, d}, {a}, {a, c}}. Define g(a) = b,g (b) = a, g(c) = d, g(d) = c. Let g^{-1}(a, b, d) = {a, b, c}, g^{-1}(a, c) = {b, d}, g^{-1}(a) = {b}.
Now, : (U, τ_R(X)) → (W, μ_R(Z)) by $g(f$(a) = c, $g(f$(b) = b, $g(f$c) = a, $g(f$(d) = d,
$g^{-1}(f^{-1}$(a)) = a, $g^{-1}(f^{-1}$(a, c)) =(a, c), $g^{-1}(f^{-1}$(a, b, d)) =(a, b, d) is not N$\hat{g}$*s-closed in U but {a, b, d} is closed in Z. Therefore, is not N$\hat{g}$*s-continuous.

Proposition 4.29 Let f : (U, τ_R(X)) → (V, σ_R(Y)) be N$\hat{g}$*s be a function. Then following are equivalent.
(i) f is N$\hat{g}$*s -continuous.
(ii) f^{-1}(A) is N$\hat{g}$*s -open for each open set A in Y.
Proof: (i) ⇒ (ii)
Suppose that f is a N$\hat{g}$*s -continuous. Let A be N-open in U. Then A^c is N-closed in V. Since f is N$\hat{g}$*s -continuous, we have f^{-1} (A^c) is N$\hat{g}$*s -closed in U. But $f^{-1}(A^c)$ = $[f^{-1}(A)]^c$. Hence f^{-1} (A) is Nĝ*s-open in U.

(ii) ⇒ (i) Suppose that f^{-1} (A) is N$\hat{g}$*s -open for each N-open set A in V. Let S be N-closed in V. Then S^c is nano open in V. By assumption, f^{-1} (S^c) is N$\hat{g}$*s -open in U and hence f^{-1}(S) is N$\hat{g}$*s -closed in U. Hence f is Nĝ*s-continuous.

Proposition 4.30 Let f : (U, τ_R(X)) → (V, σ_R(Y)) be function
(i) f is N$\hat{g}$*s-continuous.
(ii) For each u in U and for each N-open set B containing f(u), there is a N$\hat{g}$*s -open set A containing u such that f(A) ⊆ B.
(iii) f(N$\hat{g}$*s cl(A)) ⊆ Ncl(f(A)) for each subset A of U.
(iv) N$\hat{g}$*scl f^{-1}(B)) ⊆ f^{-1}(Ncl(f(B)) for each subset B of V. Then (i) ⇒ (ii) ⇒ (iii) ⇒ (i)
Proof: (i) ⇒ (ii)
Let u ∈ U and B be an open set containing f(u). Then, by (i) f^{-1}(B) is N$\hat{g}$*s -open set of U containing u. If A = f^{-1}(B), then f(A) = f(f^{-1}(B)) ⊆ B.
(ii) ⇒ (iii) Let A be a subset of a space U and f(u) ∉ Ncl(f(A)). Then there exists N-open set B of V containing f(u) such that B ∩ f (A) = ∅. Now, by (ii), there is a N$\hat{g}$*s -open set G containing u such that f(u) ∈ f (G) ⊆ B. Hence f(A) ∩ f (G) = ∅. That is, f(A ∩ G) = ∅. i.e., A ∩ G) = ∅. Therefore, u ∉ N$\hat{g}$*scl (A) and also (u) ∉ N$\hat{g}$*scl f (A). Therefore f (N$\hat{g}$*s cl(A) ⊆ Ncl(f(A))
(iii) ⇒ (iv) Let B be a subset of V such that A = f^{-1}(B). By (iii), f(N$\hat{g}$*scl(A) ⊆ Ncl(f(A) for each subset A of U. Therefore, (N$\hat{g}$*scl f^{-1}(B)) ⊆ Ncl (f(f^{-1} (B))). i.e., f (N$\hat{g}$*s cl f^{-1}(B)) ⊆ Ncl(B). i.e., N$\hat{g}$*scl f^{-1} (B) ⊆ f^{-1}(Ncl(B)).

Lemma 4.31 A subset A of a nano topological space (U, τ_R(X)) is N$\hat{g}$*s -open iff F ⊆ Nsint (A) whenever F ⊆ A and F is Nĝ-closed.

Proposition 4.32 Let B be a N$\hat{g}$*s open (or N$\hat{g}$*s -closed) subset of (V, $\tau_{R'}$(Y)) (satisfying
Nsint(B) =Nint(B). Then f^{-1} (B) is N$\hat{g}$*s -open (or N$\hat{g}$*s -closed) in (U, τ_R(X)). If f : (U, τ_R(X)) → (V, $\tau_{R'}$(Y)) is Nĝ*s-continuous and if image of a Nĝ-closed set in U under f is Nĝ closed set in V.
Proof. Let B be a N$\hat{g}$*s -open set in V. Let F ⊆ f^{-1} (B) where F is a Nĝ-closed set in U. Then f(F) ⊆ B holds. By our assumption, f(F) is Nĝ-closed set in V and B be a N$\hat{g}$*s -open set in V. Therefore, by lemma 3.7 f(F) ⊆ Nsint(B) holds. Again, by our assumption, f (F) ⊆ Nint(B) and hence F ⊆ f^{-1} (Nint(B)) holds. Since f is N$\hat{g}$*s -continuous and Nint (B) is N open in V, f^{-1}(int (B)) is N$\hat{g}$*s -open in U So, by lemma 3.7, F ⊆ Nsint (f^{-1} (Nint (B))) holds. i.e., F ⊆ Nsint (f^{-1} (Nint (B))) ⊆ Nsint (f^{-1} (B)) holds. Therefore f^{-1} (B) is N$\hat{g}$*s -open. By taking complements, we can show that if B is N$\hat{g}$*s -closed in V, then f^{-1} (B) is N$\hat{g}$*s -closed in U.

Proposition 4.33 Let f : (U, τ_R(X)) → (V, σ_R(Y)) be a function. Then the following are equivalent.
(i) f is Nĝ*s-irresolute.
(ii) f^{-1}(B) is N$\hat{g}$*s -open for each N$\hat{g}$*s -open set B in V.

Proof: (i) ⇒ (ii)
Suppose that f is N$\hat{g}$*s -irresolute. Let B be N$\hat{g}$*s -open in V. Then B^c is N$\hat{g}$*s -closed in V. Since f is N$\hat{g}$*s -irresolute, we have $f^{-1}(B^c)$ is N$\hat{g}$*s -closed in U. But $f^{-1}(B^c) = [f^{-1}(B)]^c$. Therefore f^{-1}(B) is Nĝ*s-open in U.
(ii) ⇒ (i) N$\hat{g}$*s
Suppose that f^{-1} (B) is N$\hat{g}$*s -open for each is N$\hat{g}$*s -open set B in V. Let H be is N$\hat{g}$*s closed in V. Then H^c is N$\hat{g}$*s -open in V. Therefore f^{-1} (H^c) is N$\hat{g}$*s -open in U. Therefore f^{-1} (H) is N$\hat{g}$*s -closed in U. Therefore, f is N$\hat{g}$*s -irresolute.

Proposition 4.34 If a function f : (U, τ_R(X)) → (V, σ_R(Y)) is N$\hat{g}$*s -irresolute, then f is N$\hat{g}$*s -continuous.
Proof: Let B be a N-closed set of V. But every N-closed set is N$\hat{g}$*s -closed. Therefore, B is a N$\hat{g}$*s -closed set of V. Since f is N$\hat{g}$*s -irresolute, f^{-1}(B) is N$\hat{g}$*s -closed in U. Hence, by definition 3.2 f is N$\hat{g}$*s -continuous.

Remark 4.35 The converse of Proposition 3.10 need not be true as seen from the following example.

Example 4.36 Let (U, τ_R(X)) and (V, σ_R(Y)) be a nano topological spaces where U = V = {a, b, c, d} with U/R = {{b}, {c}, {a, d}} and X = {a, c}. Then the nano topology τ_R(X) = {∅, U, {c}, {a, c, d}, {a, d}}. (τ_R(X)) = {∅, U, {a, b, d}, {b}, {b, c}}. Then N$\hat{g}$*s C (U, τ_R(X)) = {∅, U, {b}, {c}, {a, b}, {a, d}, {b, c}, {b, d},{a, b, d}, {b, c, d}}.
Let V/ R = {{a}, {b, c}, {d}} and Y = {a, c}. Then the nano topology τ_R(X) = {∅, V, {a}, {a, b, c}, {b, c}}. $(\sigma_R(Y))^c$= {∅, U, {b, c, d}, {d}, {a, d}}. Then N$\hat{g}$*sC (U, τ_R(X)) = {∅, U, {a}, {d}, {b, d},{c, d}, {a, d}, {b, c, d}, {a, b, d}, {a, c, d}}. Define f : U → V as f(a) = c, f(b) = d, f(c) = a, f(d) = b. We have f^{-1}(a) = {c}, f^{-1}(d) = {b}, f^{-1}(b, d) = {a, d}, f^{-1}(c, d) = {a, b}, f^{-1}(a, d) = {b, c}, f^{-1}(a, b, d) = {b, c, d}, f^{-1}(a, c, d) = {a, b, c}, f^{-1}(b, c, d) = {a, b, d}. f^{-1}(B) is Nĝ*s-closed for each nano-closed set B in V. Hence f: U → V is N$\hat{g}$*s continuous. But, f^{-1}(a, c, d) = {a, b, c} is not N$\hat{g}$*s -closed. Therefore f : U → V is not N$\hat{g}$*s -irresolute.

Proposition 4.37 If a function f : (U, τ_R(X)) → (V, σ_R(Y)) is N$\hat{g}$*s -irresolute, then for every subset A of U. Then f(N$\hat{g}$*scl(A)) ⊆ Nscl(f(A)).
Proof: Let A ⊆ U. We know that every Ns-closed set is N$\hat{g}$*s-closed set in V. Therefore, we have Nscl(f(A)) is N$\hat{g}$*s -closed in V. Since f is N$\hat{g}$*s -irresolute, then f^{-1}(Nscl(f(A))) is N$\hat{g}$*s-closed in U. Also A ⊆ f^{-1} (f(A)) ⊆ f^{-1} (Nscl (f(A))). Since f^{-1}(Nscl(f(A))) is N$\hat{g}$*s -closed, we have N$\hat{g}$*scl(A) ⊆ f^{-1} (Nscl(f(A))). Therefore f(N$\hat{g}$*scl(A)) ⊆ f{f^{-1} (Nscl(f (A)))} ⊆ Nscl(f (A)).

Proposition 4.38 If a function f: (U, τ_R(X)) → (V, σ_R(Y)) is bijective, N$\hat{g}$*s -continuous, Nscl (A) = Ncl(A) for all subsets B in V and if image of a Nĝ-open set is Nĝ-open under f, then f is N$\hat{g}$*s -irresolute.

Proof: Let B be N$\hat{g}$*s -closed set of V. Let f^{-1} ((B)) ⊆ A where A is Nĝ-open in U. Then $f^{-1}(f$(B)) ⊆ f (A). Since f is surjective, B ⊆ f(A). Since f(A) is Nĝ-open and since B is N$\hat{g}$*s -closed in V, we have Nscl(B) ⊆ f (A). By our assumption, Ncl(B) ⊆ f (A). Since f is injective, f^{-1}(Ncl(B)) ⊆ A. Since f is N$\hat{g}$*s -continuous and since Ncl(B) is N-closed in V, f^{-1}(Ncl(B)) is Nĝ*s-closed in U. Therefore f^{-1}((B)) is N$\hat{g}$*s -closed in U and hence f is N$\hat{g}$*s - irresolute.

Definition 4.39 A function f : (U, τ_R(X)) → (V, σ_R(Y)) is called a N$\hat{g}$*s -closed map if f (A) is N$\hat{g}$*s -closed in V for every N-closed set A of U

Definition 4.40 A function f : (U, τ_R(X)) → (V, σ_R(Y)) is called a Nĝ*s-open map if f (A) is Nĝ*s-open in V for every N-open set A of U

Proposition 4.41 If f : (U, τ_R(X)) → (V, σ_R(Y)) is N$\hat{g}$*s -irresolute and A is a N$\hat{g}$*s closed subset of U, then f (A) is N$\hat{g}$*s -closed in V.
Proof: Let f(A) ⊆ B and B is Nĝ-open in V. Then $f^{-1}(f$ (A)) ⊆ f^{-1} (B). i.e., A ⊆ f^{-1} (B). Since f is Nĝ-irresolute, f^{-1}(B) is Nĝ-open in U. Since A is N$\hat{g}$*s -closed, Ncl(A) ⊆ f^{-1} (B). So, f(Ncl(A)) ⊆ $f(f^{-1}$ (B)). i.e., f(Ncl(A)) ⊆ U. Since f is N$\hat{g}$*s -closed and Ncl(A) is N closed in U, f(Ncl(A)) is N$\hat{g}$*s -closed in V. Therefore Nscl (f (Ncl (A)) ⊆ B. Since f(A) ⊆ f (Ncl (B)), we have Nscl (f (A)) ⊆ Nscl (f(Ncl (B)) ⊆ B. Therefore f (A) is N$\hat{g}$*s -closed in V.

Proposition 4.42 If f : (U, τ_R(X)) → (V, σ_R(Y)) is N-closed and: V→ W is N$\hat{g}$*s -closed, then gof is N$\hat{g}$*s -closed.
Proof: Let A be a N-closed set of U. Since f is N-closed, f (A) is N-closed in V. Since g is N$\hat{g}$*s -closed, (A)) is N$\hat{g}$*s -closed in W. Hence gof: U→ W is N$\hat{g}$*s -closed.

Proposition 4.43 Let f : (U, τ_R(X)) → (V, σ_R(Y)) is N-closed and$g$: (V, σ_R(Y)) → (W, (Z)) be two maps such that gof : (U, τ_R(X)) → (W, μ_R(Z)) is N$\hat{g}$*s -open map, if f is N continuous and surjective.
Proof: Let B be a N-open V. Since f is N-continuous, f^{-1} (B) is N-open in U. Since f^{-1}(B) is N-open in U, go$f(f^{-1}$ (B))) is N$\hat{g}$*s -open in W. i.e., g(B) is N$\hat{g}$*s -open in W. Therefore, g is a N$\hat{g}$*s -open map.

Proposition 4.44 For any bijection f : (U, τ_R(X)) → (V, σ_R(Y)), the following are equivalent:
(i) f^{-1} : (V, σ_R(Y)) → U is Nĝ*s-continuous.
(ii) f is N$\hat{g}$*s -open.
(iii) f is N$\hat{g}$*s -closed.

Proof:

(i) ⇒ (ii) Let A be N-open in U. Then U-A is N-closed in U. Since f^{-1} is N$\hat{g}$*s - continuous, $(f^{-1})^{-1}$ (U-A) = f(U – F) = V – f(F) is N$\hat{g}$*s -closed in V. Then f(F) is N$\hat{g}$*s -open in V. Hence f is N$\hat{g}$*s -open.

(ii) ⇒ (iii) Let A be N-closed in U. Then U-A is N-open in U. Since f is N$\hat{g}$*s - open, f(U – F) = V – f (F) is N$\hat{g}$*s -open in V. Then f(F) is N$\hat{g}$*s -closed in V. Hence f is N$\hat{g}$*s closed.

(iii) ⇒ (i) Let A be N-closed in U. Since f: (U, τ_R(X)) → (V, σ_R(Y)) is N$\hat{g}$*s - closed. f (A) is N$\hat{g}$*s closed in V. i.e., $(f^{-1})^{-1}$(A) N$\hat{g}$*s -closed in V. Therefore $f^{-1}$is N$\hat{g}$*s -continuous.

5. Conclusion

In this paper, we introduced and studied the concepts of Nĝ*s-continuous and N$\hat{g}$*s - irresolute in nano topological spaces and we compare it with other nano-continuous and irresolute function and proved that composition of two Nĝ*s-continuous functions need not be a Nĝ*s-continuous functions. We also investigate some of its properties and give suitable examples for the reverse which is not true. In future this work will be extended with some real-life applications.

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