Ratio Mathematica Volume 47, 2023

Υ_G-operator in grill N-topology

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Abstract

In 2017, Lellis Thivagar et.al [4] introduced a closure operator $N\tau_G$-cl by using the local function Φ_G in grill N-topology. In this article, we introduce a new operator Υ_G in the same topological space. We study the properties of this new operator which helps us to derive a few equivalent expressions and a characterizing condition, in terms of Υ_G. Then a suitability condition for a grill in N-topological space X is formulated. Also, we discuss the characterizing condition for the discussed suitability condition. In addition, we introduce and study $\widehat{\Upsilon}_G$–sets and utilize the Υ_G -operator to define a generalized open set and their properties.

Keywords: Grill N-topology, N-topology suitable for grill, relatively G-dense, anti-co dense.
2020 AMS subject classifications: 54A05, 54A99, 54C10. [1]

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[1]Received on November 29, 2022. Accepted on April 28, 2023. Published on June 30, 2023. DOI: 10.23755/rm.v41i0.967. ISSN: 1592-7415. eISSN: 2282-8214.

1 Introduction

The grill concept is a powerful supporting tool, like nets and filters, in dealing with many a topological concept quite effectively. The idea of grill on a topological space was first introduced by Coquet [1]. Later Cattopadhyay and Thorn [2] proved the grills are always unions of ultra-filters. Further Roy and Mukherjee [6] established typical topology associated with grill on a topological space (X, τ). Lellis thivagar et.al [4] initiated the concept of grill in N-topological space and a topology $N\tau_G$ was introduced in terms of an operator Φ_G, constructed rather naturally from a grill G on a N-topological space (X, τ).

In this paper, we endeavour for an investigation in grill-associated N-topology with a new orientation. we introduce a new operator Υ_G, defined in terms of the previously introduced operator Φ_G, as a kind of dual of Φ_G. We study the properties of this new operator which helps us to derive equivalent expressions for the operator Υ_G and a characterizing condition, in terms of Υ_G. Also, a suitability condition is formed for a grill in N-topological space. Also, we discuss the characterizing condition for the discussed suitability condition. Finally, we introduce and study $\widehat{\Upsilon}_G$–sets and utilize the Υ_G -operator to define a generalized open set and their properties.

2 Prerequisites

In this section we recollect some definitions and results which are beneficial in the sequent. By a space X, we mean a grill N-topological space $(X, N\tau, G)$ with N-topology $N\tau$ and grill G on X on which no separation axioms are assumed unless explicitly stated.

Definition 2.1. *[4] A non-empty collection G of non empty subsets of a N- topological space $(X, N\tau)$ is called a grill on X if,*

(i) $A \in G$ and $A \subseteq B \subseteq X \implies B \in G$,

(ii) $A, B \subseteq X$ and $A \cup B \in G \implies A \in G$ or $B \in G$.
Then a N-topological space $(X, N\tau)$ together with a grill G is called a grill N-topological space and it is denoted by $(X, N\tau, G)$.

Remark 2.2. *[4] In $(X, N\tau)$, the following are true.*

(i) The grill $G = \mathcal{P}(A) - \{\emptyset\}$ is maximal grill in any N-topological space $(X, N\tau)$

(ii) The grill $G = \{X\}$ is the minimal grill in any N-topological space $(X, N\tau)$.

Definition 2.3. *[4] Let $(X, N\tau, G)$ be a grill N-topological space and for each $A \subseteq X$, the operator $\Phi_G(A, N\tau) = \{x \in X \mid A \cap U \in G, \forall\, U \in N\tau(x)\}$ is called the local function associated with the grill G and the N-topology $N\tau$. It is denoted as $\Phi_G(A)$. For any point x of a N-topological space $(X, N\tau)$, $N\tau(x)$ means the collection of all $N\tau$-open sets containing x.*

Theorem 2.4. *[4] Let $(X, N\tau)$ be a N-topological space. Then the following are true.*

(i) If G is any grill on X, then Φ_G is an increasing function in the sense that $A \subseteq B$ implies $\Phi_G(A, N\tau) \subseteq \Phi_G(B, N\tau)$.

(ii) If G_1 and G_2 are two grills on X with $G_1 \subseteq G_2$, then $\Phi_{G_1}(A, N\tau) \subseteq \Phi_{G_2}(A, N\tau)$ for all $A \subseteq X$.

(iii) For any grill G on X and if $A \notin G$, then $\Phi_G(A, N\tau) = \emptyset$.

Theorem 2.5. *[4] Let $(X, N\tau, G)$ be a grill N-toplogical space. Then for all $A, B \subseteq X$.*

(i) $\Phi_G(A \cup B) \supseteq \Phi_G(A) \cup \Phi_G(B)$

(ii) $\Phi_G(\Phi_G(A)) \subseteq \Phi_G(A) = N\tau\text{-}cl(\Phi_G(A)) \subseteq N\tau\text{-}cl(A)$.

Theorem 2.6. *[4] If G is a grill on a N-toplogical space $(X, N\tau)$ with $N\tau - \{\emptyset\} \subseteq G$, then for all $U \in N\tau$, $U \subseteq \Phi_G(U)$.*

Lemma 2.7. *[4] For any grill G on a N-topological space $(X, N\tau)$ and any $A, B \subseteq X, \Phi_G(A) - \Phi_G(B) = \Phi_G(A - B) - \Phi_G(B)$.*

Corolary 2.8. *[4] Let G is a grill on a N-topological space $(X, N\tau)$ and suppose $A, B \subseteq X$ with $B \notin G$. Then $\Phi_G(A \cup B) = \Phi_G(A) = \Phi_G(A - B)$.*

Proposition 2.9. *[4] Corresponding to a grill G on a N-topological space $(X, N\tau)$, the operator $N\tau_G\text{-}cl : \mathcal{P}(X) \to \mathcal{P}(X)$ defined by $N\tau_G\text{-}cl(A) = A \cup \Phi_G(A)$, for all $A \subseteq X$, satisfies Kuratowski's closure axioms and also there exists a unique topology $N\tau_G = \{U \subseteq X \mid N\tau_G\text{-}cl(U^c) = U^c\}$ which is finer than $N\tau$.*

Proposition 2.10. *[4] In a grill N-topological space $(X, N\tau, G)$, $N\tau \subseteq \beta(G, N\tau)$ $\subseteq N\tau_G$ and in particular if $G = \mathcal{P}(X) - \{\emptyset\}$ then $N\tau = \beta(G, N\tau) = N\tau_G$.*

Proposition 2.11. *[4] In a grill N-topological space $(X, N\tau, G)$ and $A \subseteq X$ such that $A \subseteq \Phi_G(A)$, then $N\tau$-$cl(A) = N\tau_G$-$cl(A) = N\tau$-$cl(\Phi_G(A)) = \Phi_G(A)$.*

Definition 2.12. *[9] In $(X, N\tau)$, $S \subseteq X$ then S is $N\tau$-dense if $N\tau$-$cl(S) = X$.*

3 Υ_G-operator via grills

In this section, we suggest a new operator is called $\Upsilon_G(A, N\tau)$ (upsilon) in grill N-topological space, and take up some basic associated results. Throughout this section $(X, N\tau, G)$ denotes a grill N-topological space.

Definition 3.1. *Let G be a grill on a N-topological space $(X, N\tau)$. We define a map $\Upsilon_G : \mathcal{P}(X) \to \mathcal{P}(X)$, given by $\Upsilon_G(A, N\tau) = X - \Phi_G(X - A)$ for any $A \subseteq X$. We shall simply write $\Upsilon_G(A)$, assuming that the grill G under consideration is understood.*

Remark 3.2. *It follows from theorem 2.5 (ii) that Υ_G is open in $(X, N\tau)$ for any subset A of X. Thus Υ_G treated as a mapping from $\mathcal{P}(X)$ to $N\tau$.*

Remark 3.3. *In view of Theorem 2.4(ii) it turns out that for two grills G_1 and G_2 on X, $G_1 \subseteq G_2 \implies \Upsilon_{G_1}(A) \supseteq \Upsilon_{G_2}(A)$. But for a given grill G on X, Υ_G is increasing in the sense that whenever $A \subseteq B \subseteq X$, then $\Upsilon_G(A) \subseteq \Upsilon_G(B)$. This is again an immediate consequence of theorem 2.4 (i); however it may so happen that $\Upsilon_G(A) \subseteq \Upsilon_G(B)$ even if $A \nsubseteq B$. The following is an example to justify our contention.*

Example 3.4. *Let $N = 3$ and $X = \{s, t, u\}$ and consider $\tau_1 O(X) = \{\emptyset, X, \{t, u\}\}$, $\tau_2 O(X) = \{\emptyset, X, \{s\}\}$ and $\tau_3 O(X) = \{\emptyset, X, \{s\}, \{t, u\}\}$. Then $3\tau O(X)$=$\{\emptyset, X, \{s\},$ $\{t, u\}\}$ is a tri topology and consider the grill $G = \{\{s\}, \{u\}, \{s, u\}, \{s, t\}, X\}$. Thus $(X, 3\tau, G)$ is a grill tri topological space on X. Now, $\Phi_G(\{s\}) = \{s\}$ and $\Phi_G(\{t\}) = \emptyset$. Then $\Upsilon_G(\{t, u\}) = X - \Phi_G(\{s\}) = \{t, u\}$ and $\Upsilon_G(\{s, u\})$ $= X - \Phi_G(\{t\}) = X$. Thus $\Upsilon_G(\{t, u\}) \subseteq \Upsilon_G(\{s, u\})$ although $\{t, u\} \nsubseteq \{s, u\}$.*

Theorem 3.5. *Let $(X, N\tau, G)$ be a grill N-topological space. Then the following statements are true:*

(i) If $S \in N\tau_G$ then $S \subseteq \Upsilon_G(S)$.

(ii) If $S, T \subseteq X$ then $\Upsilon_G(S \cap T) = \Upsilon_G(S) \cap \Upsilon_G(T)$.

(iii) If $S \subseteq X$ and $S \notin G$, then $\Upsilon_G(S) = X - \Phi_G(X)$.

(iv) If $S, T \subseteq X$ with $T \notin G$, then $\Upsilon_G(S) = \Upsilon_G(S - T) = \Upsilon_G(S \cup T)$.

(v) If $S, T \subseteq X$ with $(S - T) \cup (T - S) \notin G$, then $\Upsilon_G(S) = \Upsilon_G(T)$.

Proof. (i) In fact, $S \in N\tau_G \implies \Phi_G(X - S) \subseteq X - S$ by result 2.9,
$S \subseteq X - \Phi_G(X - S) = \Upsilon_G(S)$.
(ii) $\Upsilon_G(S \cap T) = X - \Phi_G(X - (S \cap T)) = X - \Phi_G[(X - S) \cup \Phi_G(X - T)] = [X - \Phi_G[(X - S)] \cap \Phi_G[(X - T)] = \Upsilon_G(S) \cap \Upsilon_G(T)$.
(iii) $\Upsilon_G(S) = X - \Phi_G(X - S) = X - [\Phi_G(X - S) - \Phi_G(S)] = X - [\Phi_G(X) - \Phi_G(S)] = X - \Phi_G(X)$.
(iv) $\Upsilon_G(S - T) = X - \Phi_G((X - S) \cup T) = X - [\Phi_G(X - S) \cup \Phi_G(T)] = X - \Phi_G(X - S) = X - \Upsilon_G(S)$.
(v) Let $(S-T) \cup (T-S) \notin G$ so that $S-T, T-S \notin G$. Then by using corollary 2.8 we Have $\Upsilon_G(S) = \Upsilon_G((T-(T-S)) \cup (S-T) = \Upsilon_G(T-(T-S)) = \Upsilon_G(T)$. $\square$

Remark 3.6. *From (ii) of the above theorem we see that the operator Υ_G is distributive over finite intersection. That is not necessarily true for finite union which is shown below.*

Example 3.7. *Let $N = 2$ and $X = \{1, 2, 3\}$ and consider $\tau_1 O(X)$=$\{\emptyset, X, \{1, 2\}\}$, $\tau_2 O(X)$=$\{\emptyset, X\}$. Then $2\tau O(X)$=$\{\emptyset, X, \{1, 2\}\}$. Consider the grill $G = \{\{1\}, \{1, 2\}, \{2\}, \{1, 3\}, \{2, 3\}, X\}$. Now $\Phi_G(\{1, 3\}) = \{1, 2, 3\}$ =X= $\Phi_G\{2, 3\}$ and $\Phi_G(\{3\}) = \Phi$. Then $\Upsilon_G(\{1\})$=$X - \Phi_G(\{2, 3\}) = \emptyset$, $\Upsilon_G(\{2\}) = X - \Phi_G(\{1, 3\}) = \emptyset$ and $\Upsilon_G(\{1, 2\}) = X - \Upsilon_G(\{3\}) = X$. Thus $\Upsilon_G(\{1\}) \cup \Upsilon_G(\{2\}) \neq \Upsilon_G(\{1, 2\})$.*

Next we derive two equivalent expressions for $\Upsilon_G(A)$ where $A \subseteq (X, N\tau)$.

Theorem 3.8. *In $(X, N\tau, G)$, Let $A \subseteq X$. Then the following statements are true:*
(i) $\Upsilon_G(A) = \{x \in X : \exists V \in N\tau(x)$ such that $V - A \notin G\}$.
(ii) Let $\Upsilon_G(A) = \cup\{V \in N\tau : V - A \notin G\}$.

Proof. (i) $x \in \Upsilon_G(A)$ iff $x \notin \Phi_G(X - A) \iff$ there exist $V \in N\tau(x)$ such that $V - A = V \cap (X - A) \notin G \iff x \in$ R.H.S.
(ii) Let $A^{\#} = \cup\{V \in N\tau \colon V - A \notin G\}$. Now $x \in A^{\#}$ then there exists $V \in N\tau$ with $x \in V$ such that $V - A \notin G$ which implies that there exist $V \in N\tau(x)$ such that $V - A \notin G$. Thus by (i), $x \in \Upsilon_G(A)$. From this it is clear that $\Upsilon_G(A) \subseteq A^{\#}$. □

Remark 3.9. *Let $G = \mathcal{P}(x) - \{\emptyset\}$, then by theorem 3.8 (ii) $\Upsilon_G(A) = \cup\{U \in N\tau : U - A = \emptyset\} = \cup\{U \in N\tau \colon U \subseteq A\} = N\tau$–$int(A)$, for any space $(X, N\tau)$.*

Corolary 3.10. *Let $(X, N\tau, G)$ be a grill N-topological space and $A \subseteq X$ .Then $A \cap \Upsilon_G(A) = N\tau_G$-$intA$*

Proof. We have, $x \in A \cap \Upsilon_G(A) \implies x \in A$ and $x \in \Upsilon_G(A) \implies x \in A$ and there exist $V \in N\tau(x)$ such that $V - A \notin G$ (by theorem 3.8 (i)) which implies $V - (V - A)$ is a $N\tau_G$-open neighbourhood of x such that $V - (V - A) \subseteq A \implies x \in A$. Again $x \in N\tau_G$-$intA$ implies there exist a $N\tau_G$-open neighbourhood $U - B$ of x, where $U \in N\tau$ and $B \notin G$, such that $x \in U - B \subseteq A \implies U - A \subseteq B$ and $U - A \notin G$. So by theorem 3.8 (i) $x \in \Upsilon_G(A)$. Thus $x \in A \cap \Upsilon_G(A) = N\tau_G$-$intA$. □

Theorem 3.11. *Let $(X, N\tau, G)$ be a grill N-topological space and if $A \in N\tau$ then $\Upsilon_G(A) = \cup\{S \in N\tau : S\Delta A \notin G\}$.*

Proof. Let $A^{\#} = \cup\{S \in N\tau : S\Delta A \notin G\}$. Then by Theorem 3.8 (ii), $A^{\#} \subseteq \Upsilon_G(A)$. Now, $x \in \Upsilon_G(A)$ which implies there exist $S \in N\tau(x)$ such that $S - A \notin G$ (by theorem 3.8 (i)). Let $V = S \cup A \in N\tau$. Then $V\Delta A = S - A \notin G$ and $x \in V \in N\tau$. Thus $x \in A^{\#}$. □

From the result so far, we arrive at the following simple and alternative description of the topology $N\tau_G$ in terms of our introduced operator.

Theorem 3.12. *If $(X, N\tau, G)$ is a grill N-topological space then $N\tau_G = \{S \subseteq X, S \subseteq \Upsilon_G(S)\}$.*

Proof. $T = \{S \subseteq X : S \subseteq \Upsilon_G(S)\}$. In fact, $\emptyset \subseteq \Upsilon_G(\emptyset) \implies \emptyset \in T$. $\Upsilon_G(X) = X - \Phi_G(X - X) = X - \Phi_G(\{\emptyset\}) = X \in T$. Now $A_1, A_2 \in T$ then $A_1 \subseteq \Upsilon_G(A_1)$ and $A_2 \subseteq \Upsilon_G(A_2)$ which implies $A_1 \cap A_2 \subseteq \Upsilon_G(A_1) \cap \Upsilon_G(A_2) =$

$\Upsilon_G(A_1 \cap A_2)$(by theorem 3.5(ii)). Again $\{A_i : i \in \Lambda\} \in T$ which implies $A_i \subseteq \Upsilon_G(A_i)$ for each $i \in \Lambda$. This implies $A_i \subseteq \Upsilon_G(\cup_{i\in\Lambda} A_i)$ for each $i \in \Lambda$ (by remark 3.3) hence $\cup_{i\in\Lambda} A_i \subseteq \Upsilon_G(\cup_{i\in\Lambda} A_i)$ this implies that $\cup_{i\in\Lambda} A_i \in T$. We will show that $T = N\tau_G$. Indeed, $V \in N\tau_G$, then $V \in \Upsilon_G(V)$ (by theorem 3.5 (i)). This implies that $V \in T$.
Conversely, $A \in T \implies A \subseteq \Upsilon_G(A)$ this implies $A = A \cap \Upsilon_G(A) = N\tau_G\text{-}intA$ (Remark 3.9) which implies that $A \in N\tau_G$. □

4 Suitable for a grill N-topology

In this segment, we intend to do some investigations in respect of $N\tau_G$, along with certain applications, under the assumption of such a suitability conditions imposed on the concerned grills. Throughout this section $(X, N\tau, G)$ denotes a grill N-topological space.

Definition 4.1. *Let $(X, N\tau, G)$ be a grill N-topological space and $N\tau$ is called suitable for the grill G if $A - \Phi_G(A) \notin G$, for all $A \subseteq X$.*

Example 4.2. *Let $X = \{1, 2, 3\}$. For $N = 3$, consider $\tau_1 = \{\emptyset, X, \{1\}\}, \tau_2 = \{\emptyset, X, \{2\}\}, \tau_3 = \{\emptyset, X, \{1, 2\}\}$ and $3\tau = \{\emptyset, X, \{1\}, \{2\}, \{1, 2\}, X\}$. Let $G = \{\{1\}, \{1, 2\}, \{1, 3\}, X\}$. For every $A \subseteq X$, $A - \Phi_G(A) \notin G$. Hence 3τ is suitable for G.*

Theorem 4.3. *In a grill N -topological space $(X, N\tau, G)$, the following are equivalent:*

(i) $N\tau$ is suitable for the grill G.

(ii) For any $N\tau_G$-closed subset A of X, $A - \Phi_G(A) \notin G$.

(iii) For any $A \subseteq X$ and each $x \in A$, there corresponds some $U \in N\tau(x)$ with $U \cap A \notin G$, it follows that $A \notin G$.

(iv) $A \subseteq X$ and $A \cap \Phi_G(A) = \Phi \implies A \notin G$.

Proof.

$(i) \implies (ii)$ It is obvious.

$(ii) \implies (iii)$ Let $A \subseteq X$ and suppose for every $x \in A$ there exist $U \in N\tau(x)$ such that $U \cap A \notin G$. Then $x \notin \Phi_G(A)$ so that $A \cap \Phi_G(A) = \Phi$. Now as $A \cup \Phi_G(A)$ is $N\tau_G$-closed, by (ii) we have $(A \cup \Phi_G(A)) - \Phi_G(A \cup \Phi_G(A)) \notin G$.That is $(A \cup \Phi_G(A)) - \Phi_G(A) \cup \Phi_G(\Phi_G(A)) \notin G \implies (A \cup \Phi_G(A)) - \Phi_G(A) \notin G$ (by theorem 2.5 (i)) that is $A \notin G$.

$(iii) \implies (iv)$ If $A \subseteq X$ and $A \cap \Phi_G(A) = \emptyset$ then $A \subseteq X - \Phi_G(A)$. Let $x \in A$. Then $x \notin \Phi_G(A)$ implies there exist $U \in N\tau(x)$ such that $U \cap A \notin G$. Then by (iii), $A \notin G$.

$(iv) \implies (i)$ Let $A \subseteq X$. We first claim that $(A - \Phi_G(A)) \cap \Phi_G(A - \Phi_G(A)) = \emptyset$. In fact $x \in (A - \Phi_G(A)) \cap \Phi_G(A - \Phi_G(A)) \implies x \in A - \Phi_G(A) \implies x \in A$ and $x \notin \Phi_G(A)$ implies there exist $U \in N\tau(x)$ such that $U \cap A \notin G$. Now $U \cap (A - \Phi_G(A)) \subseteq U \cap A \notin G \implies x \notin \Phi_G(A - \Phi_G(A))$, which is contradiction. Hence by (iv), $A - \Phi_G(A) \notin G$. □

Now we derive, in term of the Υ_G, a characterizing condition for a N-topology $N\tau$ to be suitable for a grill G on a N-topological space X.

Theorem 4.4. *Let G be a grill on a N-topological space X then $N\tau$ is suitable for G iff $\Upsilon_G(A) - A \notin G$ for any $A \subseteq X$.*

Proof. Let $N\tau$ be suitable for G and $A \subseteq X$, We first observe that $x \in \Upsilon_G(A) - A$ iff $\Upsilon_G(A)$ and $x \notin A$ iff there exists $U \in N\tau(x)$ such that $x \in U - A \notin G$. Thus to each $x \in \Upsilon_G(A) - A$, there exist $U \in N\tau(x)$ such that $U \cap (\Upsilon_G(A) - A) \notin G$. As $N\tau$ is suitable for G, by theorem 4.2 we have $\Upsilon_G(A) - A \notin G$. Conversely, let $A \subseteq X$, suppose that to each $x \in A$ there corresponds some $U \in N\tau(x)$ such that $U \cap A \notin G$. We need to show by virtue of theorem 4.2 that $A \notin G$. Now, by theorem 3.8 (i) we have, $\Upsilon_G(X - A) = \{x \in X\colon$ there exists $U \in N\tau(x)$ such that $U - (X - A) \notin G\} = \{x \in X\colon$ there exists $U \in N\tau(x)$ such that $U \cap A \notin G\}$. Thus $A \subseteq \Upsilon_G(X - A)$ and hence $A = \Upsilon_G(X - A) \cap A = \Upsilon_G(X - A) - (X - A) \notin G$. □

Corolary 4.5. *If the N-topology $N\tau$ of a space X is suitable for a grill G on X, then Υ_G is an idempotent operator i.e., $\Upsilon_G(\Upsilon_G(A)) = \Upsilon_G(A)$ for any $A \subseteq X$.*

Proof. Since Υ_G is $N\tau$-open in X, hence $\Upsilon_G(A) \in N\tau$ for any $A \subseteq X$ and so $\Upsilon_G(A) \in N\tau_G$. Hence by theorem 3.5(i) , $\Upsilon_G(A) \subseteq \Upsilon_G(\Upsilon_G(A))$ for any

$A \subseteq X$. Also $N\tau$ is suitable for G, so $\Upsilon_G(A) \subseteq \Upsilon_G(A \cup B)$ for some $B \notin G$. Thus $\Upsilon_G(\Upsilon_G(A)) \subseteq \Upsilon_G(A \cup B) = \Upsilon_G(A)$. $\square$

Theorem 4.6. *In $(X, N\tau, G)$, $N\tau$ is suitable for G . Let $A \subseteq X$ and V be a non-null open set such that $V \subseteq \Phi_G(A) \cap \Upsilon_G(A)$. Then $V - A \notin G$ and $V \cap A \in G$.*

Proof. $V \subseteq \Phi_G(A) \cap \Upsilon_G(A) \implies V \subseteq \Upsilon_G(A) \implies V - A \subseteq \Upsilon_G(A) - A \notin G$ by theorem 4.3 we get $V - A \notin G$. Again $V \subseteq \Phi_G(A)$ and $V \neq \emptyset \implies V \cap A \in G$. $\square$

Theorem 4.7. *In $(X, N\tau, G)$, the following assertions are similar:*
(i) $N\tau - \{\emptyset\} \subseteq G$
(ii) $\Upsilon_G(\emptyset) = \emptyset$
(iii) If A is $N\tau$-closed then $\Upsilon_G(A) - A = \emptyset$
(iv) If $A \subseteq X$ then $N\tau$-$int(N\tau$-$cl(A)) = \Upsilon_G(N\tau$-$int(N\tau$-$cl(A)))$
(v) If A is $N\tau$-regular open in X then $A = \Upsilon_G(A)$
(vi) If $V \in N\tau$ then $\Upsilon_G(V) \subseteq N\tau$-$int(N\tau$-$cl(V)) \subseteq \Phi_G(V)$

Proof. $(i) \implies (ii)$: $\Upsilon_G(\emptyset) = \cup\{V \in N\tau \colon V - \emptyset \notin G\}$ by theorem 3.8 (ii) $\Upsilon_G(\emptyset) = \cup\{V \in N\tau : V \notin G\} = \emptyset$.
$(ii) \implies (iii)$: Let $x \in \Upsilon_G(A) - A$ then there exists $V \in N\tau(x)$ such that $x \in V - A \notin G$. Since A is $N\tau$-closed, we obtain $x \in V - A \in \{U \in N\tau : U \notin G\}$, a contradiction to $\Upsilon_G(\emptyset) = \emptyset$.
$(iii) \implies (iv)$: Since $N\tau$-$int(N\tau$-$cl(A))$ is $N\tau$-open, by theorem 3.5 (i) we get $N\tau$-$int(N\tau$-$cl(A)) \subseteq \Upsilon_G(N\tau$-$int(N\tau$-$cl(A)))$. Again using (iii), we get $\Upsilon_G(N\tau$-$cl(A)) \subseteq N\tau$-$cl(A)$. By remark 3.2, $\Upsilon_G(N\tau$-$cl(A)) = N\tau$-$int(\Upsilon_G(N\tau$-$cl(A)) \subseteq N\tau$-$int(N\tau$-$cl(A))$.Since $\Upsilon_G(N\tau$-$int(N\tau$-$cl(A))) \subseteq \Upsilon_G(N\tau$-$cl(A)) \subseteq N\tau$-$int(N\tau$-$cl(A))$. Thus $N\tau$-$int(N\tau$-$cl(A)) = \Upsilon_G(N\tau$-$int(N\tau$-$cl(A)))$.
(iv) $\implies$ (v): It is trivial.
(v) $\implies$ (vi): Let $V \in N\tau$. By assumption, $\emptyset = \Upsilon_G(\emptyset) = \cup\{V \in N\tau \colon U \notin G\}$. By theorem 3.8(ii) we obtain $N\tau - \{\emptyset\} \subseteq G$. Then by theorem 2.6 we get $V \subseteq \Phi_G(V)$ and hence by Proposition 2.11, we have $\Phi_G(V) = N\tau$-$cl(V)$. Now, $V \subseteq N\tau$-$int(N\tau$-$cl(V)) \subseteq N\tau$-$cl(V) = \Phi_G(V)$. Since $\Upsilon_G(V) \subseteq \Upsilon_G(N\tau$-$int(N\tau$-$cl(V))) = N\tau$-$int(N\tau$-$cl(V)) \subseteq \Phi_G(V)$.
(vi) $\implies$ (i): If $V \in N\tau - G$, by theorem 3.5 (i), $V \subseteq \Upsilon_G(V) \subseteq \Phi_G(V) = \emptyset$. That is $V = \emptyset$ by theorem 2.4(i). $\square$

5 $\widehat{\Upsilon}_G$ -sets

In this segment, we discuss about a new open set $\widehat{\Upsilon}_G$ in grill N-topological space and investigate some of its properties.

Definition 5.1. *In $(X, N\tau, G)$ a subset S of X is called a $\widehat{\Upsilon}_G$-set if $S \subseteq N\tau$-$cl(\Upsilon_G(S))$. The group of all $\widehat{\Upsilon}_G$-sets in $(X, N\tau, G)$ is signify by $\widehat{\Upsilon}_G(X, N\tau)$.*

Proposition 5.2. *If $\{S_\alpha : \alpha \in \Delta\}$ is a group of nonempty $\widehat{\Upsilon}_G$-sets in $(X, N\tau, G)$, then $\cup_{\alpha\in\Delta}S_\alpha \in \widehat{\Upsilon}_G(X, N\tau)$.*

Proof. For each $\alpha \in \Delta$, $S_\alpha \subseteq N\tau$-$cl(\Upsilon_G(S_\alpha)) \subseteq N\tau$-$cl(\Upsilon_G(\cup_{\alpha\in\Delta}S_\alpha))$ This implies that $\cup_{\alpha\in\Delta}S_\alpha \subseteq N\tau$-$cl(\Upsilon_G(\cup_{\alpha\in\Delta}S_\alpha))$. Thus $\cup_{\alpha\in\Delta}S_\alpha \in \widehat{\Upsilon}_G(X, N\tau)$. □

Remark 5.3. *The intersection of two $\widehat{\Upsilon}_G$-sets need not be a $\widehat{\Upsilon}_G$-set and it is shown in the following example.*

Example 5.4. *Let $X = \{1, 2, 3, 4\}$. Let $N = 3$. Consider $\tau_1 = \{\emptyset, X, \{2, 3\}\}, \tau_2 = \{\emptyset, X, \{1, 2, 3\}\}$ and $\tau_3 = \{\emptyset, X, \{1\}, \{1, 2, 3\}\}$. Then $3\tau = \{\emptyset, X, \{1\}, \{1, 2, 3\}, \{2, 3\}\}$ and the grill $G = \{\{1\}, \{2\}, \{1, 3\}, \{1, 2\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{1, 2, 3\}, \{2, 3, 4\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}, X\}$. Let $A = \{1, 4\}$ and $B = \{2, 3, 4\}$ are $\widehat{\Upsilon}_G$-sets but $A \cap B$ is not a $\widehat{\Upsilon}_G$-set. For Let $A = \{1, 4\}, \Phi_G(X - A) = \{2, 3, 4\}$ and $\widehat{\Upsilon}_G(A) = \{1\}$. Hence $A \subseteq N\tau$-$cl(\widehat{\Upsilon}_G(A))$ implies that A is $\widehat{\Upsilon}_G$-set. For $B = \{2, 3, 4\}, \Phi_G(X - B) = \{1, 4\}$ and $\widehat{\Upsilon}_G(B) = \{2, 3\}$. Hence $B \subseteq N\tau$-$cl(\widehat{\Upsilon}_G(B))$ implies that B is $\widehat{\Upsilon}_G$-set. On the other hand, since $A \cap B = \{4\}, \Phi_G(X-(A\cap B)) = X$ and $\widehat{\Upsilon}_G(B) = \emptyset$. Hence $A\cap B \nsubseteq N\tau$-$cl(\widehat{\Upsilon}_G(A\cap B))$ implies that $A \cap B$ is not a $\widehat{\Upsilon}_G$-set.*

Remark 5.5. *The intersection of an $N\tau\alpha$-set and $\widehat{\Upsilon}_G$-set is a $\widehat{\Upsilon}_G$-set.*

Corolary 5.6. *In $(X, N\tau, G)$, if for any $S \in N\tau$ then $S \subseteq \Upsilon_G(S)$.*

Theorem 5.7. *Let $A \in \widehat{\Upsilon}_G(X, N\tau)$ on $(X, N\tau, G)$. If $U \in N\tau^\alpha$ then $U \cap A \in \widehat{\Upsilon}_G(X, N\tau)$.*

Proof. Assume that A is $N\tau$-open for every $A \subseteq X$, $G \cap N\tau$-$cl(A) \subseteq N\tau$-$cl(G\cap A)$. Let $U \in N\tau^\alpha$ and $A \in \widehat{\Upsilon}_G(X, N\tau)$. By corollary 5.6, we have $U\cap A \subseteq$

$N\tau$-$int(N\tau$-$cl(N\tau$-$int(U))) \cap N\tau$-$cl(\Upsilon_G(A)) \subseteq N\tau$-$int(N\tau$-$cl(\Upsilon_G(U))) \cap N\tau$-$cl(\Upsilon_G(A)) \subseteq N\tau$-$cl(N\tau$-$int(N\tau$-$cl(\Upsilon_G(U))) \cap \Upsilon_G(A) = N\tau$-$cl(N\tau$-$int(N\tau$-$cl(\Upsilon_G(U) \cap \Upsilon_G(A))) = N\tau$-$cl(\Upsilon_G(U) \cap \Upsilon_G(A)) = N\tau$-$cl(\Upsilon_G(U \cap A))$. Hence $U \cap A \in \widehat{\Upsilon}_G(X, N\tau)$. □

Corolary 5.8. *Let $A \in \widehat{\Upsilon}_G(X, N\tau)$ on $(X, N\tau, G)$. If $U \in N\tau$ then $U \cap A \in \widehat{\Upsilon}_G(X, N\tau)$.*

Definition 5.9. *In $(X, N\tau, G)$, if for every relatively nonempty open set $L \cap K$, $L \in N\tau$ and $(L \cap K) \cap E \in G$ then the set E is relatively G-dense in a set K.*

Next we prove a necessary and sufficient condition for $A \notin \widehat{\Upsilon}_G(X, N\tau)$.

Theorem 5.10. *A set $A \notin \widehat{\Upsilon}_G(X, N\tau)$ if and only if there exits $x \in A$ such that there is a neighbourhood $V_x \in N\tau(x)$ for which $X - A$ is relatively $N\tau_G$-dense in V_x.*

Proof. Let $A \notin \widehat{\Upsilon}_G(X, N\tau)$. We are to show that there exists $x \in A$ and a neighbourhood $V_x \in N\tau(x)$ satisfying that $X - A$ is relatively G-dense in V_x. Since $A \nsubseteq N\tau$-$cl(\Upsilon_G(A))$, there exits $x \in X$ such that $x \in A$ but $x \notin N\tau$-$cl(\Upsilon_G(A))$. Hence there exists a neighbourhood $V_x \in N\tau(x)$ such that $V_x \cap \Upsilon_G(A) = \emptyset$. This implies that $V_x \cap (X - \Phi_G(X - A)) = \emptyset$ and hence $V_x \subseteq \Phi_G(X - A)$. Let U be any non empty open set in V_x. Since $V_x \subseteq \Phi_G(X - A)$, therefore $U \cap (X - A) \in G$ which implies that $(X - A)$ is relatively G-dense in V_x. Converse is obvious. □

Definition 5.11. *A space $(X, N\tau, G)$ is said to be anti-co-dense grill if $N\tau$-$\{\emptyset\} \subseteq G$*

Theorem 5.12. *Let G be a anti-co dense grill on a space $(X, N\tau, G)$. Then $SO(X, N\tau_G) = \widehat{\Upsilon}_G(X, N\tau)$.*

Definition 5.13. *A set $A \subseteq X$ in $(X, N\tau, G)$, A is called a $\Upsilon_{\mathcal{A}}$-set if $A \subseteq N\tau$-$int(N\tau$-$cl(\Upsilon_G(A)))$. The collection $\Upsilon_{\mathcal{A}}$-sets in $(X, N\tau, G)$ is denoted by $N\tau^{\mathcal{A}}$. From Definitions of 5.1 and 5.13 it follows that $N\tau^{\mathcal{A}} \subseteq \widehat{\Upsilon}_G(X, N\tau)$. The collection $N\tau^{\mathcal{A}}$ forms a topology finer than $N\tau$.*

Theorem 5.14. *Let G be a anti-co dense grill on $(X, N\tau, G)$. Then the collection $N\tau^{\mathcal{A}} = \{A \subseteq X \colon A \subseteq N\tau$-$int(N\tau$-$cl(\Upsilon_G(A)))\}$ forms a N-topology on X.*

Proof. (i) It is observed that $\emptyset \subseteq N\tau\text{-}int(N\tau\text{-}cl(\Upsilon_G(\emptyset))$ and $X \subseteq N\tau\text{-}int(N\tau\text{-}cl(\Upsilon_G(X)$, and thus $\emptyset$ and $X \in N\tau^{\alpha}$.
(ii) Let $\{A_\alpha\colon \alpha \in \Delta\} \subseteq N\tau^{\mathcal{A}}$, then $\Upsilon_G(A_\alpha) \subseteq \Upsilon_G(\cup A_\alpha)$ for every $\alpha \in \Delta$. Thus $A_\alpha \subseteq N\tau\text{-}int(N\tau\text{-}cl(\Upsilon_G(A_\alpha))) \subseteq N\tau\text{-}int(N\tau\text{-}cl(\Upsilon_G((\cup A_\alpha)))$ for every $\alpha \in \Delta$, which implies that $\cup A_\alpha \subseteq N\tau\text{-}int(N\tau\text{-}cl(\Upsilon_G(\cup A_\alpha)))$. Therefore, $\cup A_\alpha \in N\tau^{\mathcal{A}}$.
(iii) Let $A, B \in N\tau^{\mathcal{A}}$. Since $\Upsilon_G(A)$ is open in $(X, N\tau)$, we obtain $A \cap B \subseteq N\tau\text{-}int(N\tau\text{-}cl(\Upsilon_G(A))) \cap N\tau\text{-}int(N\tau\text{-}cl(\Upsilon_G(A))) = N\tau\text{-}int(N\tau\text{-}cl(\Upsilon_G(A) \cap \Upsilon_G(B)))$. Therefore $A \cap B \subseteq N\tau\text{-}cl(N\tau\text{-}int(\Upsilon_G(A \cap B)))$. □

Proposition 5.15. *Let $(X, N\tau, G)$ be a grill N-topological space. Then $\Upsilon_G(A) \neq \emptyset$ if and only if A contains a nonempty $N\tau_G$-interior.*

Corolary 5.16. *Let $(X, N\tau, G)$ be a grill N-topological space. Then $\{x\} \in \widehat{\Upsilon}_G(X, N\tau)$ if and only if $\{x\} \in N\tau^{\mathcal{A}}$.*

Proof. Let $\{x\} \in \widehat{\Upsilon}_G(X, N\tau)$, therefore by proposition 5.15, $\{x\}$ is open in $(X, N\tau, G)$. Since $\{x\} \subseteq \Upsilon_G(\{x\})$ and $\Upsilon_G(\{x\})$ is $N\tau$-open in $(X, N\tau)$, therefore $\{x\} \subseteq N\tau\text{-}int(N\tau\text{-}cl(\Upsilon_G\{x\})$. □

6 Conclusion

In this paper, we introduced a new operator in grill N-topological space, using this operator some important properties and equivalent expressions are derived. We arrived a topology $N\tau_G$ using the introduced operator. In addition, suitability condition of a grill with the N-topological space X is formulated. We discuss the characterizing condition for a N-topology to be a suitable for a grill G on X. Also we introduce and study $\widehat{\Upsilon}_G$–sets and utilize the Υ_G-operator to define a generalized open set and their properties. This concept can be extended to other applicable research areas of topology such as Nano topology, Fuzzy topology, Intuitionistic topology, Digital topology and so on.

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