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p-clean properties in amalgamated rings

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Abstract

Let A be a ring. Then A is called p-clean ring if each element in A express as the sum of an idempotent and pure element. Let $f : A \to B$ be a ring homomorphism and J be an ideal of B. The amalgamation of A with B along J with respct to f is a new ring structure introduced and studied by Anna et al. in 2009. This construction is a generalization of the amalgamated duplication of a ring along an ideal and other classical constructions such as the $A + XB[X]$ and $A + XB[[X]]$ constustions. In this paper, the transfer of the notion of p-clean rings to the amalgamation of rings along ideal is studied. In particular, the necessary and sufficient conditions for amalgamation to be a p-clean ring are studied.

Keywords: pure element, amalgamation ring, p-clean ring
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1 Introduction

Throughout this paper all rings are commutative with identity. Let A and B be two rings with unity, let J be an ideal of B and let $f : A \to B$ be a ring homomorphism. Anna et al. [2009] introduced and studied the new ring structure of the following subring of $A \times B$:

$$A \bowtie^f J := \{(a, f(a) + j) \mid a \in A, j \in J\}$$

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called *the amalgamation of A with B along J with respect to f*. This new ring structure construction is a generalization of *the amalgamated duplication of a ring along an ideal*. The amalgamated duplication of a ring along an ideal was introduced and studied in [Anna [2006], Anna and Fontana [2007]]. In [Anna et al. [2009], Section 4], the authors studied the amalgamation can be in the frame of pullback constructions and also the basic properties of this construction [e.g., characterizations for $A \bowtie^f K$ to be a Noetherian ring, an integral domain, a reduced ring] and they characterized those distinguished pullbacks that can be expressed as an amalgamation. Aruldoss et al. [2022] and Vijayanand and Selvaraj [2023] studied some rings and module characterized via amalgamation construction.

The notion of regular element was first introduced by von Neumann [1936], where an element $a \in A$ is called regular if there exists $b \in A$ such that $a = aba$. A ring A is called regular ring if each element in A is regular. Many other authors interested in studying regular rings, for example see Vas [2010] and Wardayani et al. [2020]. Let the set of regular elements in A be denoted by $Reg(A)$ that is $Reg(A) = \{r \in A; r = rsr, \text{ for some } s \in A\}$. The concept of clean ring first introduced by Nicholson [1977], where the ring A is called clean ring if for each $a \in A$ there exist $e \in Id(A)$ and $u \in U(A)$ such that $a = e + u$. Many other authors interested in studying clean rings, for example see Chen and Cui [2008] and Ashrafi and Nasibi [2013]. Ashrafi and Nasibi [2013], introduced the concept of $r-$clean ring , where the ring A is called r-clean ring if for each $a \in A$ there exist $e \in Id(A)$ and $r \in Reg(A)$ such that $a = e + r$. Many authors have studied r-clean rings such as Andari [2018] and Sharma and Singh [2018]. Let A be a ring, then an element $p \in A$ is called pure element if there exists $q \in A$ such that $p = pq$ [Majidinya et al. [2016]] and the set of pure elements in A write $Pu(A) = \{p \in A : p = pq, \text{ for some } q \in A\}$. The concept of von Neumann local ring was studied by Anderson and Badawi [2012], where a ring A is called von Neumann local ring if for each $a \in A$ we have either $a \in Reg(A)$ or $1 - a \in Reg(A)$. Mohammed et al. [2021] studied the class of rings namely rings in which each element express as the sum of an idempotent and pure.

This paper aims at studying the transfer of the notion of p-clean rings to the amalgamation of rings along ideal. In particular, we study the necessary and sufficient condition for amalgamation to be a p-clean ring.

We denote by $U(A)$, $Id(A)$, $Nilp(R)$ and $Pu(A)$, the set of unit elements, the set of idempotents, the set of nilpotent elements and set of all pure elements of A, respectively.

2 p-clean ring

We start with the following definition.

Definition 2.1 (Mohammed et al. [2021]). *An element $a \in A$ is called p-clean if there exist $e \in Id(A)$ and $p \in Pu(A)$ such that $a = e + p$. A ring A is called p-clean ring if each element in A is p-clean element.*

Proposition 2.1 (Mohammed et al. [2021]).
(i) The class of p-clean is closed under homomorphic images.
(ii) If I is an ideal of a p-clean ring A, then A/I is a p-clean ring.

Proposition 2.2. *The ring A is p-clean if and only if the ring $A[[x]]$ of formal series over A is p-clean.*

Proof. If $A[[x]]$ is p-clean, then it follows by the isomorphism $A \cong A[[x]]/(x)$ and by Proposition 2.1 (i) that A is p-clean.
Conversely, suppose that A is p-clean. Let $f(x) = \sum_{i=0}^{\infty} a_i x^i \in A[[x]]$, then $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots$. Since A is p-clean, then $a_0 = p + e$, where $p \in Pu(A)$ and $e \in Id(A)$. So

$$f(x) = p + e + a_1 x + a_2 x^2 + \cdots \in Id(A[[x]]),$$

with $p \in Pu(A) \subseteq Pu(A[[x]])$ and $e + a_1 x + a_2 x^2 + \cdots \in Id(A[[x]])$. Thus, $A[[x]]$ is a p-clean ring. □

Proposition 2.3. *If A is a p-clean ring, then $A[[\{X_\alpha\}]]$ is a p-clean ring.*

Proof. Let $f(X) \in A[[\{X_\alpha\}]]$. Then $f = f_0 + f^{'}$, where $f_0 \in A$ and $f^{'} \in (\{X_\alpha\})$. Since $f_0 \in A$, we can write $f_0 = p + e$, where $p \in Pu(A)$ and $e \in Id(A)$. Now

$$f = p + e + f^{'} = (p + f^{'}) + e,$$

where $p + f^{'} \in Pu(A[[\{X_\alpha\}]])$ and $e \in Id(A) \subseteq Id(A[[\{X_\alpha\}]])$. Hence, $A[[\{X_\alpha\}]]$ is p-clean. □

Let A be a ring and M an A-module. The trivial extension of A by M is the ring $A \propto M = \{(a, m) : a \in A, m \in M\}$ under coordinatewise addition and an adjusted multiplication defined by $(a, m)(a^{'}, m^{'}) = (aa^{'}, am^{'} + a^{'}m,)$ for all $a, a^{'} \in A, m, m^{'} \in M$.

Theorem 2.1. *Let A be a ring and M an A-module. Then $A \propto M$ is p-clean if and only if A is p-clean.*

Proof. Note that $A \cong (A \propto M)/(\{0\} \times M)$ is a homomorphic image of $A \propto M$. Hence if $A \propto M$ is p-clean, so by Proposition 2.1 (i), A is p-clean.
Conversely, suppose that A is p-clean. Recall that $1_{A \propto M} = (1, 0)$ and observe that if $p \in Pu(A)$, then $(p, m) \in Pu(A \propto M)$ for each $m \in M$ and if $e \in Id(A)$, then $(e, 0)^2 = (e^2, 0) = (e, 0)$ in $A \propto M$. Hence, if $a \in A$ with $a = p + e$, where $p \in Pu(A)$ and $e \in Id(A)$, then for $m \in M$, $(a, m) = (p+e, m) = (p, m)+(e, 0)$, where $(p, m) \in Pu(A \propto M)$ and $(e, 0) \in Id(A \propto M)$. Hence, $A \propto M$ is p-clean. □

Lemma 2.1. *Let* $f = \sum_{i=0}^{n} a_i x^i \in A[x]$ *be a pure element. Then* a_0 *is pure and* a_i *is nilpotent for each* $i > 0$.

Proof. Since f is a pure element, there exists $g = \sum_{i=0}^{n} b_i x^i \in A[x]$ such that $f = fg$. Therefore, $a_0 = a_0 b_0$ and so a_0 is pure. Now let P be a prime ideal of A. Then $(A/P)[x]$ is an integral domain. Define $\phi : A[x] \to (A/P)[x]$ by $\phi(\sum_{i=0}^{k} a_i x^i) = \sum_{i=0}^{k} (a_i + P)x^i$. Clearly, ϕ is an epimorphism. Then we have $\phi(f)\phi(g) = \phi(fg) = \phi(f)$. So $deg(\phi(f)\phi(g)) = deg(\phi(f))$. Thus, $deg(\phi(f)) + deg(\phi(g)) = deg(\phi(f))$. Therefore $deg(\phi(f)) = 0$. Hence, $a_i \in P$ for $i = 1, \dots, n$. Since P is arbitrary, a_i is nilpotent for each $i > 0$. □

Theorem 2.2. $A[x]$ *is not* p*-clean.*

Proof. Let $f = \sum_{i=0}^{n} a_i x^i \in A[x]$. Suppose f is p-clean. Then $f = p + e$, where $p \in Pu(A[x])$ and $e \in Id(A[x])$. Since idempotents of A are exactly that of $A[x]$, $p = f - e$ is pure. Hence by Lemma 2.1, the element 1 should be nilpotent, which is a contradiction. □

Theorem 2.3. *For every ring* A*, we have the following statements:*
(i) If e *is a central idempotent element of* A *and* eAe *and* $(1-e)A(1-e)$ *are both* p*-clean, then so is* A*;*
(ii) Let $e_1, e_2, \cdots, e_n$ *be orthogonal central idempotents with* $e_1+e_2+\cdots+e_n = 1$. *Then* e_iRe_i *is* p*-clean for each* i*, if and only if so is* A*.*
(iii) If A *is* p*-clean, then so is the matrix ring* $M_n(A)$ *for any* $n > 1$.

Proof. (i) For convenience, write $\overline{e} = 1 - e$ for each $e \in Id(A)$. We use the Pierce decomposition of A: we have

$$A = eAe \oplus eA\overline{e} \oplus \overline{e}Ae + \overline{e}A\overline{e}.$$

Since $e, \overline{e}$ are central, we have

$$A = eAe \oplus \overline{e}A\overline{e} \cong \begin{pmatrix} eAe & 0 \\ 0 & \overline{e}A\overline{e} \end{pmatrix}.$$

So each matrix $B \in A$ is of the form $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, where $a \in eRe, b \in \overline{e}A\overline{e}$. By hypothesis, a and b are p-clean. Then $a = p_1 + e_1, b = p_2 + e_2$, where $p_1 \in Pu(eAe) \subseteq Pu(A), p_2 \in Pu(\overline{e}A\overline{e}) \subseteq Pu(A), e_1 \in Id(eAe) \subseteq Id(A), e_2 \in Id(\overline{e}A\overline{e}) \subseteq Id(A)$. So

$$B = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} p_1 + e_1 & 0 \\ 0 & p_2 + e_2 \end{pmatrix} = \begin{pmatrix} p_1 & 0 \\ 0 & p_2 \end{pmatrix} + \begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix}.$$

Since p_1 and p_2 are pure elements of A, there exist q_1, q_2 in A such that $p_1 = p_1 q_1$ and $p_2 = p_2 q_2$. Therefore, we have

$$\begin{pmatrix} p_1 & 0 \\ 0 & p_2 \end{pmatrix} \begin{pmatrix} q_1 & 0 \\ 0 & q_2 \end{pmatrix} = \begin{pmatrix} p_1 q_1 & 0 \\ 0 & p_2 q_2 \end{pmatrix} = \begin{pmatrix} p_1 & 0 \\ 0 & p_2 \end{pmatrix}.$$

So $\begin{pmatrix} p_1 & 0 \\ 0 & p_2 \end{pmatrix}$ is pure. Since $\begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix}$ is an idempotent, we have A is p-clean.

(ii) One direction of (ii) follows from (i) by induction. The other direction follows from Proposition 2.1(ii).

(iii) Follows from (ii). □

3 p-clean properties in amalgamated ring

Proposition 3.1. *Let $f : A \to B$ be a ring homomorphism and J an ideal of B. If $A \bowtie^f J$ is p-clean ring, then A and $f(A) + J$ are p-clean rings.*

Proof. Define $p_A : A \bowtie^f J \to A$ by $p_A(a, f(a) + k) = a$ and $p_B : A \bowtie^f J \to B$ by $p_B(a, f(a) + k) = f(a) + k$. Then $A \bowtie^f J/(\{0\} \times J) \cong A$ and $A \bowtie^f J/(f^{(-1)}(J) \times \{0\}) \cong f(A) + J$. Since every homomorphic image of p-clean ring is p-clean, A and $f(A) + J$ are p-clean rings. □

The converse of the above Proposition is not true.

Proposition 3.2. *Let $f : A \to B$ be a ring homomorphism and J an ideal of B. Assume that $(f(A)+J)/J$ is uniquely p-clean and B is an Integral domain. Then $A \bowtie^f J$ is p-clean ring if and only if A and $f(A) + J$ are p-clean rings.*

Proof. By Proposition 3.1, $A \bowtie^f J$ is p-clean ring implies A and $f(A) + J$ are p-clean rings. Conversely, assume that A and $f(A) + J$ are p-clean rings. Since A is p-clean, we can write $a = e + p$ with $e \in Id(A), p \in Pu(A)$. Similarly, since $f(A) + J$ is p-clean, we can write $f(a) + j = f(e_1) + j_1 + f(p_1) + j_2$ with $f(e_1) + j_1$ is an idempotent element and $f(p_1) + j_2$ is a pure element. Clearly, $\overline{f(e_1)} = \overline{f(e_1) + j_1}$ (resp., $\overline{f(e)}$) and $\overline{f(p_1)} = \overline{f(p_1) + j_2}$ (resp.,$\overline{f(p)}$) are respectively idempotent and pure element of $(f(A) + J)/J$. Then we have $\overline{f(a)} = \overline{f(e)} + \overline{f(p)} = \overline{f(e_1)} + \overline{f(p_1)}$. Since $(f(A) + J)/J$ is uniquely p-clean, $\overline{f(e)} = \overline{f(e_1)}$ and $\overline{f(p)} = \overline{f(p_1)}$. Consider $j_1', j_2' \in J$ such that $f(e_1) = f(e) + j_1'$ and $f(p_1) = f(p) + j_2'$. Then $(a, f(a) + j) = (e + p, f(e_1) + j_1 + f(p_1) + j_2) = (e, f(e) + j_1' + j_1) + (p, f(p) + j_2' + j_2)$. Clearly, $(e, f(e) + j_1' + j_1)$ is an idempotent element of $A \bowtie^f J$. Since $f(p) + j_2' + j_2$ is pure in $f(a) + J$, there exists an element $f(\alpha_0) + j_0$ such that $\overline{f(p) + j_2' + j_2} = \overline{(f(p) + j_2' + j_2)(f(\alpha_0) + j_0)}$. Since $p = pq$ for some $q \in A$, we have $\overline{f(p)f(q)} = \overline{f(p)} = \overline{f(p)f(\alpha_0)}$. Since B is an integral domain, $\overline{f(q)} = \overline{f(\alpha_0)}$. This implies $f(\alpha_0) = f(q) + j_0'$ and hence $f(\alpha_0)+j_0 = f(q)+j_0'+j_0$. Therefore, $f(p)+j_2'+j_2 = (f(p)+j_2'+j_2)(f(q)+j_0'+j_0)$. Hence, $(p, f(p) + j_2' + j_2) = (pq, (f(p) + j_2' + j_2)(f(q) + j_0' + j_0) = (p, (f(p) + j_2' + j_2))(q, (f(q) + j_0' + j_0))$. Therefore, $(p, f(p) + j_2' + j_2)$ is a pure element in $A \bowtie^f J$. Hence, $A \bowtie^f J$ is p-clean. □

Remark 3.1. *Let $f : A \to B$ be a ring homomorphism and J an ideal of B.*

1. *If $B = J$, then $A \bowtie^f B$ is p-clean if and only if A and B are p-clean since $A \bowtie^f J = A \times B$*

2. *If $f^{-1}(J) = 0$, then by [Anna et al. [2009], Proposition 5.1(3)], $A \bowtie^f J$ is p-clean if and only if $f(A) + J$ is p-clean .*

Corolary 3.1. *Let A be a ring and I an ideal such that A/I is uniquely p-clean. Then $A \bowtie^f I$ is p-clean if and only if A is p-clean.*

Theorem 3.1. *Let $f : A \to B$ be a ring homomorphism and J an ideal of B such that $f(p) + j$ is pure (in B) for each $p \in Pu(A)$ and $j \in J$. Then $A \bowtie^f J$ is p-clean if and only if A is p-clean.*

Proof. $A \bowtie^f J$ is p-clean implies A is p-clean by Proposition 3.1. Conversely, assume that A is p-clean and $f(p) + j$ is pure (in B) for each $p \in Pu(A)$ and $j \in J$. Since A is p-clean, $a = e + p$, where e and p are idempotent and pure elements, respectively in A. Since p is pure in A, then there exists $q \in B$ such that $p = pq$. Therefore $(p, f(p) + j)(q, f(q) + j) = (pq, (f(p) + j)(f(q) + j)) = (pq, f(p)f(q) + j) = (p, f(p) + j)$. Thus, $(p, f(p) + j)$ is pure in $A \bowtie^f J$. Hence, $(a, f(p) + j) = (e, f(e)) + (p, f(p) + j)$ is a sum of idempotent and pure elements in $A \bowtie^f J$. Therefore, $A \bowtie^f J$ is p-clean. □

Theorem 3.2. *Let $f : A \to B$ be a ring homomorphism and J an ideal of B. Set $\overline{A} = A/Nilp(A)$, $\overline{B} = A/Nilp(B)$. $\pi : B \to \overline{B}$, the canonical projection and $\overline{J} = \pi(J)$. Consider a ring homomorphism $\overline{f} : \overline{A} \to \overline{B}$ defined by $\overline{f}(\overline{a}) = \overline{f(a)}$. Then $A \bowtie^f J$ is p-clean (resp., uniquely p-clean) if and only if $\overline{A} \bowtie^{\overline{f}} \overline{J}$ is p-clean (resp., uniquely p-clean)*

Proof. Clearly $\overline{f}$ is well defined and ring homomorphism. Consider the map $\chi : (A \bowtie^f J)/Nilp(A \bowtie^f J) \to \overline{A} \bowtie^{\overline{f}} \overline{J}$ defined by $\chi(\overline{(a, f(a + j))}) = (\overline{a}, \overline{f}(\overline{a} + \overline{j}))$. If $\overline{(a, f(a + j)} = \overline{(b, f(b + j')}$, then $(a - b, f(a - b) + j - j') \in Nilp(A \bowtie^f J)$. Therefore, $a - b \in Nilp(A)$ and $j - j' \in Nilp(B)$. Then $\overline{a} = \overline{b}$ and $j = j'$. Hence χ is well defined. We can easily check that χ is a ring homomorphism.

Moreover,$(\overline{a}, \overline{f}(\overline{a}+\overline{j})=(0,0)$ implies that $a \in Nilp(A)$ and $j \in Nilp(B)$. Consequently, $(a, f(a)+j) \in Nilp(A \bowtie^f J)$. Hence, $\overline{(a, f(a+j)}=(0,0)$. This implies that χ is injective. Clearly, by the construction, χ is surjective. Hence, χ is an isomorphism. □

Proposition 3.3. *Let $f : A \to B$ be a ring homomorphism and let (e) be an ideal of B generated by the idempotent element e of B. Then $A \bowtie^f (e)$ is p-clean if and only if A and $f(A)+(e)$ are p-clean. In particular, if e is an element of A, then $A \bowtie^f (e)$ is p-clean if and only if A is p-clean.*

Proof. By Theorem 3.1, $A \bowtie^f (e)$ is p-clean implies A and $f(A)+(e)$ are p-clean.
Conversely, assume that A and $f(A)+(e)$ are p-clean. Let $(a, f(a)+re)$ be an element of $A \bowtie^f (e)$ with $a \in A$ and $r \in B$. Since A is p-clean. there exist an idempotent element v and pure element p in A such that $a = v+p$. Also, since $f(A)+(e)$ is p-clean, there exist an idempotent element v' and pure element p' in $f(A)+(e)$ such that $f(a)+re = v'+p'$. We have $(a, f(a)+re) = (v, f(v)+(v'-f(v))e)+(p, f(p)+(p'-f(p))e)$. On the other hand, $[f(v)+(v'-f(v))e]^2 = [f(v)(1-e)+v'e]^2 = f(v)(1-e)+v'e = f(v)+(v'-f(v))e$ and $[f(p)+(p'-f(p))e][f(q)+(q'-f(q))e] = [f(p)(1-e)+p'e][f(q)(1-e)+q'e] = f(pq)(1-e)+p'q'e = f(p)(1-e)+p'e = f(p)+(p'-f(p))e$. Then $(v, f(v)+(v'-f(v))e)$ and $(p, f(p)+(p'-f(p))e)$ are respectively idempotent and pure element in $A \bowtie^f (e)$. Hence $A \bowtie^f (e)$ is p-clean. Finally, if $A = B$ and $f = id_A$, then $A \bowtie^f (e) = A \bowtie (e)$ and $f(A)+(e) = A$. Then A is p-clean. □

Theorem 3.3. *Let $f : A \to B$ be a ring homomorphism and J an ideal of B. $f(p)+j$ is pure (in B) for each $p \in Pu(A)$ and $j \in J$. Then $A \bowtie^f J$ is a von Neumann local ring if and only if A is a von Neumann local ring.*

Proof. Note in first that a commutative ring is von Neumann local ring if and only if it is an indecomposable p-clean ring (that is a p-clean ring where $\{0,1\}$ is the set of all idempotent elements) by [Mohammed et al. [2021], Theorem 1.8]. Assume that $A \bowtie^f J$ is a von Neumann local ring. Then $A \bowtie^f J$ is an indecomposable p-clean ring. Then A must be p-clean ring. Also if $e \in Id(A)$, then $(e, f(e)) \in Id(A \bowtie^f J) = \{(0,0),(1,1)\}$. Then $Id(A) = \{0,1\}$. This implies that A is an indecomposable p-clean ring, and so A is von Neumann local ring. Conversely, assume that A is a von Neumann local ring. Again by [Mohammed et al. [2021], Theorem 1.8], A is p-clean ring. Also by Theorem 3.1, $A \bowtie^f J$ is p-clean. On the other hand, by [Chhiti et al. [2015], Lemma 2.5], $Id(A \bowtie^f J) = \{(e, f(e)) | e \in Id(A)\} = \{(0,0),(1,1)\}$. Thus $A \bowtie^f J$ is an indecomposable p-clean ring. This implies that $A \bowtie^f J$ is a von Neumann local ring. □

Corolary 3.2. *Let $f : A \to B$ be a ring homomorphism and J an ideal of B. If $f(p) + j$ is pure (in B) for each $p \in Pu(A)$ and $j \in J$. Then the following are equivalent: (i) $A \bowtie^f J$ is a von Neumann local and uniquely p-clean ring. (ii) A is a von Neumann local and uniquely p-clean ring. In particular, if A is a ring and J an ideal of B, then $A \bowtie J$ is a von Neumann local and uniquely p-clean ring if and only if A is a von Neumann local and uniquely p-clean ring.*

Proof. From Proposition 3.1 and Theorem 3.3, $A \bowtie^f J$ is a von Neumann local and uniquely p-clean ring implies that A is a von Neumann local and uniquely p-clean ring.
The converse is immediate. □

Proposition 3.4. *Let $f : A \to B$ be a ring homomorphism and let (e) be an ideal of B generated by the idempotent element e. Then $A \bowtie^f (e)$ is p-clean ring if and only if A and $f(A) = (e)$ are p-clean ring. In particular, if e is an idempotent element of A, then $A \bowtie (e)$ is p-clean ring if and only if A is p-clean ring.*

Proof. Let A and $f(A) + (e)$ are p-clean ring. We show that $A \bowtie^f (e)$ is p-clean ring. Let $(a, f(a) + re)$ be an element of $A \bowtie^f (e)$ with $a \in A$ and $r \in B$. Since A and $f(A) + (e)$ are p-clean, there exist p and v (resp., p' and v') in A (resp., $f(A) + (e)$) which are respectively pure and idempotent element such that $a = p + v$ and $f(a) = re = p' + v'$. We have $(a, f(a) + re) = (p, f(p) + (p' - f(p)e) + (v, f(v) + (v' - f(v)e))$. On the other hand, $[f(p) + (p' - f(p)e)][f(q) + (q' - f(q)e)] = [f(p)(1 - e) + p'e][f(q)(1 - e) + q'e] = f(pq)(1 - e) + p'q'e = f(p)(1-e) + p'e = f(p) + (p' - f(p)e)$. Also $[f(v) + (v' - f(v)e)]^2 = [f(v)(1 - e) + v'e]^2 = f(v)(1-e) + v'e = f(v) + (v' - f(v)e)$. Then $(p, f(p) + (p' - f(p)e)$ and $(v, f(v) + (v' - f(v)e))$ are respectively pure and idempotent in $A \bowtie^f (e)$. Consequently, $A \bowtie^f (e)$ is p-clean as desired. Finally, if $A = B$ and $f = id_A$, then $A \bowtie^f (e) = A \bowtie (e)$ and $f(A) + (e) = A$. Thus, the particular case is obvious. □

4 Conclusion

This paper studies the transfer of the notion of p-clean rings to the amalgamation of rings along ideal. In particular, the necessary and sufficient conditions for amalgamation to be a p-clean ring are studied. This study will further help in studying the properties of other ring structure such as the $A + XB[X]$ and $A + XB[[X]]$ constructions. Furthermore, this work provides an answer to the question of when $A \bowtie^f J$ is a von Neumann local ring. In future, there is a scope to study the generalization of amalgamated ring namely bi-amalgamation ring with p-clean properties.

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