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Bi-amalgamated algebra with (n, p)-weakly clean like properties

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Abstract

Let $f : A \longrightarrow B$ and $g : A \longrightarrow C$ be two ring homomorphisms and let K and K' be two ideals of B and C, respectively such that $f^{-1}(K) = g^{-1}(K')$. In this paper, we give a characterization for the bi-amalgamation of A with (B, C) along (K, K') with respect to (f, g) (denoted by $A \bowtie^{f,g} (K, K')$) to be a (n, p)-weakly clean ring.

Keywords: (n, p)-clean ring ; (n, p)- weakly clean ring; bi-amalgamation algebra along ideals.

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1 Introduction

Throughout this paper all rings are commutative with identity elements. Let A and B be two rings with unity, let K be an ideal of B and let $f : A \to B$ be a ring homomorphism. In D'Anna et al. [2009], the authors introduced and studied the new ring structure the following subring of $A \times B$:

$$A \bowtie^f K := \{(a, f(a) + k) \mid a \in A, k \in K\}$$

called *the amalgamation of* A *with* B *along* K *with respect to* f. This new ring structure construction is a generalization of *the amalgamated duplication of a ring along an ideal*. The amalgamated duplication of a ring along an ideal was introduced and studied in (D'Anna [2006], D'Anna and Fontana [2007]). In [D'Anna et al., 2009, Section 4], the authors studied the amalgamation can be in the frame of pullback constructions and also the basic properties of this construction (e.g., characterizations for $A \bowtie^f K$ to be a Noetherian ring, an integral domain, a reduced ring) and they characterized those distinguished pullbacks that can be expressed as an amalgamation.

Let $\alpha : A \longrightarrow C$, $\beta : A \longrightarrow C$ and $f : A \longrightarrow B$ be ring homomorphisms. In D'Anna et al. [2009], the authors studied amalgamated algebras within the frame of pullback $\alpha \times \beta$ such that $\alpha = \beta \circ f$ [D'Anna et al., 2009, Proposition 4.2 and 4.4]. In this motivation, the authors created the new constructions, called bi-amalgamated algebras which arise as pullbacks $\alpha \times \beta$ such that the following diagram of ring homomorphims

$$\begin{array}{ccc} A & \xrightarrow{\;f\;} & B \\ \big\downarrow g & & \big\downarrow \alpha \\ C & \xrightarrow{\;\beta\;} & D \end{array}$$

is commutative with $\alpha \circ \pi_B(\alpha \times \beta) = \alpha \circ f(A)$, where π_B denotes the canonical projection of $B \times C$ over B. Namely, let $f : A \longrightarrow B$ and $g : A \longrightarrow C$ be two ring homomorphisms and let K and K' be two ideals of B and C, respectively, such that $f^{-1}(K) = g^{-1}(K')$. The bi-amalgamation of A with (B, C) along (K, K') with respect to (f, g) is the subring of $B \times C$ given by

$$A \bowtie^{f,g} (K, K') := \{(f(a) + k, g(a) + k') | a \in A, (k, k') \in K \times K'\}$$

Following Kabbaj et al. [2013], the above definition was introduced by and studied by Kabbaj, Louartiti and Tamekkante.

Following Nicholson [1977], an element a in a ring A is called a clean if a is a sum of a unit and an idempotent in A. A ring is clean if all its elements are clean.

Clean rings were initially developed by Nicholson [1977], as a natural class of rings which have the exchange property.

This paper aims at studying the transfer of the notion of n-clean rings, (n, p)-weakly clean rings to bi-amalgamation of algebra along ideals.

We denote by $U(A)$, set of all unit elements of A.

2 (n, p)-clean ring

We start with definition of (n, p)-clean rings.

Definition 2.1. *(Chen and Qua [2014]) An element $a \in A$ is said to be (n, p)-clean if $a = u_1 + ... + u_n + x$ for some unit $u_i \in A(i = 1, ..., n)$ and $x^p = x$, where x is called p-potent element. The ring A is said to be (n, p)-clean if all of its elements are (n, p)-clean.*

Remark 2.1. *Every clean rings are $(1, 2)$-clean rings and n-clean rings are $(n, 2)$-clean rings.*

Following Chhiti [2018], the above definition is studied the authors in 2018. Now, we start with following example of $(2, 2)$-clean ring.

Example 2.1. *Let $A := \mathbb{Z}_4$, $B := \mathbb{Z}_4 \times \mathbb{Z}_4$ and $C := \mathbb{Z}_2$ and let $J := 0 \times \mathbb{Z}_4$ and $J' := \mathbb{Z}_2$ are ideals of B and C respectively. Consider the map $f : A \longrightarrow B$ is defined by $f(a) = (a, 0)$ for all $a \in A$ and the map $g : A \longrightarrow C$ is defined by $g(0) = g(2) = 0$ and $g(1) = g(3) = 1$. Hence, $A \bowtie^{f,g} (J, J') =$*

$$\Big\{ ((0,0),0), ((0,1),0), ((0,2),0), ((0,3),0), ((2,0),0), ((2,1),0),$$

$$((2,2),0), ((2,3),0), ((1,0),1), ((1,1),1), ((1,2),1), ((1,3),1), ((3,0),1),$$

$$((3,1),1), ((3,2),1), ((3,3),1) \Big\}$$

is a 2-clean ring. Therefore, by the remark 2.1 $A \bowtie^{f,g} (J, J')$ is $(2, 2)$-clean ring.

Proposition 2.1. *The class of (n, p)-clean ring is closed under homomorphic images.*

Proof. The proof is straightforward.□

Proposition 2.2. *If $A \bowtie^{f,g} (K, K')$ is a (n, p)-clean ring then $f(A) + K$ and $g(A) + K'$ are (n, p)-clean rings.*

Proof. Clearly, by Proposition 2.1 (n, p)-clean ring is a (n, p)-clean ring. Thus, in view of Kabbaj et al. [2013] [Proposition 4.1], we have the following isomorphism of rings $\frac{A \bowtie^{f,g}(K,K')}{0 \times K'} \cong f(A) + K$ and $\frac{A \bowtie^{f,g}(K,K')}{K \times 0} \cong g(A) + K'$. Hence, $f(A) + K$ and $g(A) + K'$ are (n, p)-clean rings.□

Definition 2.2. *A ring is called uniquely (n,p)-clean ring if each element in A can be written as unique way.*

Theorem 2.1. *Assume that A is (n,p)-clean ring and $\dfrac{f(A)+K}{K}$ and $\dfrac{g(A)+K'}{K'}$ are uniquely n-clean rings. Then $A \bowtie^{f,g} (K,K')$ is (n,p)-clean ring if and only if $f(A)+K$ and $g(A)+K'$ are (n,p)-clean rings.*

Proof. If $A \bowtie^{f,g} (K,K')$ is a (n,p)-clean ring, then so $f(A)+K$ and $g(A)+K'$ by Proposition 2.1. Conversely, assume that $f(A)+K$ and $g(A)+K'$ are (n,p)-clean rings. Since A is a (n,p)-clean ring, we can write $a = u_1+...+u_n+x$ for some unit $u_i \in A (i = 1,...,n)$ and $x^p = x$. On the other hand, since $f(A)+K$ is a (n,p)-clean ring, $f(a)+k = (f(x_1)+k_1)+...+(f(x_n)+k_n)+f(p)+k^*$ with $f(x_i)+k_i (i = 1,...,n)$ and $f(y)+k^*$ are respectively units and $(f(p)+k^*)^p = f(p)+k^*$ element of $f(A)+K$. It is clear that $\overline{f(x_1)} = \overline{f(x_1)+k_1}$ (resp., $\overline{f(u_1)}$),...,$\overline{f(x_n)} = \overline{f(x_n)+k_n}$ (resp., $\overline{f(u_n)}$) and $\overline{f(p)} = \overline{f(p)+k^*}$ (resp., $\overline{f(x)}$), are respectively units and an p-potent element of $\dfrac{f(A)+K}{K}$, and we have $\overline{f(a)} = \overline{f(u_1)}+...+\overline{f(u_n)}+\overline{f(p)} = \overline{f(x_1)}+...+\overline{f(x_n)}+\overline{f(x)}$. Thus, $\overline{f(u_1)} = \overline{f(x_1)}$,..., $\overline{f(u_n)} = \overline{f(x_n)}$ and $\overline{f(p)} = \overline{f(x)}$ since $\dfrac{f(A)+K}{K}$ is an uniquely (n,p)-clean ring. Consider $k_1^*,...,k_n^*,k_l^* \in K$ such that $f(x_1) = f(u_1)+k_1^*$,..., $f(x_n) = f(u_n)+k_n^*$ and $f(x) = f(p)+k_l^*$ and also since $g(A)+K'$ is a (n,p)-clean ring, $g(a)+k' = (g(x_1')+k_1')+...+(g(x_n')+k_n')+g(p')+k'^*$ with $g(x_i')+k_i' (i = 1,...,n)$ and $g(p')+k'^*$ are respectively units and p-potent element of $g(A)+K'$. It is clear that $\overline{g(x_1')} = \overline{g(x_1')+k_1'}$ (resp., $\overline{g(u_1)}$),...,$\overline{g(x_n')} = \overline{g(x_n')+k_n'}$ (resp., $\overline{g(u_n)}$) and $\overline{g(p')} = \overline{g(p')+k'^*}$ (resp., $\overline{g(x)}$), are respectively units and an p-potent element of $\dfrac{g(A)+K'}{K'}$, and we have $\overline{g(a)} = \overline{g(u_1)}+...+\overline{g(u_n)}+\overline{g(p)} = \overline{g(x_1')}+...+\overline{g(x_n')}+\overline{g(p')}$. Thus, $\overline{g(u_1)} = \overline{g(x_1')}$,..., $\overline{g(u_n)} = \overline{g(x_n')}$ and $\overline{g(x)} = \overline{g(p')}$ since $\dfrac{g(A)+K'}{K'}$ is an uniquely (n,p)-clean ring. Consider $k_1'^*,...,k_n'^*,k_l'^* \in K$ such that $g(x_1') = g(u_1)+k_1'^*$,..., $g(x_n') = g(u_n)+k_n'^*$ and $g(y') = g(p)+k_l'^*$. We have $(f(a)+k, g(a)+k') = \{(f(u_1)+k_1^*+k_1)+...+(f(u_n)+k_n^*+k_n)+(f(p)+k_l^*+k^*), (g(u_1)+k_1'^*+k_1')+...+(g(u_n)+k_n'^*+k_n')+(g(p')+k_l'^*+k'^*)\} = (f(u_1)+k_1^*+k_1, g(u_1)+k_1'^*+k_1')+...+(f(u_n)+k_n^*+k_n, g(u_n)+k_n'^*+k_n')+(f(p)+k_l^*+k^*, g(p')+k_l'^*+k'^*)$. It is clear that $(f(p)+k_l^*+k^*, g(p')+k_l'^*+k'^*)$ is an p-potent in $A \bowtie^{f,g} (K,K')$. Hence, we have only to prove that $(f(u_1)+k_1^*+k_1, g(u_1)+k_1'^*+k_1')$ is invertible in $A \bowtie^{f,g} (K,K')$. Since $f(u_1)+k_1^*+k_1$ is invertible in $f(A)+K$, there exists an element $f(\beta)+k_0$ such that $(f(u_1)+k_1^*+k_1)(f(\beta)+k_0) = 1$. Thus $\overline{f(u_1)f(\beta)} = \overline{1}$. Then $\overline{f(\beta)} = \overline{f(u_1^{-1})}$. So there exists $k_0^* \in K$ such that $f(\beta) = f(u_1^{-1})+k_0^*$. Similarly, $g(u_1)+k_1'^*+k_1'$ is invertible in $g(A)+K'$. Hence, $(f(u_1)+k_1^*+k_1, g(u_1)+k_1'^*+k_1')(f(u_1^{-1})+k_0^*+k_0, g(u_1^{-1})+k_0'^*+k_0') =$

$(f(u_1) + k_1^* + k_1, g(u_1) + k_1'^* + k_1')(f(\beta) + k_0, g(\gamma) + k_0') = (1, 1)$. Therefore, $(f(u_1) + k_1^* + k_1, g(u_1) + k_1'^* + k_1')$ is invertible in $A \bowtie^{f,g} (K, K')$. Similarly, each term are invertible. This completes the proof.□

Proposition 2.3. *Let $f : A \to B$ and $g : A \to C$ be two surjective ring homomorphisms, let K and K' be two ideals of B and C respectively such that $f^{-1}(K) = g^{-1}(K') = I_0$ and let A is a (n, p)-clean ring and A/I_0 is an uniquely (n, p)-clean ring. Then $A \bowtie^{f,g} (K, K')$ is a (n, p)-clean ring.*

Proof. It is clear that B and C are (n, p)-clean rings. So, since $f(A)+K = B$ and $g(A) + K' = C$, we conclude that $A \bowtie^{f,g} (K, K')$ is a (n, p)-clean ring by Theorem 2.1.□

Let A be a commutative ring with identity and let M be a unitary A-module. The idealization of M in A(or trivial extension of A by M) is the commutative ring $A \propto M = \{(a, m)|a \in R, m \in M\}$ under the usual addition and the multiplication defined as $(a_1m_1)(a_2m_2) = (a_1a_2, a_1m_2 + a_2m_1)$ for all $(a_1, m_1), (a_2, m_2) \in A \propto M$.

Theorem 2.2. *Consider n and p two positive integers ($p \geq 2$). Let $f : A \longrightarrow C$, $g : A \longrightarrow C$ be two ring homomorphisms. Assume that A is (n, p)-clean ring. Let K be an ideal of B such that $f(u) + k$ is invertible (in B) for each $u \in U(A)$ and $k \in K$ and K' be an ideal of C such that $g(u) + k'$ is invertible (in C) for each $u \in U(A)$ and $k' \in K'$. Then $A \bowtie^{f,g} (K, K')$ is a (n, p)-clean ring if and only if $f(A) + K$ and $g(A) + K'$ are (n, p)-clean ring.*

Proof. In light of Proposition 2.1, homomorphic image of (n, p)-clean ring is (n, p)-clean ring. Thus, in view of Kabbaj et al. [2013] [Proposition 4.1], we have the following isomorphism of rings $\frac{A \bowtie^{f,g}(K,K')}{0 \times K'} \cong f(A) + K$ and $\frac{A \bowtie^{f,g}(K,K')}{K \times 0} \cong g(A) + K'$. Hence, $f(A) + K$ and $g(A) + K'$ are (n, p)-rings. Conversely, we assume that A is (n, p)-clean and K be an ideal of B such that $f(u)+k$ is invertible (in B) and K' be an ideal of C such that $g(u) + k'$ is invertible (in C). Then there exist $v_1 \in B$ such that $(f(u_1 + k)v_1) = 1$ and there exist $v_2 \in C$ such that $(g(u_1 + k')v_2) = 1$. Hence, $(f(u_1) + k)(f(u_1^{-1}) - v_1f(u_1^{-1})k) = f(u_1)f(u_1^{-1}) + kf(u_1^{-1}) - (f(u_1) + k)v_1f(u_1^{-1})k) = 1 + kf(u_1^{-1}) - f(u_1^{-1})k = 1$ and $(g(u_1) + k')(g(u_1^{-1}) - v_2g(u_1^{-1})k') = g(u_1)g(u_1^{-1}) + k'g(u_1^{-1}) - (g(u_1) + k')v_2g(u_1^{-1})k') = 1 + k'g(u_1^{-1}) - g(u_1^{-1})k' = 1$. Thus, $(f(u_1) + k, g(u_1) + k')$ is invertible in $A \bowtie^{f,g} (K, K')$. Hence, $(f(a) + k, g(a) + k') = (f(u_1 + u_2 + ... + u_n + x) + k, g(u_1 + u_2 + ... + u_n + x) + k') = (f(u_1) + k, g(u_1) + k') + (f(u_2), g(u_2)) + ... + (f(u_n), g(u_n)) + (f(x), g(x))$, where $(f(u_1) + k, g(u_1) + k') \in U(A \propto M), (f(u_i), g(u_i) \in U(A \propto M)(i = 2, 3, ..., n)$ and $(f(x), g(x)^p = (f(x), g(x))$. Consequently, $A \bowtie^{f,g} (K, K')$ is (n, p)-clean ring.□

3 (n, p)-weakly clean ring

Now we introduce the new class of ring.

Definition 3.1. *A ring A is called (n, p)-weakly clean ring if $a = u_1 + ... + u_n + x$ or $a = u_1 + ... + u_n - x$ for some unit $u_i \in A(i = 1, ..., n)$ and $x^p = x$, where x is p-potent element.*

If the above representation is unique, we say that A is uniquely (n, p)-weakly clean ring.

Note that (n, p)-clean ring is weakly (n, p)-clean ring. In this section we study the tranfer of (n, p)-weakly clean ring property to the ring $A \bowtie^{f,g} (K, K')$ is defined above. We establishes necessary and sufficient conditions for $A \bowtie^{f,g} (K, K')$ to be (n, p)-weakly clean.

Proposition 3.1. *The class of (n, p)-weakly clean is closed under homomorphic images.*

Proof. The proof is straightforward.□

Proposition 3.2. *A ring A is called uniquely (n, p)-weakly clean ring if the representation of a (n, p)-weakly clean element in a unique way.*

Our first main result gives a necessary and sufficient conditions for $A \bowtie^{f,g} (K, K')$ to be (n, p)-weakly clean ring.

Theorem 3.1. *Let $f : A \to B$ and $g : A \to C$ be two ring homomorphisms, let K and K' be two ideals of B and C respectively such that $f^{-1}(K) = g^{-1}(K') = I_0$. Assume that the following conditions hold:*

a) A is a (n, p)-weakly clean ring and A/I_0 is an uniquely (n, p)-weakly clean ring.

b)$f(A) + K$ and $g(A) + K'$ are (n, p)-weakly clean rings and atmost one of them is not a (n, p)-clean ring. Then $A \bowtie^{f,g} (K, K')$ is a (n, p)-weakly clean ring.

Proof. Without loss of generality, we assume that $f(A) + K$ is a n-weakly clean ring and $g(A) + K'$ is a n-clean ring. Let $a \in A$. Then a can be written as $a = u_1 + ... + u_n + x$ or $a = u_1 + ... + u_n - x$ for some $u_i \in A(i = 1, ..., n)$ and p-potent $x \in A$. Since A is (n, p)-weakly clean ring and since $f(A) + K$ is a (n, p)-weakly clean ring, $f(a)+k = (f(x_1)+k_1)+...+(f(x_n)+k_n)+(f(p)+k^*)$ or $f(a)+k = (f(x_1)+k_1)+...+(f(x_n)+k_n)-(f(p)+k^*)$ with $f(x_i)+k_i(i = 1, ..., n)$ and $f(p) + k^*$ are respectively units and p-potent element of $f(A) + K$. Therefore, $f(a) = f(u_1) + ... + f(u_n) + f(x)$ or $f(a) = f(u_1) + ... + f(u_n) - f(x)$. Then, in $f(A) + K/K$ we have: $\overline{f(a)} = \overline{f(u_1)} + ... + \overline{f(u_n)} + \overline{f(x)}$

or $\overline{f(a)} = \overline{f(u_1)} + ... + \overline{f(u_n)} - \overline{f(x)}$. It is clear that $\overline{f(x_1)} = \overline{f(x_1) + k_1}$ (resp., $\overline{f(u_1)}$),...,$\overline{f(x_n)} = \overline{f(x_n) + k_n}$ (resp., $\overline{f(u_n)}$) and $\overline{f(p)} = \overline{f(p) + k^*}$ (resp., $\overline{f(x)}$), are respectively units and p-potent element of $\dfrac{f(A)+K}{K}$, and we have $\overline{f(a)} = \overline{f(u_1)} + ... + \overline{f(u_n)} + \overline{f(x)} = \overline{f(x_1)} + ... + \overline{f(x_n)} + \overline{f(p)}$ or $\overline{f(a)} = \overline{f(u_1)} + ... + \overline{f(u_n)} - \overline{f(x)} = \overline{f(x_1)} + ... + \overline{f(x_n)} - \overline{f(p)}$. Thus, $\overline{f(u_1)} = \overline{f(x_1)}$,..., $\overline{f(u_n)} = \overline{f(x_n)}$ and $\overline{f(x)} = \overline{f(p)}$ since $\dfrac{f(A)+K}{K}$ is an uniquely (n, p)-weakly clean ring. Consider $k_1^*, ..., k_n^*, k_l^* \in K$ such that $f(x_1) = f(u_1) + k_1^*$,..., $f(x_n) = f(u_n) + k_n^*$ and $f(p) = f(x) + k_l^*$. Hence, $f(a) + j = (f(u_1) + k_1^* + k_1) + ... + (f(u_n) + k_n^* + k_n) + (f(x) + k_l^* + k^*)$ or $f(a) + j = (f(u_1) + k_1^* + k_1) + ... + (f(u_n) + k_n^* + k_n) - (f(x) + k_l^* + k^*)$. Thus, using the same technique of the previous part $g(a) + k' = (g(u_1) + k_1'^* + k_1') + ... + (g(u_n) + k_n'^* + k_n') + (g(p) + k_l'^* + k'^*)$ or $g(a) + k' = (g(u_1) + k_1'^* + k_1') + ... + (g(u_n) + k_n'^* + k_n') - (g(p) + k_l'^* + k'^*)$ since $g(A) + K'/K' \cong A/I_0$ is an uniquely n-weakly clean ring. This implies that $(f(a)+k, g(a)+k') = \{(f(u_1)+k_1^*+k_1)+...+(f(u_n)+k_n^*+k_n)+(f(x)+k_l^*+k^*), (g(u_1)+k_1'^*+k_1')+...+(g(u_n)+k_n'^*+k_n')+(g(x)+k_l'^*+k'^*)\} = (f(u_1)+k_1^*+k_1, g(u_1)+k_1'^*+k_1')+...+(f(u_n)+k_n^*+k_n, g(u_n)+k_n'^*+k_n')+(f(x)+k_l^*+k^*, g(x)+k_l'^* + k'^*)$. Now, the same argument follows from Theorem 2.1 in the remaining case, $f(a) + j = (f(u_1) + k_1^* + k_1) + ... + (f(u_n) + k_n^* + k_n) - (f(x) + k_l^* + k^*)$. Let $g(a)+k' = (g(u_1)+k_1'^*+k_1')+...+(g(u_n)+k_n'^*+k_n')-(g(x)+k_l'^*+k'^*)$. We have $(f(a)+k, g(a)+k') = \{(f(u_1)+k_1^*+k_1)+...+(f(u_n)+k_n^*+k_n)+(f(x)+k_l^* + k^*), (g(u_1) + k_1'^* + k_1') + ... + (g(u_n) + k_n'^* + k_n') + (g(x) + k_l'^* + k'^*)\} = (f(u_1) + k_1^* + k_1, g(u_1) + k_1'^* + k_1') + ... + (f(u_n) + k_n^* + k_n, g(u_n) + k_n'^* + k_n') - (f(x) + k_l^* + k^*, g(x) + k_l'^* + k'^*)$. In all cases, $(f(a) + k, g(a) + k')$ is a (n, p)-weakly clean elements of $A \bowtie^{f,g} (J, J')$. This completes the proof. □

4 Conclusions

Through the above we have studied the characterization for the bi-amalgamation of A with (B, C) along (K, K') with respect to (f, g) to be a $(n, p$- clean ring along with an example. Further, we have studied the necessary and sufficient conditions for $A \bowtie^{f,g} (K, K')$ to be a (n, p)-weakly clean ring.

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