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Co-even Geodetic Number of a Graph

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Abstract

Let $G = (V, E)$ be a graph with vertex set V and edge set E. If S is a set of vertices of G, then $I[S]$ is the union of all sets $I[u, v]$ for $u, v \in S$. If $I[S] = V(G)$, then S is a geodetic set for G. The geodetic number $g(G)$ is the minimum cardinality of a geodetic set. A geodetic set S is called co- even geodetic set if the degree of vertex v is even number for all $v \in V - S$. The cardinality of a smallest co- even geodetic set of G, denoted by $g_{coe}(G)$ is the co- even geodetic number of G. In this paper, we find the co-even geodetic number of certain graphs and complement graphs.

Keywords: geodetic set, co-even geodetic set, co-even geodetic number

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1. Introduction

By a graph $G = (V, E)$, we mean a finite undirected connected graph without loops or multiple edges. As usual $n = |V|$ and $m = |E|$ denote the number of vertices and edges of a graph G respectively. The minimum and maximum degree $\delta(G)$ and $\Delta(G)$, respectively. In case where $\Delta(G) = \delta(G)$, G is called a regular graph. The distance $d(x, y)$ is the length of a shortest $x - y$ path in G. It is known that the distance is a metric on the vertex set of G. An $x - y$ path of length $d(x, y)$ is called an $x - y$ geodesic. For any vertex u of G,the eccentricity of u is $e(u) = max\{d(u, v) : v \in V\}$. A vertex v is an eccentric vertex of u if $e(u) = d(u, v)$. The neighborhood of a vertex v is the set $N(v)$ consisting of all vertices u which are adjacent with v. A vertex v is an extreme vertex of G if the subgraph induced by its neighbors is complete. The closed interval $I[x, y]$ consists of all vertices lying on some $x - y$ geodesic of G, while for $S \subseteq V$, $[S] = \cup_{x,y \in S} I[x, y]$. A set S of vertices is a geodetic set if $I[S] = V$ and the minimum cardinality of a geodetic set is the geodetic number $g(G)$. In this paper, we study the co-even geodetic number and is denoted by $g_{coe}(G)$ also we discuss the co-even geodetic number of some standard graphs.

2. Co-even geodetic number of a graph

Definition 2.1 A geodetic set $\boldsymbol{S}$ is called co-even geodetic set if the degree of vertex $\boldsymbol{v}$ is even number for all $\boldsymbol{v} \in \boldsymbol{V} - \boldsymbol{S}$. The cardinality of a smallest co-even geodetic set of $\boldsymbol{G}$, denoted by $\boldsymbol{g_{coe}(G)}$ is the co-even geodetic number of $\boldsymbol{G}$.

Example 2.2

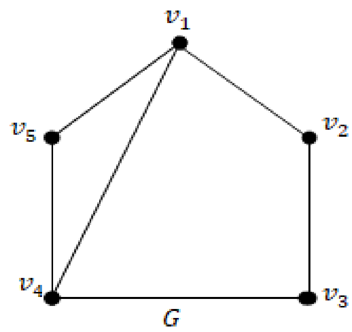


Figure 2.1

In figure 2.1, $S = \{v_1, v_3, v_4, v_5\}$ is a co-even geodetic set. Here, the vertices v_1 and v_4 has odd degree. These two vertices do not make a geodetic set and no 3- element subset of G is a co-even geodetic set. Then it is clear that $g_{coe}(G) = 4$.

Remark In figure 2.1, $S = \{v_1, v_3, v_5\}$ is the minimum geodetic set of G. ie) $g(G) = 3$. Thus, the geodetic number and co-even geodetic number of a graph G can be different.

Proposition 2.3 Let G be a graph and S is a co-even geodetic set . Then,
i) All vertices of odd degrees belong to every co-even geodetic set.
ii) $deg(v) \geq 2$ for all $v \in V - S$.

Proposition 2.4 If G is p- regular graph, then $g_{coe}(G) = \begin{cases} n & if \;\; p \; is \; odd \\ g(G) & if \;\; p \;\; is \;\; even \end{cases}$

Theorem 2.5 If G be a graph of order n, then $2 \leq g(G) \leq g_{coe}(G) \leq n$.
Proof: A geodetic set needs atleast two vertices. Therefore, $g(G) \geq 2$. Clearly, every co-even geodetic set is a geodetic set of G, $g(G) \leq g_{coe}(G)$. Also, all the vertices of G is the co-even geodetic set of G.ie) $g_{coe}(G) \leq n$.

Remark 2.6 The bounds of the theorem 2.5 are sharp. The co-even geodetic number of paths P_n with n vertices is 2. In this case, the smallest bounds is obtained. Also, K_n with n vertices have the co-even geodetic number is n. Then the upper bound is obtained.

Theorem 2.7 If G is a non trivial connected graph with $n \geq 2$.If $g_{coe}(G) = 2$ then $g(G) = 2$.
Proof. It is follows from theorem 2.5.

Remark 2.8 The converse part of above theorem is need not be true for all graphs. In Figure 2.2, The minimum geodetic number is 2 and the minimum co-even geodetic number is 3.

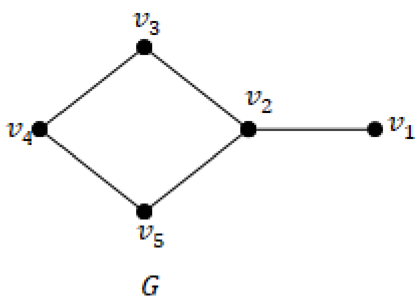
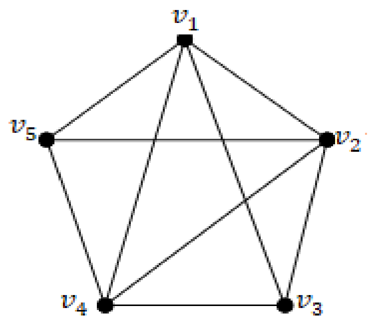


Figure 2.2

Corollary 2.9 Let G be the non-trivial connected graph, $g(G) = 2$ then $g_{coe}(G) = 2$.
Proof. Case (i) If $G = K_2$
It is easy to see $g(K_2) = 2$ then $g_{coe}(K_2) = 2$.
Case (ii) All the vertices of G should be even degree.
Consider the even Cycle C_{2n}. All vertices have even degree for C_{2n}.We know that $g(C_{2n}) = 2$. Further more, $g_{coe}(C_{2n}) = 2$.

Case (iii) A graph with exactly two odd degree vertices which only belongs to the minimum geodetic set.



G

Figure 2.3

For example, In Figure 2.3, the vertices v_3 and v_5 have odd degree and v_1, v_2, v_4 have even degree. The minimum geodetic number of G is 2. Also, it is easily seen that $g_{coe}(G) = 2$.

Remark All the graphs are not satisfied for the corollary 2.9 except the above three type graphs.

Observation. 2.10 $g_{coe}(C_n) = g(C_n)$, where C_n is a cycle of order n.
Proof. Every cycle is the 2- regular graph .by the proposition 2.4, we get $g_{coe}(C_n) = g(C_n)$.

Theorem 2.11 For the Wheel graph W_n $(n \geq 4)$, then

$$g_{coe}(W_n) = \begin{cases} n-1 & if\ \ n\ \ is\ \ odd \\ n & if\ \ n\ \ is\ \ even \end{cases}$$

Proof. Case (i) n is odd

Let $W_n = K_1 + C_{n-1}$ and u be the vertex of K_1. It is easy to see that the $n-1$ vertices has odd degree except the vertex u. By the proposition 2.3, $n-1$ vertices belong to the co-even geodetic set S. Also, the vertex $u \in V - S$, which has even degree. Hence $|S| = n-1$.

Case (ii) n is even.

Every vertex of W_n has odd degree. By the proposition 2.3, All the vertices of W_n belongs to the co-even geodetic set. Therefore, $g_{coe}(W_n) = n$.

Corollary 2.12 For the wheel graph with $n \geq 4$ then $g_{coe}(W_n) = 2\alpha_0(W_n) - 2$.

Proof. We prove this theorem by two cases.

Case (i) n is even

We have $g_{coe}(W_n) = n$ if n is even and $\alpha_0(W_n) = \frac{n+2}{2}$.

We have $g_{coe}(W_n) = n$. $g_{coe}(W_n) + 2 = n + 2$. Then $\frac{g_{coe}(W_n)+2}{2} = \frac{n+2}{2}$

$$\frac{g_{coe}(W_n)}{2} + 1 = \alpha_0(W_n)$$
$$g_{coe}(W_n) = 2\alpha_0(W_n) - 2$$

Case (ii) n is odd

Since $g_{coe}(W_n) = n - 1$ if n is odd and $\alpha_0(W_n) = \frac{n+1}{2}$

We have $g_{coe}(W_n) = n - 1$

$$\frac{g_{coe}(W_n) + 1}{2} = \frac{n - 1 + 1}{2}$$
$$\frac{g_{coe}(W_n)}{2} = \frac{n + 1}{2} - 1$$
$$g_{coe}(W_n) = 2\alpha_0(W_n) - 2.$$

Theorem 2.13 If G is the double fan graph $F = P_n + \overline{K_2}$ with $n \geq 5$, then $g_{coe}(G) = 4$.

Proof

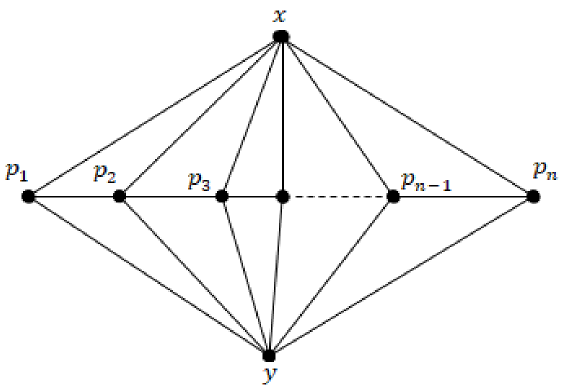


Figure 2.4

Let $p_1, p_2, \ldots, p_n$ be the vertices of path P_n and let x and y be the two vertices of $\overline{K_2}$. All the vertices of path P_n is adjacent to x and y. Now, the double fan graph $F = P_n + \overline{K_2}$ have the $n + 2$ vertices. We prove this theorem by two cases.

Case (i) n is odd

If n is odd then the end vertices of P_n and the vertices of $\overline{K_2}$ have the odd degree. By the proposition 2.3, these four vertices p_1, p_n ,x,y belongs to co-even geodetic set. Also all the vertices of F lies on any geodesic of the co-even geodetic set. Thus $g_{coe}(P_n + \overline{K_2}) = 4$.

Case (ii) n is even

If n is even then all the vertices of F is even degree except the vertices p_1 and p_n belongs to co-even geodetic set. All the vertices of F does not lies the $p_1 - p_n$ geodesic. So we chosen the vertices x and y in the co-even geodetic set. Now the set $S = \{p_1, p_n, x, y\}$ is the co-even geodetic set as well as all the vertices of $V - S$ has even degree. Therefore, $g_{coe}(P_n + \overline{K_2}) = 4$.

Corollary 2.14 For the double fan graph $F = P_n + \overline{K_2}$ with $n \geq 5$ then,

$$g_{coe}(P_n + \overline{K_2}) = \begin{cases} 2\alpha_0(P_n + \overline{K_2}) - n + 1 & if\ n\ is\ odd \\ 2\alpha_0(P_n + \overline{K_2}) - n & if\ n\ is\ even \end{cases}$$

Theorem 2.15 For the ladder graph L_n then, $g_{coe}(L_n) = 2n - 2$.

Proof

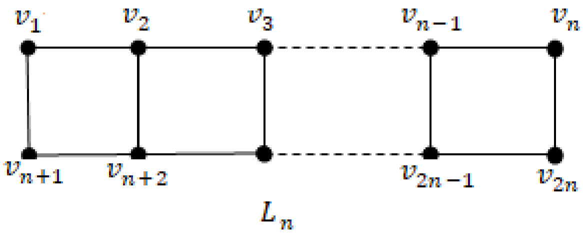


Figure 2.5

The ladder graph $L_n$with $2n$ vertices. The geodetic number of L_n is 2. $S = \{v_1, v_{2n}\}$ or $\{v_n, v_{n+1}\}$ is the minimum geodetic set of L_n, which is not a co-even geodetic set. Because some vertices of $V - S$ has odd degree. Therefore, the odd degree vertices $\{v_2, v_3, \ldots, v_{n-1}, v_{n+2}, \ldots, v_{2n+1}\}$ is belong to the co-even geodetic set of L_n. Therefore, all the vertices of L_n except two vertices make the co-even geodetic set. Hence $g_{coe}(L_n) = 2n - 2$.

Theorem2.16 For the Cone graph $C_m + \bar{K}_n$ then $g_{coe}(C_m + \bar{K}_n) = \begin{cases} n\ if\ n\ is\ even, & m \geq 5 \\ m\ if\ m\ is\ even, & n\ is\ odd \\ m + n\ if\ m\ is\ odd, & n\ is\ odd \end{cases}$

Proof. The Cone graph $C_m + \bar{K}_n$ is adding with cyclic graph C_m and empty graph $\bar{K}_n$. The cone graph has $m + n$ vertices. We prove this theorem by three cases.

Case (i) If n is even

In this case, we prove with two subcases.

Sub Case (i) If n is even, m is odd

For the Cone graph $C_m + \bar{K}_n$, only n vertices have odd degree. By the proposition 2.3, n- vertices belongs to the co-even geodetic set. Now, every vertex belongs to any geodesic of the co-even geodetic set. Hence $g_{coe}(C_m + \bar{K}_n) = n$.

Sub Case (ii) If n is even, m is even

Both the vertices of $C_m + \overline{K}_n$ has even degree. Now, n- vertices forms a co-even geodetic set of $C_m + \overline{K}_n$. Hence $g_{coe}(C_m + \overline{K}_n) = n$.

Case (ii) If m is even and n is odd

Let m is even number of vertices and n is odd number of vertices. Here, $C_m + \overline{K}_n$ has m- even vertices have odd degree and n-odd vertices have even degree. Then it follows from the sub case (i) we get $g_{coe}(C_m + \overline{K}_n) = m$.

Case (iii) If both m and n are odd

For all the vertices of $C_m + \overline{K}_n$ have odd degree. Then it follows from the subcase (i). Thus, we get, $g_{coe}(C_m + \overline{K}_n) = m + n$. Hence proved.

3.Co-even geodetic number of Complement of a graph

Theorem 3.1 If P_n is a path graph with $n \geq 5$,then $g_{coe}(\overline{P_n}) = \begin{cases} 4 & if \quad n \ is \ odd \\ n-2 & if \quad n \ is \ even \end{cases}$

Proof. Let u and v be the end vertices of P_n. The vertices u and v are adjacent to $n-2$ vertices in $\overline{P_n}$. The remaining vertices are adjacent to $n-3$ vertices in $\overline{P_n}$.

Case (i) If n is odd

Since u and v are adjacent to $n-2$ vertices in $\overline{P_n}$. Clearly, u and v are odd vertices. Therefore $\{u , v\} \in S$. Also, $\{u , v\}$ is not a geodetic set. Consider a vertex x, which is adjacent to v and non adjacent to u. Obviously, $n-3$ vertices lie on the $x-u$ geodesic. Choose a vertex y there exist $y \in V(\overline{P_n})$ such that $y \notin I[x , u]$. Also no 3-element subset contains the co-even geodetic set. Hence, $S = \{u , v , x , y\}$ is the minimum co-even geodetic set.

Case (ii) If n is even

For n is even, clearly, u and v are even degree vertices. Remaining $n-2$ vertices are adjacent to $n-3$ vertices. Obviously, $n-2$ vertices is odd vertices. Also, every vertex lies on the any geodesic of $n-2$ vertices. Therefore, the minimum co-even geodetic number is $n-2$. ie) $g_{coe}(\overline{P_n}) = n-2$.

Theorem 3.2 For any Gear graph G_n with $n \geq 3$ then $g_{coe}(\overline{G_n}) = n + 1$.

Proof. For the Gear graph G_n ,if n is odd, then $\overline{G_n}$ has $n+1$ odd vertices. By the proposition 2.3, $n+1$ vertices belong to co-even geodetic set. Moreover, if n is even, then the graph $\overline{G_n}$ has n vertices have odd degree. These n vertices containing the co-even geodetic set. It is easy to see that all vertices do not lies any geodesic of co-even geodetic set. So we add one more vertex in co-even geodetic set. Obviously, $g_{coe}(\overline{G_n}) = n + 1$.

Theorem 3.3 For the complement of the cycle $\overline{C_n}$ with $n \geq 5$, then $g_{coe}(\overline{C_n}) = \begin{cases} 3 & if \quad n \ is \ odd \\ n & if \quad n \ is \ even \end{cases}$

Proof. This theorem follows from the Theorem 3.1

4. Conclusions

In this paper, we obtained co-even geodetic number of some kind of graphs and complement of some graphs. Also, we see the relation between vertex covering and co-even geodetic number of some graphs.

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