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Split Domination Decomposition of Path Graphs

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Abstract

A decomposition (G_1, G_2, G_3, …, G_n) of G is said to be a split domination decomposition (SDD), if the following conditions are satisfied:(i) each G_i is connected(ii)$\gamma_s(G_i) = i$, $1 \leq i \leq n$. In this paper, we prove that path, path corona and subdivision of path graph admit SDD.

Keywords: split domination, decomposition, split domination decomposition.

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1. Introduction

Graph theory in mathematics refers to the study of graphs. The theory of domination is one of the rapidly developing areas in graph theory. The concept of split domination was developed by Veerabhadrappa R. Kulli and Bidarahalli Janakiram [2]. Another important concept in graph theory is decomposition of graphs. Decompositions are imposed by applying several conditions on G_i in the decompositions by several authors based on their studies.

We introduce a new concept split domination decomposition of a graph which is motivated by the concepts of Linear path decomposition [3] and Connected Domination Decomposition [4]. We have considered here simple undirected graphs without loops or multiple edges. The order and size of the graph are indicated by p and q respectively. Terms not defined here are used in the sense of Frank Harary [1].

2. Preliminaries

Definition2.1: A dominating set D of a graph G = (V, E) is a set of vertices such that each vertex of G is either in D or has at least one neighbor in D.

Definition 2.2: A dominating set D of a graph G = (V, E) is a split dominating set if the induced sub graph <V-D> is disconnected. The split domination number $\gamma_s(G)$ of G is the minimum cardinality of a split dominating set.

Definition2.3: If $G_1, G_2, G_3, \ldots, G_n$ are edge disjoint sub graphs of G such that $E(G) = E(G_1) \cup E(G_2) \cup \ldots \cup E(G_n)$, then $(G_1, G_2, G_3, \ldots, G_n)$ is said to be decomposition of G.

Definition2.4: The corona $P_p \odot K_1$ is the graph constructed from a copy of P_p, where for each vertex $u \in V(P_p)$, a new vertex u' and a pendent edge uu' are added. It is denoted by P_p^+ and is called comb.

Definition 2.5: A subdivision of a graph G is a graph obtained by inserting a new vertex in each edge of G and is denoted by S(G).

3. Split Domination Decomposition

Definition 3.1: A Decomposition $(G_1, G_2, G_3, \ldots, G_n)$ of G is said to be a split domination decomposition (SDD), if the following conditions are satisfied:

(i). each G_i is connected

(ii). $\gamma_s(G_i) = i$, $1 \leq i \leq n$.

Example 3.2:

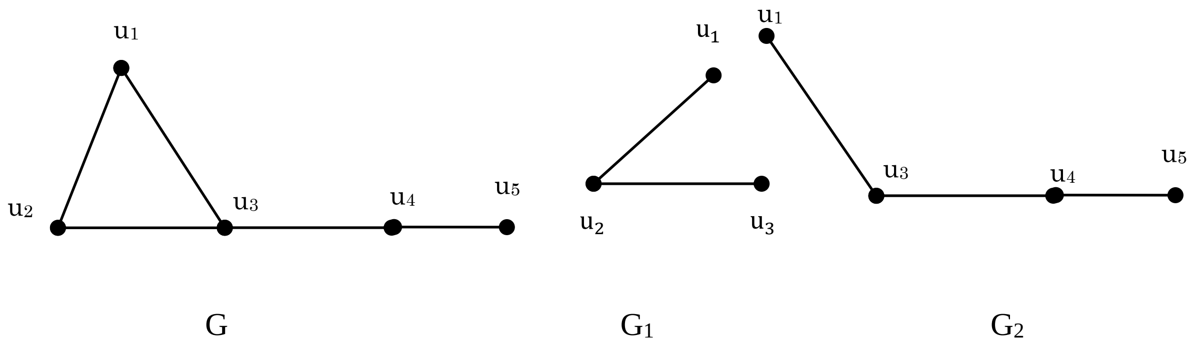

Figure 1: Graph G and its SDD (G_1, G_2)

Remark 3.3: A path with 3 vertices have split domination number 1 and path having 3k-2,3k-1 and 3k vertices have the split domination number k, k.≥2.

Theorem 3.4: A pathP_p, $\frac{3n^2-3n+6}{2} \leq p \leq \frac{3n^2+n+2}{2}$ admits split domination decomposition (G_1, G_2, G_3, …, G_n) if and only if$\sum_{i=1}^{n} \gamma_s(G_i) = \frac{n(n+1)}{2}$.

Proof: Let $P_p=u_1u_2 \ldots u_p$ be a path of order p.

Assume that P_p , $\frac{3n^2-3n+6}{2} \leq p \leq \frac{3n^2+n+2}{2}$ admits split domination decomposition (G_1, G_2, G_3,…, G_n).

Clearly $\gamma_s(G_i) = i$, $1 \leq i \leq n$.

Therefore $\gamma_s(G_1) + \gamma_s(G_2) + \cdots + \gamma_s(G_n) = 1 + 2 + \cdots + n$

$$\sum_{i=1}^{n} \gamma_s(G_i) = \frac{n(n+1)}{2}$$

Conversely, assume that $\sum_{i=1}^{n} \gamma_s(G_i) = \frac{n(n+1)}{2}$.

Clearly $\gamma_s(G_i) = i$, $1 \leq i \leq n$.

Therefore P_p admits split domination decomposition (G_1, G_2, G_3, …,G_n).

Next, we have to find the bound for p.

By remark 3.3, the subgraphs G_1, G_2, G_3, …, $G_n$of P_p having minimum possible vertices are

$G_1 = u_1u_2u_3$

$G_2 = u_3u_4u_5u_6$

$G_3 = u_6u_7u_8u_9u_{10}u_{11}u_{12}$

⋮

$G_n = u_mu_{m+1} \ldots u_p$ Where$m = \frac{3n^2-9n+12}{2}$, $p = \frac{3n^2-3n+6}{2}$

Clearly $|V(P_p)| = |V(G_1)| + |V(G_2)| + \cdots + |V(G_n)|$

$p = (3+4+\ldots+3n-2)-(n-1) = \frac{3n^2-3n+6}{2}$

Next, the maximum possible vertices of subgraphs $G_1, G_2, G_3, \ldots, G_n$ of P_p are

$G_1 = u_1u_2u_3$

$G_2 = u_3u_4u_5u_6u_7u_8$

$G_3 = u_8u_9u_{10}u_{11}u_{12}u_{13}u_{14}u_{15}u_{16}$

$\vdots$

$G_n = u_mu_{m+1}\ldots u_p$ Where $m = \frac{3n^2-5n+4}{2}$, $p = \frac{3n^2+n+2}{2}$

Clearly $|V(P_p)| = |V(G_1)| + |V(G_2)| + \cdots + |V(G_n)|$

$p = (3+6+\ldots+3n)-(n-1) = \frac{3n^2+n+2}{2}$

Therefore $\frac{3n^2-3n+6}{2} \leq p \leq \frac{3n^2+n+2}{2}$.

Corollary 3.5: If $\frac{3n^2+n+2}{2} < p < \frac{3n^2+3n+6}{2}$, then P_p does not admit split domination decomposition.

Theorem 3.6: P_p^+ admits split domination decomposition ($G_1, G_2, G_3, \ldots, G_n$) if and only if P_p has $\frac{n^2+n}{2}$ $(n > 1)$ vertices.

Proof: Let $P_p = u_1u_2\ldots u_p$ be a path with p vertices.

If we join the vertices $u_1', u_2', \ldots, u_p'$ to $u_1, u_2, \ldots, u_p$ respectively, then we get P_p^+.

Assume that P_p has $\frac{n^2+n}{2}$ $(n > 1)$ vertices.

To prove P_p^+ admits split domination decomposition ($G_1, G_2, G_3, \ldots, G_n$).

Suppose $p = \frac{n^2+n}{2}$

$G_1 = <\{u_1, u_2, u_1'\}>$

$G_2 = <\{u_2, u_3, u_2', u_3'\}>$

$G_3 = <\{u_3, u_4, u_5, u_6, u_4', u_5', u_6'\}>$

$\vdots$

$G_n = <\{u_l, u_{l+1}, \ldots, u_p, u_{l+1}', \ldots, u_p'\}>$

Notice that the minimum split dominating set of G_n has n vertices and P_p has $1+2+3+\ldots+n = \frac{n(n+1)}{2} = \frac{n^2+n}{2}$ vertices.

Clearly $\gamma_s(G_i) = i$, $1 \leq i \leq n$.

Therefore ($G_1, G_2, G_3, \ldots, G_n$) is a split domination decomposition of P_p^+.

Conversely, suppose P_p^+ admits split domination decomposition.

To prove P_p has $\frac{n^2+n}{2}$, $(n > 1)$ vertices.

Suppose not,

Case (i): $|V(P_p)| > \frac{n^2+n}{2}$

We join m vertices in P_p where m=1, 2, 3, …, or n. Constructing ($G_1, G_2, G_3, \ldots, G_n$) in the above, we have remaining m vertices where m=1, 2, 3, …, or n. We cannot arrange the m vertices in the minimum split dominating set of G_i otherwise ($G_1, G_2, G_3, \ldots, G_n$) would not be a split domination decomposition for P_p^+. If these m vertices alone to give a sub graph G_{k_m} , then $(G_1, G_2, \ldots, G_n, G_{k_m})$ would not be a split domination decomposition for P_p^+ which is a contradiction.

Case (ii): $|V(P_p)| < \frac{n^2+n}{2}$. We eliminate m vertices in P_p where $m = 1, 2, 3, \ldots$ or $n-1$. Constructing($G_1, G_2, G_3, \ldots, G_n$) in the above, we have remaining m vertices where $m = n-1$, n-2, .., or n-(n-1). We cannot arrange the m vertices in the minimum split dominating set of G_i otherwise ($G_1, G_2, G_3, \ldots, G_n$) would not be a split domination decomposition for P_p^+. If these m vertices alone to give a sub graph G_{k_m} , then$(G_1, G_2, \ldots, G_{n-1}, G_{k_m})$ would not be a split domination decomposition for P_p^+ which is a contradiction. Therefore P_p has $\frac{n^2+n}{2}$, $(n > 1)$ vertices.

Note 3.7: In general, if P_p admits split domination decomposition, then S (P_p) need not admit split domination decomposition and vice-versa. So we cannot use the range of p as in theorem 3.4 to S (P_p).

Theorem 3.8: Let P_p be a (p, q) -path. Subdivision of the path graph S (P_p) admits split domination decomposition ($G_1, G_2, G_3, \ldots, G_n$)if and only if $\frac{2n^2-6n+14}{2} \leq p \leq \frac{n^2+5n-8}{2}$.

Proof: Let $P_p = u_1 u_2 \ldots u_p$ be a path with p vertices. Then S (P_p) has $2p-1$ vertices.
Assume that S (P_p) admits split domination decomposition. Now we can find the range of p if and only if S (P_p) admits split domination decomposition.
From note 3.7, we can't apply the range of p in P_p as in theorem 3.4 toS (P_p). Hence using the range of p in P_p to S (P_p), the following table shows the probabilities for S (P_p) admits split domination decomposition.

No. of decompositions (n)	2	3	4	5	6	7
No. of vertices in $P_p(p)$	4	7 8	11 12 13 14	17 18 19 20 21	25 26 27 28 29	34 35 36 37 38 39
No. of vertices in $S(P_p)$	7	13 15	21 23 25 27	33 35 37 39 41	49 51 53 55 57	67 69 71 73 75 77

Table:1

Using Newton's forward difference formula, we have to find the upper and lower bound of p for P_p such that S (P_p) admitssplit domination decomposition.
To find the lower bound of p, using table-1.
By Newton's forward formula,$p = p_0 + \frac{u}{1!}\Delta p_0 + \frac{u(u-1)}{2!}\Delta^2 p_0 + \cdots$
Where $u = \frac{n-n_0}{h} = n-3$ ($n_0 = 3$ and$h = 1$)
Here $p_0 = 7, \Delta p_0 = 4, \Delta^2 p_0 = 2, \Delta^3 p_0 = 0$
Therefore$p = \frac{2n^2-6n+14}{2}$
Next, to find the upper bound of p, using table-1.
By Newton's forward formula,$p = p_0 + \frac{u}{1!}\Delta p_0 + \frac{u(u-1)}{2!}\Delta^2 p_0 + \cdots$
Where $u = \frac{n-n_0}{h} = n-3$ ($n_0 = 3$ and $h = 1$)
Here $p_0 = 8, \Delta p_0 = 6, \Delta^2 p_0 = 1, \Delta^3 p_0 = 0$
Therefore $p = \frac{n^2+5n-8}{2}$
Therefore S(P_p) admits split domination decomposition, if $\frac{2n^2-6n+14}{2} \le p \le \frac{n^2+5n-8}{2}$.
Conversely, Assume that$\frac{2n^2-6n+14}{2} \le p \le \frac{n^2+5n-8}{2}$.
To prove S (P_p) admit split domination decomposition.
Suppose not,
Consider the lower bound of p, if we eliminate one vertex fromP_p,then the corresponding S(P_p) will not admit split domination decomposition.
Hence $p = \frac{2n^2-6n+14}{2} - 1 < \frac{2n^2-6n+14}{2}$.which is a contradiction.
Consider the upper bound of p, if we join one vertex to P_p, then the corresponding S (P_p) will not admit split domination decomposition.
Hence$p = \frac{n^2+5n-8}{2} + 1 > \frac{n^2+5n-8}{2}$.which is a contradiction.
Therefore S (P_p) admits split domination decomposition.

4. Conclusion

In this paper, we deal that path, path corona and subdivision of path graph admits split domination decomposition. Further investigations could also be done to get the condition at which some graphs admit split domination decomposition.

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