Ratio Mathematica Volume 44, 2022

Connected $2-$ Dominating Sets and Connected $2-$ Domination Polynomials of the Complete Bipartite Graph $k_{2,m}$.

Y. A. Shiny[1]
T. Anithababy[2]

Abstract

Let $G=(V,E)$ be a simple graph. Let $D_{c_2}(G,j)$ be the family of connected $2-$dominating sets in G with cardinality j and $d_{c_2}(G,j)=|D_{c_2}(G,j)|$. Then the polynomial $D_{c_2}(G,x)=\sum_{j=\gamma_{c_2}(G)}^{|V(G)|} d_{c_2}(G,j)x^j$, is called the $2-$domination polynomial of G where $\gamma_{c_2}(G)$ is the connected $2-$ domination number of G.Let $D_{c_2}(k_{2,m},j)$ be the family of connected $2-$dominating sets of the Complete bipartite graph $k_{2,m}$ with cardinality j and let $d_{c_2}(k_{2,m},j)=|D_{c_2}(k_{2,m},j)|$. Then the connected $2-$ domination polynomial $D_{c_2}(k_{2,m},x)$ of $k_{2,m}$ is defined as $D_{c_2}(k_{2,m},x)=\sum_{j=\gamma_{c_2}(k_{2,m})}^{|V(k_{2,m})|} d_{c_2}(k_{2,m},j)x^j$, where $\gamma_{c_2}(k_{2,m},j)$ is the connected 2 – domination number of $k_{2,m,}$. In this paper, we obtain a recursive formula for $d_{c_2}(k_{2,m},j)$.Using this recursive formula, we construct the connected $2-$domination polynomial $D_{c_2}(k_{2,m},x)=\sum_{j=3}^{m+2} d_{c_2}(k_{2,m},j)x^j$, where $d_{c_2}(k_{2,m},j)$ is the number of connected $2-$dominating sets of $k_{2,m}$ of cardinality j and some properties of this polynomial have been studied.

Keywords: Dominating, Connected and cardinality

2010 Mathematical classification number: 05C69, 54D05[3].

---

[1]Reg. No.: 19213042092006, Research Scholar (Full time), Research Department of Mathematics, Women's Christian College, Nagercoil. Affiliated by Manonmaniam Sundaranar University, Tamil Nadu, India, shinyjebalin@gmail.com

[2]Assistant Professor, Research Department of Mathematics, Women's Christian College, Nagercoil. Affiliated by Manonmaniam Sundaranar University, Tamil Nadu, India. anithasteve@gmail.com

[3]Received on June 7th, 2022. Accepted on Aug 10th, 2022. Published on Nov 30th, 2022. doi: 10.23755/rm.v44i0.889. ISSN: 1592-7415. eISSN: 2282-8214.

1. Introduction

Let $G = (V, E)$ be a simple graph of order, $|V| = m$. For any vertex $v \epsilon V$, the open neighbourhood of v is the set $N(v) = \{u \epsilon V / uv \epsilon E\}$ and the closed neighbourhood of V is the set $N[v] = N(v) \cup \{v\}$. For a set $S \subseteq V$, the open neighbourhood of S is $N(S) = U_{v \epsilon S} N(v)$ and the closed neighbourhood of S is $N(S) \cup S$.A set $D \subseteq V$ is a dominating set of G, if $N[D] = V$ or equivalently, every vertex in $V - D$ is adjacent to atleast one vertex in D.The domination number of a graph G is defined as the cardinality of a minimum dominating set D of vertices in G and is denoted by $\gamma(G)$. A dominating set D of G is called a connected dominating set if the induced sub-graph $< D >$ is connected. The connected domination number of a graph G is defined as the cardinality of a minimum connected dominating set D of vertices in G and is denoted by $\gamma_c(G)$.

A graph $G = (V, E)$is called a bipartite graph if its vertices V can be partitioned into two subsets V_1 and V_2 such that each edge of G connects a vertex of $V_1$to a vertex of V_2. If G contains every edge joining a vertex of V_1 and a vertex of V_2 then G is called a complete bipartite graph. It is denoted by $k_{m,n}$, where m and n are the numbers of vertices in V_1 and V_2 respectively. Let $k_{2,m}$ be the Complete bipartite graph with $m + 2$ vertices. Throughout this paper let us take $V(k_{2,m}) = \{v_1, v_2, v_3, \ldots, v_{m+1}, v_{m+2}\}$ and $E(k_{2,m}) = \{(v_1, v_3), (v_1, v_4), (v_1, v_5), \ldots, (v_1, v_{m+1}), (v_1, v_{m+2}), (v_2, v_3), (v_2, v_4), (v_2, v_5), \ldots, (v_2, v_{m+1}), (v_2, v_{m+2})$.

As usual we use $\binom{m}{j}$ for the combination m to j. Also, we denote the set $\{1, 2, \ldots \ldots \ldots, 2m - 1, 2m\}$ by $[2m]$, throughout this paper.

2. Connected 2 – Dominating Sets of the Complete Bipartite Graph $k_{2,m}$

In this section, we state the connected 2 – domination number of the complete bipartite graph $k_{2,m}$ and some of its properties.

Definition 2.1. Let G be a simple graph of order m with no isolated vertices. A subset $D \subseteq V$ is a 2− dominating set of the graph G if every vertex $v \epsilon V - D$ is adjacent to atleast two vertices in D. A 2− dominating set is called a connected 2− dominating set if the induced subgraph <D> is connected.

Definition 2.2. The cardinality of a minimum connected 2 – dominating sets of G is called the connected 2 – domination number of G and is denoted by $\gamma_{C_2}(G)$.

Lemma 2.3 For all $m \in z^+$, $\binom{m}{j} = 0$ if $j > m$ or $j < 0$.

Theorem 2.4 $d_{c2}(k_{2,m}, j) = \left\{ \binom{m+2}{j} - \binom{m+1}{j} - \binom{m}{j-1} \right.$ for $3 \le j \le m + 2$

Proof: Let the partite sets of $k_{2,m}$ be $V_1 =\{v_{1,}v_{2,}\}$ and $V_2 = \{v_{3,}v_{4,},...,v_{m+1,}v_{m+2}\}$. Since the subgraph induced by the vertex set as $\{v_{1,}v_{2,}\ \}$ is not connected, every connected $2-$dominating set of $k_{2,m}$ must contain the vertex $\{v_1\}$ or $\{v_2\}$ or $\{v_{1,}v_{2,}\}$. When $3\le j \le m$, every connected $2-$dominating set must contain $\{v_{1,}v_{2,}\}$. Since, $|V(k_{2,m})| = m+2$, $k_{2,m}$ contains $\binom{m+2}{j}$ number of subsets of cardinality j. Since, the subgraphs induced by $\{v_{1,}v_{2,}\}$ and $\{v_{3,}v_{4,},...,v_{m+1,}v_{m+2}\}$ are not connected, each time $\binom{m+1}{j}$ number of subsets of $k_{2,m}$ of cardinality j and $\binom{m}{j-1}$ number of subsets of $k_{2,m}$ of cardinality $j-1$ are not connected $2-$dominating sets. Hence, $k_{2,m}$ contains $\binom{m+2}{j}-\binom{m+1}{j}-\binom{m}{j-1}$ number of subsets of connected $2-$dominating sets, when $3\le j \le m$.

Therefore, $d_{c2}(k_{2,m},j) = \binom{m+2}{j}-\binom{m+1}{j}-\binom{m}{j-1}$ for all $3\le j \le m$.

When the cardinality is $m+1$, every subset of $k_{2,m}$ containing $\{v_1\}$ or $\{v_2\}$ are connected $2-$dominating sets. Therefore, two more sets are connected $2-$dominating sets when the cardinality is $m+1$.

Hence, $d_{c2}(k_{2,m},j) = \binom{m+2}{j}-\binom{m+1}{j}-\binom{m}{j-1}+2$, when $j = m+1$.

Since, there is only one subset of $k_{2,m}$ with cardinality $m+2$ and that set is a connected $2-$dominating set. we get $d_{c2}(k_{2,m},j) = \binom{m+2}{j}$ when $j = m+2$.

Theorem 2.5. Let $k_{2,m}$ be the complete bipartite graph with $m \ge 3$. Then

(i) $d_{c2}(k_{2,m},j) = d_{c2}(k_{2,m-1},j) + d_{c2}(k_{2,m-1},j-1)$

(ii) $d_{c2}(k_{2,m},j) = d_{c2}(k_{2,m-1},j) + 1$ if $j = 3$.

(iii) $d_{c2}(k_{2,m},j) = d_{c2}(k_{2,m-1},j) + d_{c2}(k_{2,m-1},j-1) - 2$ if $j = m$.

Proof:

(i) By Theorem 2.4, we have,

$d_{c2}(k_{2,m},j) = \binom{m+2}{j}-\binom{m+1}{j}-\binom{m}{j-1}$ for all $3 \le j \le m+2$.

$d_{c2}(k_{2,m-1},j) = \binom{m+1}{j}-\binom{m}{j}-\binom{m-1}{j-1}$

$d_{c2}(k_{2,m-1},j-1) = \binom{m+1}{j-1}-\binom{m}{j-1}-\binom{m-1}{j-2}$.

Consider, $d_{c2}(k_{2,m-1},j)+d_{c2}(k_{2,m-1},j-1) = \binom{m+1}{j}-\binom{m}{j}-\binom{m-1}{j-1}+\binom{m+1}{j-1}-\binom{m}{j-1}-\binom{m-1}{j-2}$.

$= \binom{m+1}{j}+\binom{m+1}{j-1}-\left[\binom{m}{j}+\binom{m}{j-1}\right]-\left[\binom{m-1}{j-1}+\binom{m-1}{j-2}\right]$

$= \binom{m+2}{j} - \binom{m+1}{j} - \binom{m}{j-1}$.

$= d_{c2}(k_{2,m}, j)$.

Therefore,

$d_{c2}(k_{2,m}, j) = d_{c2}(k_{2,m-1}, j) + d_{c2}(k_{2,m-1}, j-1)$ for all $4 \leq j \leq m+2$ and $j \neq m$.

(i) When $j = 3, d_{c2}(k_{2,m}, 3) = \binom{m+2}{3} - \binom{m+1}{3} - \binom{m}{2}$ by Theorem 2.4

$= \binom{m+1}{2} - \binom{m}{2}$

$= \binom{m}{1}$

Consider, $d_{c2}(k_{2,m-1}, 3) = \binom{m+1}{3} - \binom{m}{3} - \binom{m-1}{2} = \binom{m}{2} - \binom{m-1}{2}$

$= \binom{m-1}{1}$

$= m - 1$.

That is, $d_{c2}(k_{2,m-1}, 3) = d_{c2}(k_{2,m}, 3) - 1$.

Therefore, $d_{c2}(k_{2,m}, 3) = d_{c2}(k_{2,m-1}, 3) + 1$.

Hence, $d_{c2}(k_{2,m}, j) = d_{c2}(k_{2,m-1}, j) + 1 \; if \; j = 3$.

(ii) When $j = m$,

$d_{c2}(k_{2,m}, m) = \binom{m+2}{m} - \binom{m+1}{m} - \binom{m}{m-1}$, by Theorem 2.2.

$= \binom{m+1}{m-1} - \binom{m}{m-1}$.

$= \binom{m}{m-2}$.

Consider, $d_{c2}(k_{2,m-1}, m) + d_{c2}(k_{2,m-1}, m-1) = \binom{m+1}{m} - \binom{m}{m} - \binom{m-1}{m-1} +$
$2 + \binom{m+1}{m-1} - \binom{m}{m-1} - \binom{m-1}{m-2}$

$= \binom{m+1}{m} - \binom{m+1}{m-1} - \left[\binom{m}{m} + \binom{m}{m-1}\right] - \left[\binom{m-1}{m-1} + \binom{m-1}{m-2}\right] + 2$

$= \binom{m+2}{m} - \binom{m+1}{m} - \binom{m}{m-1} + 2$

$= \binom{m+1}{m-1} - \binom{m}{m-1} + 2$

$= \binom{m}{m-2} + 2$.

$= d_{c2}(k_{2,m}, m) + 2$.

Therefore, $d_{c2}(k_{2,m}, m) = d_{c2}(k_{2,m-1}, m) + d_{c2}(k_{2,m-1}, m-1) + 2$.

Hence, $d_{c2}(k_{2,m}, j) = d_{c2}(k_{2,m-1}, j) + d_{c2}(k_{2,m-1}, j-1) - 2$ when $j = m$.

3. Connected 2 −Domination Polynomials of the Complete Bipartite Graph $k_{2,m}$.

Definition 3.1. Let $d_{c2}(k_{2,m}, j)$ be the number of connected 2 –dominating sets of the Complete bipartite Graph $k_{2,m}$ with cardinality j.Then, the connected 2 − domination

Polynomial of $k_{2,m}$ is defined as $D_{c_2}(k_{2,m,}x) = \sum_{j=\gamma_{c_2}(k_{2,m})}^{|V(k_{2,m})|} d_{c_2}(k_{2,m,}\text{j})x^j$, where $\gamma_{c_2}(k_{2,m})$ is the connected 2 – domination number of $k_{2,m}$.

Remark 3.2 $\gamma_{c_2}(k_{2,m})$ =3.

Proof. Let $k_{2,m}$ be the complete bipartite graph with partite sets V_1 ={$v_{1,}v_2$} and V_2 = {$v_{3,}v_{4,}$,...,$v_{m+1,}v_{m+2}$}. Let $v_{1,}v_2 \epsilon$ V ($k_{2,m}$) and $v_{1,}v_2$ are adjacent to all the other vertices $v_{3,}v_4$,...,$v_{m+1,}v_{m+2}$ of $k_{2,m}$. Also Since, v_1 and v_2 are not connected, every connected 2 –dominating set must contain the vertices $v_{1,}v_2$ and one more vertex from {v_3, v_4,...,$v_{m+1,}v_{m+2}$}. Therefore, the minimum cardinality is 3. Hence, γ_{c_2} ($k_{2,m}$) =3.

Theorem 3.3 Let $k_{2,m}$ be the complete bipartite graph with $m \geq 3$.
Then $D_{c2}(k_{2,m,}x)$=$(1 + x)$ $D_{c2}(k_{2,m-1,}x)$+$x^3 - 2x^m$.

Proof:

From the definition of connected 2 − domination Polynomial, we have,

$D_{c2}(k_{2,m,}x) = \sum_{j=3}^{m+2} d_{c_2}(k_{2,m,}\text{j})x^j$.

$= d_{c2}(k_{2,m}, 3)x^3 + \sum_{j=4}^{m-1} d_{c_2}(k_{2,m,}\text{j})x^j + d_{c_2}(k_{2,m,}m)x^m + \sum_{j=m+1}^{m+2} d_{c_2}(k_{2,m,}\text{j})x^j$.

$$= [d_{c_2}(k_{2,m-1,}3) + 1]x^3 + \sum_{j=4}^{m+2} [d_{c_2}(k_{2,m-1,}\text{j}) + d_{c_2}k_{2,m-1,}\, j - 1]x^j$$

$$+[d_{c_2}(k_{2,m-1,}m) + d_{c_2}(k_{2,m-1,}m - 1) - 2]x^m$$, by Theorem 2.5

$$= d_{c2}(k_{2,m-1}, 3)x^3 + x^3 + \sum_{j=4}^{m+2} d_{c_2}(k_{2,m-1,}\text{j})\, x^j$$

$$+ \sum_{j=4}^{m+2} d_{c_2}(k_{2,m-1,}\text{j} - 1)\, x^j - 2x^m.$$

$= \sum_{j=3}^{m+2} d_{c_2}(k_{2,m-1,}\text{j})\, x^j + x\sum_{j=4}^{m+2} d_{c_2}(k_{2,m-1,}\text{j} - 1)\, x^{j-1} + x^3 - 2x^m$.

$= D_{c2}(k_{2,m-1,}x) + xD_{c2}(k_{2,m-1,}x) + x^3 - 2x^m$.

Hence, $D_{c2}(k_{2,m,}x) = (1 + x)D_{c2}(k_{2,m-1,}x) + x^3 - 2x^m$, for every $m \geq 3$.

Example 3.4 Let $k_{2,7}$ be the complete bipartite graph with order 9 as given in Figure 2.1.

$k_{2,7}$:

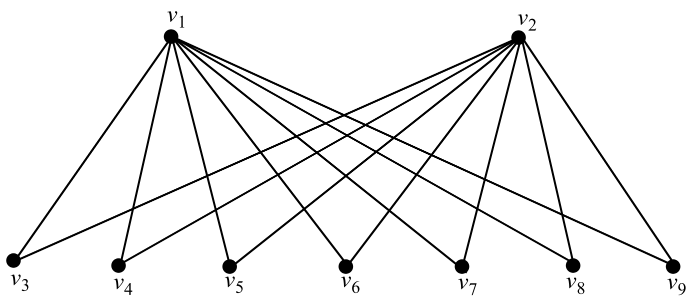

Figure 2.1

$D_{c_2}(K_{2,6,}x)$ = $6x^3 + 15x^4$+$20x^5 + 15x^6 + 8x^7 + x^8$.

By Theorem 3.3, we have,
$D_{c_2}(K_{2,7,}x)$ = (1 + x)($6x^3$+$15x^4$+$20x^5$+$15x^6$+$8x^7$+x^8)+ $x^3 - 2x^7$.
= $6x^3$+$15x^4$+$20x^5$+$15x^6$+$8x^7$+x^8 +$6x^4$+$15x^5$+$20x^6$+$15x^7$+$8x^8$+x^9 + $x^3 - 2x^7$.
= $7x^3$+$21x^4$+$35x^5$+$35x^6$+$21x^7$+$9x^8$+x^9
We obtain $d_{c_2}(k_{2,m,}j)$ for 3≤ m ≤ 15 and 3≤ j ≤ 15 as shown in Table 1.

j \ m	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
3	0	3	5	1												
4	0	4	6	6	1											
5	0	5	10	10	7	1										
6	0	6	15	20	15	8	1									
7	0	7	21	35	35	21	9	1								
8	0	8	28	56	70	56	28	10	1							
9	0	9	36	84	126	126	84	36	11	1						
10	0	10	45	120	210	252	210	120	45	12	1					
11	0	11	55	165	330	462	462	330	165	55	13	1				
12	0	12	66	220	495	792	924	792	495	220	66	14	1			
13	0	13	78	286	715	1287	1716	1716	1287	715	286	78	15	1		
14	0	14	91	364	1001	2002	3003	3432	3003	2002	1001	364	91	16	1	
15	0	15	105	455	1365	3003	5005	6435	6435	5005	3003	1365	455	107	17	1

Table 1.$d_{c_2}(k_{2,m,}j)$, the number of connected $2-$ dominating sets of $k_{2,m}$ with cardinality j.

In the following Theorem, we obtain some properties of $d_{c2}\ (K_{2,m,}j)$.

Theorem: 3.5 The following properties hold for the coefficients of $D_{c_2}(K_{2,m,}x)$ for all m.

(i) $d_{c_2}(k_{2,m,}3) = m$, for all $m \geq 3$.
(ii) $d_{c_2}(k_{2,m,}m + 2) = 1$, for all $m \geq 3$.
(iii) $d_{c_2}(k_{2,m,}m + 1) = m + 2$, for all $m \geq 3$.
(iv) $d_{c_2}(k_{2,m,}m) = \binom{m+2}{2} - \binom{m+1}{1} - m$.
(v) $d_{c_2}(k_{2,m,}m - 1) = \binom{m+2}{3} - \binom{m+1}{2} - \binom{m}{2}$, for all $m \geq 4$.
(vi) $d_{c_2}(k_{2,m,}m - 2) = \binom{m+2}{4} - \binom{m+1}{3} - \binom{m}{3}$, for all $m \geq 5$.
(vii) $d_{c_2}(k_{2,m,}m - 3) = \binom{m+2}{5} - \binom{m+1}{4} - \binom{m}{4}$, for all $m \geq 6$.

(viii) $d_{c_2}\left(k_{2,m,}m-i\right)=\binom{m+2}{i+2}-\binom{m+1}{i+1`}-\binom{m}{i}$, for all $m\geq 4$ and $i\geq 1$.

Proof:

(i) $d_{c_2}\left(k_{2,m,}3\right)=m$.

We prove this by induction on m.

When $m=3$, $d_{c_2}\left(k_{2,m,}3\right)=3$.

Therefore, the result is true for $m=3$.

Now, suppose that the result is true for all numbers less than 'm' and we prove it for m.

By Theorem 2.6,

$d_{c_2}\left(k_{2,m,}3\right)=d_{c_2}\left(k_{2,m-1,}3\right)+1=m-1+1\ =m$.

(ii) $d_{c_2}\left(k_{2,m,}m+2\right)=1$, for all $m\geq 3$.

Since, there is only one connected $2-$ dominating set of cardinalities

$m+\ 2,\ d_{c_2}\left(k_{2,m,}m+2\right)=1$.

(iii) $d_{c_2}\left(k_{2,m,}m+1\right)=m+2$, for all m≥4.

Since, $d_{c_2}\left(k_{2,m,}m+1\right)=\{[m+2]-x/x\varepsilon[m+2]\}$,we have the result.

(iv), (v), (vi), (vii) and (viii) follows from Theorem 2.4.

4. Conclusion

In this paper, the connected $2-$ domination polynomials of the complete bipartite graph $K_{2,m}$ has been derived by identifying its connected $2-$ dominating sets. It also helps us to characterize the connected. Connected $2-$ dominating sets of cardinality j. We can generalize this study to any of complete bipartite graph $K_{n,m}$ and some interesting properties can be obtained.

References

[1] Alikhani. S, peng. Y. H." Introduction to Domination Polynomial of a Graph," Ars Combinatoria, Available as arxiv:0905.2251|v| [math.co]14 May 2009.

[2] Alikhani. S, and Hamzeh Torabi," On Domination Polynomials of Complete Partite Graphs" World Applied Sciences Journal,9(1):23−24,2010.

[3] Sahib. Sh. Kahat. Abdul Jalil M. Khalaf and Roslan Hasni," Dominating sets and Domination Polynomials of Wheels". Asian Journal of Applied Sciences (ISSN:2321−0893), volume 02− Issue 03, June 2014.

[4] Sahib Shayyal Kahat, Abdul Jalil M. Khalaf and Roslan Hasni," Dominating sets and Domination Polynomials of Stars". Australian Journal of Basics and Applied Science,8(6) June 2014, pp 383−386.

[5] A. Vijayan, T. Anithababy, G. Edwin," Connected Total Dominating Sets and

Connected Total Domination Polynomials of Stars and Wheels", IOSR Journal of Mathematics, Volume II, pp 112−121.

[6] A. Vijayan, T. Anithababy, G. Edwin, "Connected Total Dominating Sets and Connected Total Domination Polynomials of Fan Graphs $F_{2,n}$". International Journal of Mathematical Sciences and Engineering Applications (IJMSEA), Vol.10, No.1(April 2016), pp.135−146, ISSN:0473 −9424.