Connected Hub Sets and Connected Hub Polynomials of the Lollipop Graph $L_{p,1}$

T. Angelinshiny[1]
T. Anitha Baby[2]

Abstract

Let G be a graph with vertex set $V(G)$. The number of vertices in G is the order of G and is denoted by $|V(G)|$. The connected hub polynomial of G denoted by $H_c(G,y)$ is defined as $H_c(G,y)=\sum_{k=\hbar_c(G)}^{|V(G)|} h_c(G,k)y^k$ where $h_c(G,k)$ denotes the number of connected hub sets of G of cardinality k and $\hbar_c(G)$ denotes the connected hub number of G. Let $L_{p,1}$ denotes the Lollipop graph with $p+1$ vertices. The connected hub polynomial of $L_{p,1}$ denoted by $H_c\left(L_{p,1},y\right)$ is defined as,$H_c\left(L_{p,1},y\right)=\sum_{k=\hbar_c(L_{p,1})}^{|V(L_{p,1})|} h_c\left(L_{p,1},k\right) y^k$where $h_c\left(L_{p,1},k\right)$ denotes the number of connected hub sets of $L_{p,1}$ of cardinality k, and $\hbar_c\left(L_{p,1}\right)$ denotes the connected hub number of $L_{p,1}$.In this paper, we derive a recursive formula for $\mathrm{h_c}\left(\mathrm{L_{p,1}},\mathrm{k}\right)$. From this recursive formula, we construct the connected hub polynomial of $\mathrm{L_{p,1}}$ as,$\mathrm{H_c}\left(\mathrm{L_{p,1}},\mathrm{y}\right)=\sum_{\mathrm{k}=1}^{\mathrm{p}+1}\mathrm{h_c}\left(\mathrm{L_{p,1}},\mathrm{k}\right)\mathrm{y^k}$Also we study some properties of this polynomial.

Keywords: Lollipop Graph, connected hub set, connected hub number, Connected hub polynomial.

Mathematics Subject Classification Code: 05C31, 05C99[3]

---

[1]Research Scholar (Reg. No. 20213282092009), Department of Mathematics, Women's Christian College, Nagercoil, Tamil Nadu, India. Affiliated by Manonmaniam Sundaranar University, Abishekapatti, Tirunelveli - 627 012. Mail Id: angelinshinyt@gmail.com

[2] Assistant Professor, Department of Mathematics, Women's Christian College, Nagercoil, Tamil Nadu, India. Affiliated by Manonmaniam Sundaranar University, Abishekapatti, Tirunelveli - 627 012. Mail Id: anithasteve@gmail.com

[3]Received on June 19th, 2022. Accepted on Sep 1st, 2022. Published on Nov 30th, 2022. doi: 10.23755/rm.v44i0.886. ISSN: 1592-7415. eISSN: 2282-8214.

1. Introduction

If any two distinct vertices of a graph G are adjacent, then G is a complete graph. If a tree has two nodes of vertex degree 1 and other nodes of vertex degree 2, then it is a path graph. A complete graph of order p is denoted by K_p and a path graph of order q is denoted by P_q. Join a complete graph K_p to a path graph P_q with a bridge. The resulting graph is a Lollipop graph $L_{p,q}$.

2. Connected Hub Sets of the Lollipop Graph $L_{p,1}$

In this section, we give the connected hub number of the Lollipop graph $L_{p,1}$ and some of the properties of the connected hub sets of the Lollipop graph $L_{p,1}$.

Definition 2.1 Join a complete graph K_p to a path graph P_1 with a bridge. The resulting graph is a Lollipop graph $L_{p,1}$.

Definition 2.2 Let $G = (V, E)$ be a connected graph. A subset H of V is called a hub set of G if for any two distinct vertices $u, v \in V - H$, there exists a $u - v$ path P in G, such that all the internal vertices of P are in H. The minimum cardinality of a hub set of G is called the hub number of G and is denoted by $\hbar(G)$.

Definition 2.3 A hub set H of G is called a connected hub set if the induced subgraph $< H >$ is connected. The minimum cardinality of a connected hub set of G is called connected hub number of G and is denoted by $\hbar_c(G)$.

Theorem 2.4

$$h_c(L_{p,1,}k) = \begin{cases} \binom{p+1}{k} - \binom{p}{k} + 1 \; when \; k = 1 \; and \; p-1 \\ \binom{p+1}{k} - \binom{p}{k} \quad if \; 2 \leq k \leq p+1 \; and \; k \neq p-1 \end{cases}$$

Proof. Let $L_{p,1}$ be the Lollipop graph with $p+1$ vertices and $p \geq 4$. Let $v_1, v_2, v_3 \ldots v_p, v_{p+1}$ be the vertices of $L_{p,1}$, in which the degree of the vertices $v_1, v_2, v_3, \ldots, v_{p-1}$ is $p-1$, the degree of the vertex v_p is p and the degree of the vertex v_{p+1} is 1. Since, $L_{p,1}$ contains $p+1$ vertices, the number of subsets of $L_{p,1}$ with cardinality k is $\binom{p+1}{k}$. Also, Since, the subgraph with vertex set $\{v_1, v_2, v_3 \ldots v_{p-1}\}$ is not adjacent to v_{p+1} every hub set must contain the vertices v_p or v_{p+1}. Therefore, every time $\binom{p}{k}$ number of subsets of $L_{p,1}$ of cardinality k are not connected hub sets.
Thus, $L_{p,1}$ have $\binom{p+1}{k} - \binom{p}{k}$ number of connected hubs sets of cardinalities k.
When the cardinality is $p-1$, the set which contains v_{p-1} is also a connected hub set.
When the cardinality is 1, $\{v_p\}$ and $\{v_{p+1}\}$ are the only connected hub sets.

Hence, $h_c(L_{p,1,}k) = \begin{cases} \binom{p+1}{k} - \binom{p}{k} + 1 \; when \; k = 1, \; p-1 \\ \binom{p+1}{k} - \binom{p}{k} \quad if \; 2 \le k \le p+1 \; and \; k \ne \; p-1 \end{cases}$

Theorem 2.5 Let $L_{p,1}$ be the Lollipop graph with $p \ge 4$.Then
(i) $h_c(L_{p,1,}k) = \binom{p}{k-1}$ for all $2 \le k \le p+1$ and $k \ne p-1$
(ii) $h_c(L_{p,1,}k) = \binom{p}{k-1} + 1$ when $k = 1$ and $p-1$.
(iii) $h_c(L_{p,1,}k) =$
$\begin{cases} h_c(L_{p-1,1,}\,k) + h_c(L_{p-1,1,}k-1) \; if \; 1 \le k \le p+1 \; and \; k \ne 2\,, p-2 \\ h_c(L_{p-1,1,}\,k) + \; h_c(L_{p-1,1,}k-1) - 1 \; if \; k = 2 \; and \; p-2 \end{cases}$

Proof: (i) From Theorem 2.4, we have
$h_c(L_{p,1,}k) = \binom{p+1}{k} - \binom{p}{k}$
We know that, $\binom{p+1}{k} - \binom{p}{k} = \binom{p}{k-1}$
Therefore, $h_c(L_{p,1,}k) = \binom{p}{k-1}$ for all $2 \le k \le p+1$ and $k \ne p-1$.
(ii) From Theorem 2.4, we have
$h_c(L_{p,1,}\,k) = \binom{p+1}{k} - \binom{p}{k} + 1 \;\; when \; k = 1 \; and \; p-1.$
We know that, $\binom{p+1}{k} - \binom{p}{k} = \binom{p}{k-1}$
Therefore, $h_c(L_{p,1,}k) = \binom{p}{k-1} + 1 \; when \; k = 1 \; and \; p-1.$
(iii) From (i) $h_c(L_{p,1,}k) = \binom{p}{k-1}$ for all $2 \le k \le p+1$ and $k \ne p-1$.
$h_c(L_{p-1,1,}\,k) = \binom{p-1}{k-1}$
and $h_c(L_{p-,1,}k-1) = \binom{p-1}{k-2}$
Consider, $h_c(L_{p-1,1,}\,k) + h_c(L_{p-,1,}k-1) = \binom{p-1}{k-1} + \binom{p-1}{k-2} = \binom{p}{k-1} =$
$h_c(L_{p,1,}k)$
Therefore, $h_c(L_{p,1,}k) = h_c(L_{p-1,1,}\,k) + h_c(L_{p-,1,}k-1)$ for $1 \le k \le p+1 \; and \; k \ne$
$2, p-2$
When $k = 2$,
$h_c(L_{p,1,}\,2) = \binom{p}{1}$
$h_c(L_{p-1,1,}\,2) = \binom{p-1}{1}$
and $h_c(L_{p-1,1,}\,1) = \binom{p-1}{0} + 1$, by (ii)
Consider, $h_c(L_{p-1,1,}\,2) + h_c(L_{p-1,1,}\,1) = \binom{p-1}{1} + \binom{p-1}{0} + 1 = \binom{p}{1} + 1$

$= h_c(L_{p,1,}\ 2) + 1$
That is, $h_c(L_{p,1,}\ 2) = h_c(L_{p-1,1,}\ 2) + h_c(L_{p-1,1,}\ 1) - 1$.
Therefore, $h_c(L_{p,1,}k) = h_c(L_{p-1,1,}\ k) + h_c(L_{p-,1,}k-1) - 1$ when $k = 2$.
When $k = p - 2$,

$$h_c(L_{p,1,}\ p-2) = \binom{p}{p-3}$$

$$h_c(L_{p-1,1,}\ p-2) = \binom{p-1}{p-3}$$

and $h_c(L_{p-1,1,}\ p-3) = \binom{p-1}{p-4} + 1$, by (ii)

Consider, $h_c(L_{p-1,1,}\ p-2) + h_c(L_{p-1,1,}\ p-3) = \binom{p-1}{p-3} + \binom{p-1}{p-4} + 1 = \binom{p}{p-3} + 1 = h_c(L_{p,1,}\ p-2) + 1$

That is, $h_c(L_{p,1,}\ p-2) = h_c(L_{p-1,1,}\ p-2) + h_c(L_{p-1,1,}\ p-3) - 1$.

Therefore, $h_c(L_{p,1,}k) = h_c(L_{p-1,1,}\ k) + h_c(L_{p-,1,}k-1) - 1$ when $k = p - 2$.
Hence,

$$h_c(L_{p,1,}k) = \begin{cases} h_c(L_{p-1,1,}\ k) + h_c(L_{p-1,1,}k-1)\ if\ 1 \le k \le p+1\ and\ k \neq 2\ , p-2 \\ h_c(L_{p-1,1,}\ k) + \ h_c(L_{p-1,1,}k-1) - 1\ if\ k = 2\ and\ p-2 \end{cases}$$

3. Connected Hub Polynomials of the Lollipop Graph $L_{p,1}$.

Definition 3.1 Let $H_c(L_{p,1,}, k)$ denotes the family of connected hub sets of the Lollipop graph $L_{p,1,}$ of cardinality k and $h_c(L_{p,1,}, k) = |H_c(L_{p,1,}, k)|$. Then, the connected hub polynomial of $L_{p,1}$ denoted by $H_c(L_{p,1,}, y)$ is defined as $H_c(L_{p,1,}, y) = \sum_{k=\hbar_c(L_{p,1,})}^{p+1} h_c(L_{p,1,}, k)\, y^k$
where $\hbar_c(L_{p,1,})$ is connected hub number of $L_{p,1,}$.

Remark 3.2 $\hbar_c(L_{p,1,}) = 1$.
Proof: Label the vertices of $L_{p,1}$ by $v_1, v_{2,}v_3, \dots, v_p, v_{p+1}$ in which the degree of the vertices $v_1, v_2, v_3, \dots, v_{p-1}$ is $p - 1$, the degree of the vertex v_p is p and the degree of the vertex v_{p+1} is 1. Since, any two vertices of $v_1, v_2, v_{3,} \dots, v_p$ are adjacent there is a path between any two vertices of $v_1, v_2, v_{3,} \dots, v_p$. Also, v_p is the internal vertex for all the path between the vertices of $\{v_1, v_{2,}v_3, \dots, v_{p-1}\}$ and v_{p+1}. Hence $\{v_p\}$ and $\{v_{p+1}\}$ are two connected hub sets of cardinalities 1.
Hence, $\hbar_c(L_{p,1}) = 1$.

Theorem 3.3 $H_c(L_{p,1,}y) = (1+y)H_c(L_{p-1,1,}\, y) - y^2 - y^{p-2}$ with initial value $H_c(L_{4,1,}y) = 2y + 4y^2 + 7y^3 + 4y^4 + y^5$.

Proof. We have, $H_c(L_{p,1,}y) = \sum_{k=1}^{p+1} h_c(L_{p,1}, k)y^k$

$H_c(L_{p,1,}y) = \sum_{\substack{k=1\\k\neq 2,p-2}}^{p+1} h_c(L_{p,1}, k)y^k + h_c(L_{p,1}, 2)y^2 + h_c(L_{p,1}, p-2)y^{p-2}$

$$= \sum_{\substack{k=1\\k\neq 2,p-2}}^{p+1} [h_c(L_{p-1,1}, k) + h_c(L_{p-1,1}, k-1)]y^k + [h_c(L_{p-1,1}, 2) + h_c(L_{p-1,1}, 1) - 1]y^2 + [h_c(L_{p-1,1}, p-2) + h_c(L_{p-1,1}, p-3) - 1]y^{p-2}$$

$= \sum_{k=1}^{p+1}[h_c(L_{p-1,1}, k) + h_c(L_{p-1,1}, k-1)]y^k - y^2 - y^{p-2}$

$= \sum_{k=1}^{p+1} h_c(L_{p-1,1}, k)y^k + \sum_{k=1}^{p+1} h_c(L_{p-1,1}, k-1)\, y^k - y^2 - y^{p-2}$

$= \sum_{k=1}^{p+1} h_c(L_{p-1,1}, k)y^k + y\sum_{k=1}^{p+1} h_c(L_{p-1,1}, k-1)\, y^{k-1} - y^2 - y^{p-2}$

$= H_c(L_{p-1,1}, y) + yH_c(L_{p-1,1}, y) - y^2 - y^{p-2}$

$= (1+y)H_c(L_{p-1,1}, y) - y^2 - y^{p-2}$

Hence, $H_c(L_{p,1,}y) = (1+y)H_c(L_{p-1,1}, y) - y^2 - y^{p-2}$ with initial value $H_c(L_{4,1,}x) = 2y + 4y^2 + 7y^3 + 4y^4 + y^5$.

Example 3.4 Consider the Lollipop graph $L_{5,1}$ be with order 6 given in Figure 1.

$L_{5,1}$:

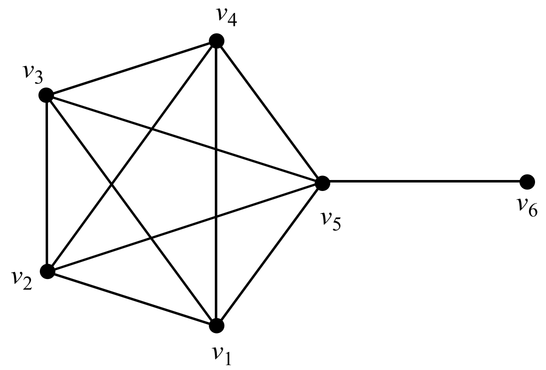

Figure 1

$H_c(L_{5,1,}y) = 2y + 5y^2 + 10y^3 + 11y^4 + 5y^5 + y^6$

By Theorem 3.3, we have,

$H_c(L_{5,1,}y) = (1+y)H_c(L_{4,1,}\, y) - y^2 - y^3$

$= (1+y)(2y + 4y^2 + 7y^3 + 4y^4 + y^5) - y^2 - y^3$

$= 2y + 5y^2 + 10y^3 + 11y^4 + 5y^5 + y^6$

Theorem 3.5 Let $L_{p,1}$ be the Lollipop graph with $p \geq 4$.Then

(i) $H_c(L_{p,1,}y) = \sum_{k=1}^{p+1}\binom{p+1}{k} y^k - \sum_{k=1}^{p+1}\binom{p}{k} y^k - y^2 - y^{p-2}$.

(ii) $H_c(L_{p,1,}\ y) = \sum_{k=1}^{p+1} \binom{p}{k-1} y^k - y^2 - y^{p-2}$.

Proof. Proof is obvious.

$h_c(L_{p,1,}\ k)$ for $4 \le p \le 14$ and $1 \le k \le 15$.

k / p	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
4	2	4	7	4	1										
5	2	5	10	11	5	1									
6	2	6	15	20	16	6	1								
7	2	7	21	35	35	22	7	1							
8	2	8	28	56	70	56	29	8	1						
9	2	9	36	84	126	126	84	37	9	1					
10	2	10	45	120	210	252	210	120	46	10	1				
11	2	11	55	165	330	462	462	330	165	56	11	1			
12	2	12	66	220	495	792	924	792	495	220	67	12	1		
13	2	13	78	286	715	1287	1716	1716	1287	715	286	79	13	1	
14	2	14	91	364	100	2002	3003	3432	3003	2002	1001	364	92	14	1

Table 1

Theorem 3.6 The coefficients of $H_c(L_{p,1}, y)$ satisfy the following properties.

(i) $h_c(L_{p,1,}\ p+1) = 1$, for every $p \ge 4$.

(ii) $h_c(L_{p,1}, p) = p$, for every $p \ge 4$.

(iii) $h_c(L_{p,1}, p-1) = \frac{1}{2}(p^2 - p + 2)$, for every $p \ge 4$.

(iv) $h_c(L_{p,1}, p-2) = \frac{1}{6}(p^3 - 3p^2 + 2p)$, for every $p \ge 4$.

(v) $h_c(L_{p,1}, p-3) = \frac{1}{24}(p^4 - 6p^3 + 11p^2 - 6p)$, for every $p \ge 6$.

(vi) $h_c(L_{p,1}, 1) = 2$, for every $p \ge 4$.

(vii) $h_c(L_{p,1}, 2) = p$, for every $p \ge 4$.

4. Conclusion

In this paper, we identified the connected hub sets of $L_{p,1}$ and using the connected hub sets we derived the connected hub polynomial of $L_{p,1.}$ We can generalize this study to derive the connected hub polynomial of any Lollipop graph $L_{p,q}$.

References

[1] Walsh, Matthew, "The Hub Number Of A Graph", Int. J. Math. Comput. Sci 1, No. 1 (2006): 117-124.

[2] Veettil, Ragi Puthan, And T. V. Ramakrishnan, "Introduction To Hub Polynomial Of Graphs", Malaya Journal Of Matematik (Mjm) 8, No. 4, 2020 (2020): 1592-1596.

[3] S. Alikhani And Y.h. Peng, "Dominating Sets And Domination Polynomials Of Paths", International Journal Of Mathematics And Mathematical Sciences, 2009.

[4] S. Alikhani And Y.h. Peng, "Introduction To Domination Polynomial Of A Graph ", Arxiv Preprint Arxiv:0905.2251, 2009.

[5] Sahib.sh. Kahat, Abdul Jalil Khalaf And Roslan Hasni, "Dominating Sets And Domination Polynomials Of Wheels", Asian Journal Of Applied Sciences (Issn:2321-0893), Volume 02- Issue 03, June 2014.

[6] Sahib.sh. Kahat, Abdul Jalil M. Khalaf And Roslan Hasni, "Dominating Sets And Domination Polynomials Of Stars", Australian Journal Of Basics And Applied Science,8(6) June 2014, 383-386.

[7] A. Vijayan, T. Anitha Baby, G. Edwin, "Connected Total Dominating Sets And Connected Total Domination Polynomials Of Stars And Wheels", Iosr Journal Of Mathematics, Volume II, 112-121.