Ratio Mathematica Volume 47, 2023

Superior domination polynomial of cycles

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Abstract

Superior domination polynomial $SD(G,x) = \sum_{t=\gamma_{sd}(G)}^{n} |sd(G,t)|x^t$ is a polynomial in which the power of the variable denotes the cardinality of a superior dominating set and the total number of sets of same cardinality forms the coefficient of the variable. In this paper we find the $SD(G,S_n)$ of stars and $SD(G,C_n)$ of cycles and properties of the coefficients are discussed. The $SD(G,x)$ different standard graphs are obtained and the roots of the polynomial are tabulated.
Keywords: Superior distance, superior domination, neighbourhood vertex, superior domination polynomial.
2020 AMS subject classifications: 05C12, 05C69, 05C31.[1]

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[1]Received on September 15, 2022. Accepted on December 15, 2022. Published on March 1, 2023. DOI: 10.23755/rm.v41i0.819. ISSN: 1592-7415. eISSN: 2282-8214.

1 Introduction

The graph $G = (V, E)$ is a finite, undirected, simple, ordred pair where $V(G)$ is a set of vertices and $E(G)$ is the set of edges. In 2009 Saeid Alikhani and Yee-hock Peng[1] conceptualized the concept of domination polynomial. Domination is a vast arena in graph theory, Ore[8] coined the term domination in graphs. A vast literature about domination can be found in domination in graphs[3].

There are different types of distances in graph theory one being superior distance, Kathiresan and Marimuthu[7] were the pioneers of superior distance in graphs. The same authors[6] put forth the concept of superior domination in 2008. A Mohamed Ismayil and Tejaskumar R[4] introduced eccentric domination polynomial which was the hybrid idea of combining eccentric domination[5] and domination polynomial.

In this paper, a distance based domination polynomial called superior domination polynomial is introduced by coalescence of superior domination and domination polynomial. Standard formulas to find the coeffcients or the superior dominating sets of stars S_n and cycles C_n for any value of n. Theorems realted to properties of these coefficients are stated and proved. Superior domination polynomial $SD(G, x)$ of different standard graphs are calculated, their roots are tabulated. For all the undefined terminologies and basic concepts of graphs refer the book Graph theory by Frank Harary[2].

2 Preliminaries

Definition 2.1. *[1]. Let $\mathcal{D}(G, i)$ be the family of dominating sets of a graph G with cardinality i and let $d(G, i) = |\mathcal{D}(G, i)|$. Then the domination polynomial $D(G, x)$ of G is defined as $D(G, x) = \sum_{i=\gamma(G)}^{|V(G)|} d(G, i)x^i$, where $\gamma(G)$ is the domination number of G.*

Definition 2.2. *[7]. Let $D_{uv} = N[u] \cup N[v]$. A D_{uv}-walk is defined as a $u - v$ walk in G that contains every vertex of D_{uv}. The superior distance $d_D(u, v)$ from u to v is the length of a shortest $D_{u,v}$ walk.*

Definition 2.3. *[7]. The superior neighbour of a vertex u is given by $d_D(u) = min\{d_D(u, v) : v \in V(G) - \{u\}\}$. A vertex $v(\neq u)$ is called a superior neighbour of u if $d_D(u, v) = d_D(u)$.*

Definition 2.4. *[6]. A vertex u is said to be a superior dominate a vertex v if v is a superior neighbour of u.*

Definition 2.5. *[6]. A set S of vertices of G is called a superior dominating set of G if every vertex $V(G) - S$ is superior dominated by some vertex in S. A superior dominating set G of minimum cardinality is a minimum superior dominating set and its cardinality is called superior domination number of G and denoted by $\gamma_{sd}(G)$.*

Theorem 2.1. *[6]. For a cycle C_n the superior domination number is given by*

$$\gamma_{sd}(C_n) = \begin{cases} \frac{n}{3}, \text{ if } n \equiv 0(mod\, 3) \\ \frac{n+2}{3}, \text{ if } n \equiv 1(mod\, 3) \\ \frac{n+1}{3}, \text{ if } n \equiv 2(mod\, 3) \end{cases}$$

3 Superior Domination Polynomial of Graphs

In this section, we defined superior domination polynomial, properties and results related to superior domination polynomial are observed, stated and proved.

Definition 3.1. *Superior domination polynomial is given by $SD(G, x) = \sum_{t=\gamma_{sd}(G)}^{n} |sd(G, t)|x^t$ where $|sd(G, t)|$ is the number of distinct superior dominating set with cardinality t and $\gamma_{sd}(G)$ is the superior domination number.*

Example 3.1. .

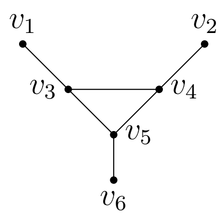


Figure 1: Net graph

Vertex	Minimum superior distance d_D	Superior neighbour
v_1	3	v_2, v_6
v_2	3	v_1, v_6
v_3	4	v_1
v_4	4	v_2
v_5	4	v_6
v_6	3	v_1, v_2

From figure-1 we get $\{v_3, v_4, v_5\}$ *is a superior dominating set with cardinality 3,* $\{v_1, v_3, v_4, v_5\}$, $\{v_2, v_3, v_4, v_5\}$, $\{v_3, v_4, v_5, v_6\}$ *are superior dominating sets of cardinality 4,* $\{v_1, v_2, v_3, v_4, v_5\}$, $\{v_1, v_3, v_4, v_5, v_6\}$, $\{v_2, v_3, v_4, v_5, v_6\}$ *are superior dominating sets of cardinality 5 and* $\{v_1, v_2, v_3, v_4, v_5, v_6\}$ *is superior dominating set with cardinality 6. Therefore superior domination polynomial is given by* $SD(G, x) = x^6 + 3x^5 + 3x^4 + x^3$.

Theorem 3.1. *For a complete graph* K_n *the superior domination polynomial is given by* $SD(K_n, x) = (1 + x)^n - 1$.

Proof. The degree of every vertex $v \in K_n$ is $n - 1$. For any two vertices u and v the number of vertices on their D_{uv}-walk is given by $|V(K_n)|$. Since $|N[u]| = n$ and $|N[v]| = n$ both the vertices have common neighbours and both u and v are incident to each other. Therefore a $D_{u,v}$-walk between u and v contains all vertices of K_n and all the vertices of K_n forms the superior neighbour of any $v \in V(K_n)$ other than itself. By the definition of superior distance, the distance between any two vertices is $n - 1$. Now by the definition of superior domination, for every vertex of $V(K_n) - S$ is superior dominated by some vertex in S which is a superior dominating set and every vertex of $V(K_n) - S$ has a superior neighbour in S. Therefore $SD(K_n, x) = (1 + x)^n - 1$.

Theorem 3.2. *If two graphs are isomorphic then* $SD(G_1, x) = SD(G_2, x)$.

Proof. Let G_1 and G_2 be any two isomorphic graphs. Then there exist a one-one and onto function between the vertex sets such that $f : V(G_1) \to V(G_2)$ such that V_m and V_n are superior neighbours in G_1 if and only if $f(V_m)$ and $f(V_n)$ are superior neighbour of some vertex in G_2. Therefore $|SD(G_1, n)| = |SD(G_2, n)| \, \forall \, n$. Therefore $SD(G_1, x) = SD(G_2, x)$.

Example 3.2. *In the figure 2 and 3 both the tetrahedral graph and complete graph* K_4 *are isomorphic to each other.*

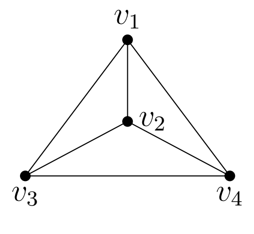


Fig:2-Tetrahedral graph T_g

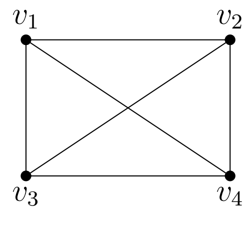


Fig:3-Complete graph K_4

$SD(T_g, x) = x^4 + 4x^3 + 6x^2 + 4x$.
$SD(K_4, x) = x^4 + 4x^3 + 6x^2 + 4x$.
Hence $T_g \cong K_4$ *implies* $SD(T_g, x) = SD(K_4, x)$.

Definition 3.2. *Superior domination polynomial of a star graph S_n is given by $SD(S_n, x) = \sum_{t=\gamma_{sd}(S_n)}^{n} |sd(S_n, t)|x^t$ where $|sd(S_n, t)|$ is the number of distinct superior dominating sets with cardinality t and $\gamma_{sd}(S_n)$ is the superior domination number of a star graph.*

Theorem 3.3. *For a star graph S_n of order n where $n \geq 3$, the following are true.*

1. $|sd(S_n, t)| = |sd(S_{n-1}, t-1)| + |sd(S_{n-1}, t)|$, $t \in Z^+$, $t \leq n$.
2. $SD(S_n, x) = xSD(S_{n-1}, x) + SD(S_{n-1}, x)$.
3. $SD(S_n, x) = x(x+1)^{n-1}$.

Proof.

1. Let $V(S_n) = \{v_1, v_2, \ldots v_n\}$. All the pendant vertices form the superior neighbours of central vertex v_1 since deg$(v_1) = \Delta(S_n) = n-1$. Here we have ${}^{(n-1)}C_{t-1}$ superior dominating sets of cardinality t. Therefore $|sd(S_n, t)| = ^{(n-1)} C_{t-1}$, $|sd(S_{n-1}, t-1)| = ^{(n-2)} C_{t-2}$ and $|sd(S_{n-1}, t)| = ^{(n-2)} C_{t-1}$.
   But ${}^{(n-1)}C_{t-1} = ^{(n-2)} C_{t-2} + ^{(n-2)} C_{t-1}$.
   Therefore $|sd(S_n, t)| = |sd(S_{n-1}, t-1)| + |sd(S_{n-1}, t)|$.

2. By theorem-3.3-(1) we have
   $|sd(S_n, t)| = |sd(S_{n-1}, t-1)| + |sd(S_{n-1}, t)|$.
   When $t = 1$, $|sd(S_n, 1)| = |sd(S_{n-1}, 0)| + |sd(S_{n-1}, 1)|$.
   $\implies x|sd(S_n, 1)| = x|sd(S_{n-1}, 0)| + x|sd(S_{n-1}, 1)|$.
   When $t = 2$, $|sd(S_n, 2)| = |sd(S_{n-1}, 1)| + |sd(S_{n-1}, 2)|$.
   $\implies x^2|sd(S_n, 2)| = x^2|sd(S_{n-1}, 1)| + x^2|sd(S_{n-1}, 2)|$.
   When $t = 3$, $|sd(S_n, 3)| = |sd(S_{n-1}, 2)| + |sd(S_{n-1}, 3)|$.
   $\implies x^3|sd(S_n, 3)| = x^3|sd(S_{n-1}, 2)| + x^3|sd(S_{n-1}, 3)|$.
   When $t = 4$, $|sd(S_n, 4)| = |sd(S_{n-1}, 3)| + |sd(S_{n-1}, 4)|$.
   $\implies x^4|sd(S_n, 4)| = x^4|sd(S_{n-1}, 3)| + x^4|sd(S_{n-1}, 4)|$.
   $\vdots$
   When $t = n-1$, $|sd(S_n, n-1)| = |sd(S_{n-1}, n-2)| + |sd(S_{n-1}, n-1)|$.
   $\implies x^{n-1}|sd(S_n, n-1)| = x^{n-1}|sd(S_{n-1}, n-2)| + x^{n-1}|sd(S_{n-1}, n-1)|$.
   When $t = n$, $|sd(S_n, n)| = |sd(S_{n-1}, n-1)| + |sd(S_{n-1}, n)|$.
   $\implies x^n|sd(S_n, n)| = x^n|sd(S_{n-1}, n-1)| + x^n|sd(S_{n-1}, n)|$.
   Hence $x|sd(S_n, 1)| + x^2|sd(S_n, 2)| + x^3|sd(S_n, 3)| + x^4|sd(S_n, 4)| + \cdots + x^{n-1}|sd(S_n, n-1)| + x^n|sd(S_n, n)| = x|sd(S_{n-1}, 0)| + x|sd(S_{n-1}, 1)| + x^2|sd(S_{n-1}, 1)| + x^2|sd(S_{n-1}, 2)| + x^3|sd(S_{n-1}, 2)| + x^3|sd(S_{n-1}, 3)| + x^4|sd(S_{n-1}, 3)| + x^4|sd(S_{n-1}, 4)| + \cdots + x^{n-1}|sd(S_{n-1}, n-2)| + x^{n-1}|sd(S_{n-1}, n-$

$1)| + x^n|sd(S_{n-1}, n-1)| + x^n|sd(S_{n-1}, n)|$.

$= x|sd(S_{n-1}, 0)| + x^2|sd(S_{n-1}, 1)| + x^3|sd(S_{n-1}, 2)| + x^4|sd(S_{n-1}, 3)| + \cdots + x^{n-1}|sd(S_{n-1}, n-2)| + x^n|sd(S_{n-1}, n-1)| + x|sd(S_{n-1}, 1)| + x^2|sd(S_{n-1}, 2)| + x^3|sd(S_{n-1}, 3)| + x^4|sd(S_{n-1}, 4)| + \cdots + x^{n-1}|sd(S_{n-1}, n-1)| + x^n|sd(S_{n-1}, n)|$.

$= x[x|sd(S_{n-1}, 1)| + x^2|sd(S_{n-1}, 2)| + x^3|sd(S_{n-1}, 3)| + x^4|sd(S_{n-1}, 4)| + \cdots + x^{n-1}|sd(S_{n-1}, n-1)|] + x|sd(S_{n-1}, 1)| + x^2|sd(S_{n-1}, 2)| + x^3|sd(S_{n-1}, 3)| + x^4|sd(S_{n-1}, 4)| + \cdots + x^{n-1}|sd(S_{n-1}, n-1)| + x^n|sd(S_{n-1}, n)|$.
Since $|sd(S_{n-1}, 0)| = |sd(S_{n-1}, n)| = 0$.

$= x\sum_{t=1}^{n-1} |sd(S_{n-1}, t)|x^t + \sum_{t=1}^{n-1} |sd(S_{n-1}, t)|x^t$.
$SD(S_n, x) = x\,SD(S_{n-1}, x) + SD(S_{n-1}, x)$.

3. We prove this by mathematical induction.
When $n = 3$,

$$\begin{aligned} SD(S_n, x) &= x(x+1)^{n-1} \\ &= x(x+1)^{3-1} \\ &= x(x+1)^2 \end{aligned}$$

The result is true for $n = 3$.
When $n = 4$,

$$SD(S_n, x) = x(x+1)^3$$

The result is true for $n = 4$.
Assume the result is true for all natural numbers less than n.

$$\begin{aligned} SD(S_{n-1}, x) &= x(x+1)^{(n-1)-1} \\ &= x(x+1)^{n-2} \end{aligned}$$

Now we prove the result for n.

$$\begin{aligned} SD(S_n, x) &= x\,SD(S_{n-1}, x) + SD(S_{n-1}, x) \quad \textit{using theorem3.3-(2)} \\ &= x[x(x+1)^{n-2}] + x(x+1)^{n-2} \\ &= x(x+1)^{n-2}[x+1] \\ &= x(x+1)^{n-2+1} \\ &= x(x+1)^{n-1} \end{aligned}$$

$\therefore$ The result is true for all n.

Table: $|sd(S_n, t)|$ is the number of superior dominating sets of S_n with cardinality t where $1 \le t \le 15$.

n \ t	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	0														
2	2	1													
3	1	2	1												
4	1	3	3	1											
5	1	4	6	4	1										
6	1	5	10	10	5	1									
7	1	6	15	20	15	6	1								
8	1	7	21	35	35	21	7	1							
9	1	8	28	56	70	56	28	8	1						
10	1	9	36	84	126	126	84	36	9	1					
11	1	10	45	120	210	252	210	120	45	10	1				
12	1	11	55	165	330	462	462	330	165	55	11	1			
13	1	12	66	220	495	792	924	792	495	220	66	12	1		
14	1	13	78	286	715	1287	1716	1716	1287	715	286	78	13	1	
15	1	14	91	364	1001	2002	3003	3423	3003	2002	1001	364	91	14	1

Theorem 3.4. *The following properties for the co-efficients of $SD(S_n, x)$ holds.*

1. $|sd(S_n, 1)| = 1$ *for all* $n > 2$.
2. $|sd(S_n, n)| = 1$ *for all* $n \geq 2$.
3. $|sd(S_n, n-1)| = n - 1$ *for all* $n > 2$.
4. $|sd(S_n, n-2)| = \dfrac{(n-1)(n-2)}{2}$ *for all* $n \geq 3$.
5. $|sd(S_n, n-3)| = \dfrac{(n-1)(n-2)(n-3)}{6}$ *for all* $n \geq 4$.
6. $|sd(S_n, n-4)| = \dfrac{(n-1)(n-2)(n-3)(n-4)}{24}$ *for all* $n \geq 5$.
7. $|sd(S_n, t)| = |sd(S_n, n-t+1)|$ *for all* $n \geq 3$.
8. *If* $SD_n = \sum_{t=1}^{n} |sd(S_n, t)|$ *for all* $n \geq 3$ *then* $SD_n = 2(SD_{n-1})$ *with initial condition* $SD_3 = 4$.
9. SD_n *=Total number of superior dominating sets in* $S_n = 2^{n-1}$ *for all* $n \geq 3$.

Proof.

1. Let $V(S_n) = \{v_1, v_2, \ldots v_n\}$. In a star graph S_n all the vertices form a superior neighbour of central vertex v_1 except itself. Therefore the only set with single cardinality $D = \{v_1\}$ forms the superior dominating set of every star graph S_n where $n > 2$. Therefore $|sd(S_n, 1)| = 1$ for all $n > 2$.

2. The whole set of vertices $V(S_n)$ forms the superior dominating set $|sd(S_n, n)| = 1$ for all $n \geq 2$.

3. By mathematical induction on n. The result is true for $n = 3$,
since $|sd(S_3, 3-1)| = |sd(S_3, 2)| = 2$.
Assume the result is true for all natural numbers less than n.
Now we prove it for n ie, $|sd(S_{n-1}, n-2)| = n-2$.
By theorem-3.3-(1) and 3.4-(2)

$$\begin{aligned} |sd(S_n, n-1)| &= |sd(S_{n-1}, n-2)| + |sd(S_{n-1}, n-1)| \\ &= (n-2) + 1 \\ &= n-1 \end{aligned}$$

$\therefore$ The result is true for all n.

4. By mathematical induction on n.
For $n = 3$, $|sd(S_3, 1)| = 1$.
For $n = 4$, $|sd(S_4, 2)| = 3$.
Assume the result is true for all natural numbers less than n,
ie, for $n = n-1$, $|sd(S_{n-1}, n-3)| = \frac{(n-2)(n-3)}{2}$
Now we prove it for n. By theorem-3.3-(1) and 3.4-(3),

$$\begin{aligned} |sd(S_n, n-2)| &= |sd(S_{n-1}, n-3)| + |sd(S_{n-1}, n-2)| \\ &= \frac{(n-2)(n-3)}{2} + (n-2) \\ &= \frac{(n-2)(n-3) + 2(n-2)}{2} \\ &= \frac{(n-2)(n-3+2)}{2} \\ &= \frac{(n-2)(n-1)}{2} \\ &= \frac{(n-1)(n-2)}{2} \end{aligned}$$

$\therefore$ The result is true for all n.

5. By mathematical induction on n.
For $n = 4$, $|sd(S_4, 1)| = 1$.
For $n = 5$, $|sd(S_5, 2)| = 4$.
Assume the result is true for all natural numbers less than n.
For $n = n-1$, $|sd(S_{n-1}, n-4)| = \frac{(n-2)(n-3)(n-4)}{6}$

Now we prove it for n. By theorem-3.3-(1) and 3.4-(4),

$$\begin{aligned} |sd(S_n, n-3)| &= |sd(S_{n-1}, n-4)| + |sd(S_{n-1}, n-3)| \\ &= \frac{(n-2)(n-3)(n-4)}{6} + \frac{(n-2)(n-3)}{2} \\ &= \frac{(n-2)(n-3)(n-4+3)}{6} \\ &= \frac{(n-1)(n-2)(n-3)}{6} \end{aligned}$$

$\therefore$ The result is true for all n.

6. By mathematical induction on n.
   The result is true for $n = 5$ since $|sd(S_5, 1)| = 1$
   For $n = 6$, $|sd(S_6, 2)| = 5$.
   Assume the result is true for all natural numbers less than n.
   For $n = n-1$, $|sd(S_{n-1}, n-5)| = \frac{(n-2)(n-3)(n-4)(n-5)}{24}$
   Now we prove it for n. By theorem-3.3-(1) and 3.4-(5),

$$\begin{aligned} |sd(S_n, n-4)| &= |sd(S_{n-1}, n-5)| + |sd(S_{n-1}, n-4)| \\ &= \frac{(n-2)(n-3)(n-4)(n-5)}{24} + \frac{(n-2)(n-3)(n-4)}{6} \\ &= \frac{(n-2)(n-3)(n-4)(n-5+4)}{24} \\ &= \frac{(n-1)(n-2)(n-3)(n-4)}{24} \end{aligned}$$

   $\therefore$ The result is true for all n.

7. By mathematical induction on n.
   The result is true for $n = 3$ since $|sd(S_3, 1)| = |sd(S_3, 3-1+1)| = |sd(S_3, 3)| = 1$.
   For $n = 4, t = 2$, $|sd(S_4, 2)| = |sd(S_4, 4-2+1)| = |sd(S_4, 3)| = 3$.
   Assume the result is true for all natural numbers less than n.
   For $n = n-1$, $|sd(S_{n-1}, t-1)| = |sd(S_{n-1}, (n-t+1))|$
   Now we prove it for n. By theorem-3.3 we have,

$$\begin{aligned} |sd(S_n, t)| &= |sd(S_{n-1}, t-1)| + |sd(S_{n-1}, t)| \\ &= |sd(S_{n-1}, (n-1-(t-1)+1))| + |sd(S_{n-1}, (n-1-(t)+1))| \\ &= |sd(S_{n-1}, (n-1-t+1+1))| + |sd(S_{n-1}, (n-1-t+1))| \\ &= |sd(S_{n-1}, (n-t+1))| + |sd(S_{n-1}, (n-t))| \\ &= |sd(S_n, (n-t+1))| \end{aligned}$$

   $\therefore$ The result is true for all n.

8. $SD_n = \sum_{i=1}^{n} |sd(S_n, t)|$
   By theorem-3.3 we have

$$\begin{aligned} SD_n &= \sum_{i=1}^{n}[|sd(S_{n-1}, t-1)| + |sd(S_{n-1}, t)|] \\ &= \sum_{i=1}^{n-1}|sd(S_{n-1}, t)| + \sum_{i=1}^{n-1}|sd(S_{n-1}, t)| \\ &= SD_{n-1} + SD_{n-1} \\ SD_n &= 2[SD_{n-1}] \end{aligned}$$

9. By mathematical induction on n.
   When $n = 3$,
   $SD_3 = 2^{3-1} = 2^2 = 4$.
   $SD_4 = 2^{4-1} = 2^3 = 8$.
   Assume the result is true for all natural numbers less than n. Now we prove it for n. Therefore $SD_{n-1} = 2^{n-1-1} = 2^{n-2}$
   Now

$$\begin{aligned} SD_n &= 2[SD_{n-1}] \text{ from theorem-3.4-(8)} \\ &= 2[2^{n-2}] \\ &= 2^{n-2+1} \\ &= 2^{n-1} \end{aligned}$$

   $\therefore$ The result is true for all n.
   Hence the theorem.

4 Superior domination polynomial of cycle

Let $sd(C_n, m)$ be the superior dominating set of cycle C_n with cardinality m. K.M. Kathiresan and G. Marimuttu[6] proved theorem-2.1, for our convenience we reframe theorem-2.1 as $\gamma_{sd}(C_n) = \lceil \frac{n}{3} \rceil \ \forall \ n$. Hereafter we denote the vertex set $V(G) = \{v_1, v_2, \ldots v_n\} = [n]$.

Lemma 4.1. *For a cycle C_n, $sd(C_n, m) = \emptyset$ if $m > n$ or $m < \lceil \frac{n}{3} \rceil$*

Proof. Let C_n be a cycle, a superior dominating set D has the minimum cardinality among the minimum superior dominating set with cardinality $\lceil \frac{n}{3} \rceil$ by theorem-2.1. Therefore there is no proper subset of D which forms a superior dominating set. Hence

$$sd(C_n, m) = \emptyset \ \ where \ \ m < \lceil \frac{n}{3} \rceil = |D| \tag{1}$$

There can not exists a superior dominating set greater than the order of the graph. Therefore

$$sd(C_n, m) = \emptyset \ \ if \ \ m > n \tag{2}$$

From equation-(1) and (2) we obtain the result.

Observation 4.1. *If a cycle C_n contains a maximal simple path of length $3k-1$, $3k$ or $3k+1$ then every dominating set of C_n must contain at least k, $k+1$ or $k+1$ vertices respectively.*

Lemma 4.2. *Let L be a subset of the vertex set, $L \subseteq [n]$. If L is in $sd(C_{n-4}, m-1)$ or $sd(C_{n-5}, m-1) \ni L \cup \{v\} \in sd(C_n, m)$ for $v \in [n]$ then $L \in sd(C_{n-3}, m-1)$.*

Proof. Let $L \in sd(C_{n-4}, m-1)$ and $L \cup \{v\} \in sd(C_n, m)$ for $v \in [n]$ then by lemma-4.3 we consider $\{1, n-4\}, \{2, n-4\}$ and $\{1, n-5\}$ as a subset of L. Then $L \in sd(C_{n-3}, m-1)$ suppose $L \in sd(C_{n-5}, m-1)$ and $L \cup \{v\} \in sd(C_n, m)$ for $v \in [n]$. Then by lemma-4.3 $\{1, n-5\}$ must be a subset of L. Hence $L \in sd(C_{n-3}, m-1)$.

Lemma 4.3. .

1. *If $sd(C_{n-1}, m-1) = sd(C_{n-3}, m-1) = \emptyset$ then $sd(C_{n-2}, m-1) = \emptyset$.*
2. *If $sd(C_{n-1}, m-1) \neq \emptyset$ and $sd(C_{n-3}, m-1) \neq \emptyset$ then $sd(C_{n-2}, m-1) \neq \emptyset$.*
3. *If $sd(C_{n-1}, m-1) = sd(C_{n-2}, m-1) = sd(C_{n-3}, m-1) = \emptyset$ then $sd(C_n, m) = \emptyset$.*

Proof.

1. Since $sd(C_{n-1}, m-1) = sd(C_{n-3}, m-1) = \emptyset$, by lemma-4.1, $m-1 > n-1$ or $m-1 < \lceil \frac{n-3}{3} \rceil$. In both cases we have $sd(C_{n-2}, m-1) = \emptyset$.
2. Suppose that $sd(C_{n-2}, m-1) = \emptyset$ by lemma-4.1, $m-1 > n-2$ or $m-1 < \lceil \frac{n-2}{3} \rceil$. If $m-1 > n-2$ then $m-1 > n-3$. Hence $sd(C_{n-3}, m-1) = \emptyset$, a contradiction. Hence $m-1 < \lceil \frac{n-2}{3} \rceil$. So $m-1 < \lceil \frac{n-1}{3} \rceil$, $sd(C_{n-1}, m-1) = \emptyset$, also a contradiction.
3. Suppose $sd(C_n, m) \neq \emptyset$. Let $L \in sd(C_n, m)$ such that at least one vertex labelled as v_n or v_{n-1} is in L. If $v_n \in L$, then by observation-4.1 at least one vertex labelled as v_{n-1}, v_{n-2} or v_{n-3} is in L. If $v_{n-1} \in L$ or $v_{n-2} \in L$, then $L - \{v_n\} \in sd(C_{n-1}, m-1)$, a contradiction. If $v_{n-3} \in L$, then $L - \{v_n\} \in sd(C_{n-2}, m-1)$ a contradiction. Now suppose that $v_{n-1} \in L$. Then by observation-4.1 at least one vertex labelled v_{n-2}, v_{n-3} or v_{n-4} is in L. If $v_{n-2} \in L$ or $v_{n-3} \in L$, then $L - \{v_{n-1}\} \in sd(C_{n-2}, m-1)$, a contradiction. If $v_{n-4} \in L$ then $L - \{v_{n-1}\} \in sd(C_{n-3}, m-1)$, a contradiction. Therefore $sd(C_n, m) = \emptyset$.

Lemma 4.4. *Suppose that $sd(C_n, m) \neq \emptyset$ then we have*

1. $sd(C_{n-1}, m-1) = sd(C_{n-2}, m-1) = \emptyset$ *and* $sd(C_{n-3}, m-1) \neq \emptyset$ *if and only if* $n = 3k$ *and* $m = k$ *for some* $k \in N$.

2. $sd(C_{n-2}, m-1) = sd(C_{n-3}, m-1) = \emptyset$ *and* $sd(C_{n-1}, m-1) \neq \emptyset$ *if and only if* $m = n$.

3. $sd(C_{n-1}, m-1) = \emptyset$, $sd(C_{n-2}, m-1) \neq \emptyset$ *and* $sd(C_{n-3}, m-1) \neq \emptyset$ *if and only if* $n = 3k+2$ *and* $m = \lceil \frac{3k+2}{3} \rceil$ *for some* $k \in N$.

4. $sd(C_{n-1}, m-1) \neq \emptyset$, $sd(C_{n-2}, m-1) \neq \emptyset$ *and* $sd(C_{n-3}, m-1) = \emptyset$ *if and only if* $m = n-1$.

5. $sd(C_{n-1}, m-1) \neq \emptyset$, $sd(C_{n-2}, m-1) \neq \emptyset$ *and* $sd(C_{n-3}, m-1) \neq \emptyset$ *if and only if* $\lceil \frac{n-1}{3} \rceil + 1 \leq m \leq n-2$.

Proof.

1. Since $sd(C_{n-1}, m-1) = sd(C_{n-2}, m-1) = \emptyset$. By lemma-4.1 $m-1 > n-1$ or $m-1 < \lceil \frac{n-2}{3} \rceil$. If $m-1 > n-1$ then $m > n$ by lemma-4.1 $sd(C_n, m) = \emptyset$ a contradiction. Therefore $m < \lceil \frac{n-2}{3} \rceil + 1$ since $sd(C_n, m) \neq \emptyset$ together we have $\lceil \frac{n}{3} \rceil \leq m \leq \lceil \frac{n-2}{3} \rceil + 1$. Hence $n = 3k$ and $m = k$ for $k \in N$.
   Conversely suppose if $n = 3k$, $m = k$ for $k \in N$ then by lemma-4.1 $sd(C_{n-1}, m-1) = sd(C_{n-2}, m-1) = \emptyset$ and $sd(C_{n-3}, m-1) \neq \emptyset$.

2. Since $sd(C_{n-2}, m-1) = sd(C_{n-3}, m-1) = \emptyset$ by lemma-4.1 $m-1 > n-2$ or $m-1 < \lceil \frac{n-3}{3} \rceil$. If $m-1 < \lceil \frac{n-3}{3} \rceil$ then $m-1 < \lceil \frac{n-1}{3} \rceil$. Hence $sd(C_{n-1}, m-1) = \emptyset$ a contradiction. So we have $m > n-1$ also since $sd(C_{n-1}, m-1) \neq \emptyset$ we have $m-1 \leq n-1$. Hence $m = n$.
   Conversely suppose if $m = n$ then by lemma-4.1 we have $sd(C_{n-2}, m-1) = sd(C_{n-3}, m-1) = \emptyset$ and $sd(C_{n-1}, m-1) \neq \emptyset$.

3. Since $sd(C_{n-1}, m-1) = \emptyset$ by lemma-4.1 $m-1 > n-1$ or $m-1 < \lceil \frac{n-1}{3} \rceil$. If $m-1 > n-1$ then $m-1 > n-2$ and by lemma-4.1 $sd(C_{n-2}, m-1) = sd(C_{n-3}, m-1) = \emptyset$ a contradiction so we must have $m < \lceil \frac{n-1}{3} \rceil + 1$. But also we have $m-1 \geq \lceil \frac{n-2}{3} \rceil$ because $sd(C_{n-2}, m-1) \neq \emptyset$. Hence we have $\lceil \frac{n-2}{3} \rceil + 1 \leq m < \lceil \frac{n-1}{3} \rceil + 1$. Therefore $n = 3k+2$ and $m = k+1 = \lceil \frac{3k+2}{3} \rceil$ for some $k \in N$.
   Conversely suppose if $n = 3k+2$, $m = \lceil \frac{3k+2}{3} \rceil$ for some $k \in N$ then by lemma-4.1 $sd(C_{n-1}, m-1) = sd(C_{3k+1}, k) = \emptyset$, $sd(C_{n-2}, m-1) \neq \emptyset$ and $sd(C_{n-3}, m-1) \neq \emptyset$.

4. Since $sd(C_{n-3}, m-1) = \emptyset$ by lemma-4.1 we have $m-1 > n-3$ or $m-1 < \lceil \frac{n-3}{3} \rceil$ since $sd(C_{n-2}, m-1) \neq \emptyset$ by lemma-4.1 we have $\lceil \frac{n-2}{3} \rceil + 1 \leq m \leq n-1$. Therefore $m-1 < \lceil \frac{n-3}{3} \rceil$ is not possible. Hence we must have $m-1 > n-3$. Then $m = n-1$ or n but $m \neq n$ as $sd(C_{n-2}, m-1) \neq \emptyset$. Therefore $m = n-1$.
   Conversely suppose if $m = n-1$ then by lemma-4.1 $sd(C_{n-1}, m-1) \neq \emptyset$ $sd(C_{n-2}, m-1) \neq \emptyset$ and $sd(C_{n-3}, m-1) = \emptyset$.

5. Since $sd(C_{n-1}, m-1) \neq \emptyset$, $sd(C_{n-2}, m-1) \neq \emptyset$ and $sd(C_{n-3}, m-1) \neq \emptyset$ then by applying lemma-4.1 we have $\lceil \frac{n-1}{3} \rceil \leq m-1 \leq n-1$, $\lceil \frac{n-2}{3} \rceil \leq m-1 \leq n-2$ and $\lceil \frac{n-3}{3} \rceil \leq m-1 \leq n-3$ so $\lceil \frac{n-1}{3} \rceil \leq m-1 \leq n-3$ and hence $\lceil \frac{n-1}{3} \rceil + 1 \leq m \leq n-2$.
   Conversely suppose if $\lceil \frac{n-1}{3} \rceil + 1 \leq m \leq n-2$ then by lemma-4.1 we have the result.

Theorem 4.1. *For every $n \geq 4$ and $m \geq \lceil \frac{n}{3} \rceil$,*

1. *If $sd(C_{n-1}, m-1) = sd(C_{n-2}, m-1) = \emptyset$ and $sd(C_{n-3}, m-1) \neq \emptyset$ then $sd(C_n, m) = sd(C_n, \frac{n}{3}) = \{\{1, 4, \ldots n-2\}, \{2, 5, \ldots n-1\}, \{3, 6, \ldots n\}\}$.*

2. *If $sd(C_{n-2}, m-1) = sd(C_{n-3}, m-1) = \emptyset$ and $sd(C_{n-1}, m-1) \neq \emptyset$ then $sd(C_n, m) = sd(C_n, n) = \{[n]\}$.*

3. *If $sd(C_{n-1}, m-1) = \emptyset$, $sd(C_{n-2}, m-1) \neq \emptyset$ and $sd(C_{n-3}, m-1) \neq \emptyset$ then*
   $sd(C_n, m) = \{\{1, 4, \ldots n-4, n-1\}, \{2, 5, \ldots n-3, n\}, \{3, 6, \ldots n-2, n\}\} \bigcup$
   $$\{S \bigcup \begin{cases} \{n-2\}, \text{ if } 1 \in S \\ \{n-1\}, \text{ if } 1 \notin S, 2 \in S \mid S \in sd(C_{n-3}, m-1) \\ \{n\}, \text{otherwise} \end{cases}$$
   where $S \subseteq V(C_n)$.

4. *If $sd(C_{n-3}, m-1) = \emptyset$, $sd(C_{n-2}, m-1) \neq \emptyset$ and $sd(C_{n-1}, m-1) \neq \emptyset$ then $sd(C_n, m) = \{[n] - \{P\} | P \in [n]\}$*

5. *If $sd(C_{n-1}, m-1) \neq \emptyset$, $sd(C_{n-2}, m-1) \neq \emptyset$ and $sd(C_{n-3}, m-1) \neq \emptyset$ then*
   $sd(C_n, m) = \{\{n\} \bigcup S \mid S \in sd(C_{n-1}, m-1) \bigcup$
   $$\{S_1 \begin{cases} \{n\}, \text{ if } n-2 \text{ or } n-3 \in S_1 \text{ for } S_1 \in sd(C_{n-2}, m-1) \text{ or } sd(C_{n-1}, m-1) \\ \{n-1\}, \text{ if } n-2 \notin S_1, n-3 \notin S_1 \text{ or } S_1 \in sd(C_{n-1}, m-1) \cap sd(C_{n-2}, m-1) \end{cases} \Bigg\} \bigcup$$

$$\{S_2 \begin{cases} \{n-2\}, \ \textit{if } 1 \in S_2 \textit{ for } S_2 \in sd(C_{n-3}, m-1) \textit{ or } S_2 \in sd(C_{n-3}, m-1) \cap sd(C_{n-2}, m-1) \\ \{n-1\}, \ \textit{if } n-3 \in S_2 \textit{ or } n-4 \in S_2 \textit{ for } S_2 \in sd(C_{n-3}, m-1) \textit{ or } sd(C_{n-2}, m-1) \end{cases} \}$$

where S, S_1 and S_2 are subsets of $V(C_n)$.

Proof.

1. $sd(C_{n-1}, m-1) = sd(C_{n-2}, m-1) = \emptyset$ by lemma-4.4-(1) $n = 3k$, $m = k$ for some $k \in N$. Hence $sd(C_n, m) = sd(C_n, \frac{n}{3}) = \{\{1, 4, 7, \ldots n-2\}, \{2, 5, 8, \ldots n-1\}, \{3, 6, 9, \ldots n\}\}$.

2. $sd(C_{n-2}, m-1) = sd(C_{n-3}, m-1) = \emptyset$ and $sd(C_{n-1}, m-1) \neq \emptyset$ by lemma-4.4-(2) $m = n$. Therefore $sd(C_n, m) = (C_n, n) = \{[n]\}$.

3. $sd(C_{n-1}, m-1) = \emptyset$, $sd(C_{n-2}, m-1) \neq \emptyset$ and $sd(C_{n-3}, m-1) \neq \emptyset$ by lemma-4.4-(3) $n = 3k+2$, $m = k+1$ for some $k \in N$ we denote the families
$\{\{1, 4, \ldots 3k-2, 3k+1\}, \{2, 5, \ldots 3k-1, 3k+2\}, \{3, 6, \ldots 3k, 3k+3\}\}$
and
$$\{S \bigcup \begin{cases} \{3k\}, \ \textit{if } 1 \in S \\ \{3k+1\}, \ \textit{if } 1 \notin S, 2 \in S \,|\, S \in sd(C_{3k-1}, k) \\ \{3k+2\}, \textit{otherwise} \end{cases}$$
by L_1 and L_2 respectively. We shall prove that $sd(C_{3k+2,k+1}) = L_1 \cup L_2$. Since $sd(C_k, 3k) = \{\{1, 4, 7, \ldots 3k-2\}, \{2, 5, 8, \ldots 3k-1\}, \{3, 6, 9, \ldots 3k\}\}$ then $L_1 \subseteq sd(C_{3k+2}, k+1)$. Also it is obvious that $L_2 \subseteq sd(C_{3k+2}, k+1)$. Hence $L_1 \cup L_2 \subseteq sd(C_{3k+2}, k+1)$.

   Now let $L \in sd(C_{3k+2}, k+1)$ then by observation-4.1 at least one of the vertices labelled $3k+2, 3k+1$ or $3k$ is in L. Suppose that $3k+2 \in L$ then by observation-4.1 at least one of vertices say $1, 2, 3, 3k+1, 3k$ or $3k-1$ are in L. If $3k+1$ and at least one of $\{1, 2, 3\}$ and also $3k$ and at least one of $\{1, 2\}$ are in L. Then $L - \{3k+2\} \in sd(C_{3k+1}, k)$ a contradiction. If $\{3, 3k\}$ or $2, 3k-1$ is a subset of L, then $L = S \cup \{3k+2\}$ for some $S \in sd(C_{3k}, k)$ therefore $L \in L_1$ if $\{1, 3k-1\}$ is a subset of L then $L - \{3k+2\} \in sd(C_{3k+1}, k)$ a contradiction. If $\{3, 3k-1\}$ is a subset of L and $\{3k, 3k+1\}$ is not a subset of L then $L - \{3k+2\} \in sd(C_{3k-1}, k)$ hence $L \in L_2$. If $3k+1$ or $3k$ is in L we have the result.

4. By lemma-4.4-(4) $m = n-1$ therefore $sd(C_n, m) = sd(C_n, n-1) = \{[n] - \{x\} | x \in [n]\}$.

5. $sd(C_{n-1}, m-1) \neq \emptyset$, $sd(C_{n-2}, m-1) \neq \emptyset$ and $sd(C_{n-3}, m-1) \neq \emptyset$. First suppose that $S \in sd(C_{n-1}, m-1)$ then $S \cup \{n\} \in sd(C_n, m)$. So

$L_1 = \{\{n\} \cup S | S \in sd(C_{n-1}, m-1)\} \subseteq sd(C_n, m)$. Now suppose that $sd(C_{n-2}, m-1) \neq \emptyset$. Let $S_1 \in sd(C_{n-2}, m-1)$ we denote
$\{S_1 \bigcup \begin{cases} \{n\}, \text{ if } n-2 \text{ or } n-3 \in S_1 \text{ for } S_1 \in sd(C_{n-2}, m-1) \text{ or } sd(C_{n-1}, m-1) \\ \{n-1\}, \text{ if } n-2 \notin S_1, n-3 \notin S_1 \text{ or } S_1 \in sd(C_{n-1}, m-1) \cap sd(C_{n-2}, m-1) \end{cases}$
simply by L_2 by observation-4.1 at least one vertices labeled $n-3, n-2$ or 1 is in S_1 if $n-2$ or $n-3$ is in S_1. Then $S_1 \cup \{n\} \in sd(C_n, m)$ otherwise $S_1 \cup \{n-1\} \in sd(C_n, m)$. Hence $L_2 \subseteq sd(C_n, m)$ there we shall consider $sd(C_{n-3}, m-1) \neq \emptyset$. Let $S_2 \in sd(C_{n-3}, m-1)$ we denote
$\{S_2 \bigcup \begin{cases} \{n-2\}, \text{ if } 1 \in S_2 \text{ for } S_2 \in sd(C_{n-3}, m-1) \text{ or } S_2 \in sd(C_{n-3}, m-1) \cap sd(C_{n-2}, m-1) \\ \{\{n-1\}, \text{ if } n-3 \in S_2 \text{ or } n-4 \in S_2 \text{ for } S_2 \in sd(C_{n-3}, m-1) \text{ or } sd(C_{n-2}, m-1) \end{cases}$
Simply by L_3. If $n-3$ or $n-4$ is in S then $S \cup \{n-1\} \in sd(C_n, m)$, otherwise $S_2 \cup \{n-2\} \in sd(C_n, m)$. Hence $L_3 \subseteq L$. Therefore we proved that $L_1 \cup L_2 \cup L_3 \subseteq sd(C_n, m)$.
Now suppose that $L \in sd(C_n, m)$ so by observation-4.1 L contains at least one of the vertices say $n, n-1$ or $n-2$. If $n \in L$ so by observation-4.1 at least one of the vertices labelled $n-1, n-2$ or $n-3$ and $1, 2,$ or 3 in L. If $n-2 \in L$ or $n-3 \in L$ then $L = S \cup \{n\}$ for some $S \in sd(C_{n-2}, m-1)$. Hence $L \in L_2$ otherwise $L = S \cup \{n-1\}$ sor some $S \in sd(C_{n-2}, m-1)$. Hence $L \in L_2$. If $n-1$ or $n-2$ is in L, we have the result.

Theorem 4.2. *If $sd(C_n, m)$ is the family of superior dominating sets of C_n with cardinality m then* $|sd(C_n, m)| = |sd(C_{n-1}, m-1)| + |sd(C_{n-2}, m-1)| + |sd(C_{n-3}, m-1)|$

Proof. We consider the five cases in theorem-4.1 we write theorem-4.1 in the following form.

1. If $sd(C_{n-1}, m-1) = sd(C_{n-2}, m-1) = \emptyset$ and $sd(C_{n-3}, m-1) \neq \emptyset$ then $sd(C_n, m) = \{\{n-2\} \bigcup S_1, \{n-1\} \bigcup S_2, \{n\} \bigcup S_3 | 1 \in S_1, 2 \in S_2, 3 \in S_3, S_1, S_2, S_3 \in sd(C_{n-3}, m-1)\}$.

2. If $sd(C_{n-2}, m-1) = sd(C_{n-3}, m-1) = \emptyset$ and $sd(C_{n-1}, m-1) \neq \emptyset$ then $sd(C_n, m) = \{\{n\} \bigcup S | S \in sd(C_{n-1}, m-1)\}$.

3. If $sd(C_{n-1}, m-1) = \emptyset$, $sd(C_{n-2}, m-1) \neq \emptyset$ and $sd(C_{n-3}, m-1) \neq \emptyset$ then
$sd(C_n, m) = \{\{n\} \bigcup S_1, \{n-1\} \bigcup S_2$ or $S_1, S_2 \in sd(C_{n-2}, m-1), 1 \in S_2\} \bigcup$
$$(S \bigcup \begin{cases} \{n-2\}, \text{ if } 1 \in S \\ \{n-1\}, \text{ if } 1 \in S, 2 \in S \\ \{n\}, \text{otherwise} \end{cases}) \text{ where } S \in sd(C_{n-3}, m-1)$$

4. If $sd(C_{n-3}, m-1) = \emptyset$ and $sd(C_{n-2}, m-1) \neq \emptyset$, $sd(C_{n-1}, m-1) \neq \emptyset$ then $sd(C_n, m) = \{\{n\} \bigcup S_1, \{n-1\} \bigcup S_2$ or $S_1 \in sd(C_{n-1}, m-1), S_2 \in sd(C_{n-2}, m-1)\}$.

5. If $sd(C_{n-1}, m-1) \neq \emptyset$, $sd(C_{n-2}, m-1) \neq \emptyset$ and $sd(C_{n-3}, m-1) \neq \emptyset$ then $sd(C_n, m) = \{\{n\} \bigcup S | S \in sd(C_{n-1}, m-1)\} \bigcup$
$\{S_1 \bigcup \begin{cases} \{n\}, \text{ if } n-2 \text{ or } n-3 \in S_1 \text{ for } S_1 \in sd(C_{n-2}, m-1) \text{ or } sd(C_{n-2,m-1}) \\ \{n-1\}, \text{ if } n-2 \notin S_1, n-3 \notin S_1 \text{ or } S_1 \in sd(C_{n-1}, m-1) \bigcap sd(C_{n-2}, m-1) \end{cases}\} \bigcup$
$\{S_2 \bigcup \begin{cases} \{n-2\}, \text{ if } 1 \in S_2 \text{ for } S_2 \in sd(C_{n-3}, m-1) \text{ or } S_2 \in sd(C_{n-3}, m-1) \bigcap sd(C_{n-2}, m-1) \\ \{n-2\}, \text{ if } n-3 \in S_2 \text{ or } n-4 \in S_2 \text{ for } S_2 \in sd(C_{n-3}, m-1) \text{ or } sd(C_{n-2}, m-1) \end{cases}\}$
where $S_1 \in sd(C_{n-2}, m-1)$ or $sd(C_{n-1}, m-1)$ and $S_2 \in \{sd(C_{n-3}, m-1)$ or $sd(C_{n-2}, m-1)\} \bigcap sd(C_{n-1}, m-1)$. Hence we have $|sd(C_n, m)| = |sd(C_{n-1}, m-1)| + |sd(C_{n-2}, m-1)| + |sd(C_{n-3}, m-1)|$.

Definition 4.1. *Let $sd(C_n, m)$ be the family of superior dominating sets of a cycle C_n with cardinality n. Then the superior domination polynomial $SD(C_n, x)$ of C_n is defined as $SD(C_n, x) = \sum_{m=\lceil \frac{n}{3} \rceil}^{n} |sd(C_n, m)| x^m$ where $sd(C_n, m)$ is the number of distinct superior dominating sets of same cardinality.*

Theorem 4.3. *For every $n \geq 4$ $SD(C_n, x) = x[SD(C_{n-1}, x) + SD(C_{n-2}, x) + SD(C_{n-3}, x)]$ with initial values $SD(C_1, x) = x$, $SD(C_2, x) = x^2 + 2x$, $SD(C_3, x) = x^3 + 3x^2 + x$.*

Table: The co-efficients of $SD(C_n, x)$ for $1 \leq n \leq 16$

n \ t	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	0															
2	0	0														
3	3	3	1													
4	0	6	4	1												
5	0	5	10	5	1											
6	0	3	14	15	6	1										
7	0	0	14	28	21	7	1									
8	0	0	8	38	48	28	8	1								
9	0	0	3	36	81	75	36	9	1							
10	0	0	0	25	102	150	110	45	10	1						
11	0	0	0	11	99	231	253	154	55	11	1					
12	0	0	0	3	72	282	456	399	208	66	12	1				
13	0	0	0	0	39	273	663	819	598	273	78	13	1			
14	0	0	0	0	14	210	786	1372	1372	861	350	91	14	1		
15	0	0	0	0	3	125	765	1905	2590	2178	1200	440	105	15	1	
16	0	0	0	0	0	56	608	2214	4096	4560	3312	1628	544	120	16	1

Theorem 4.4. *The following properties hold for co-efficients of $SD(C_n, x)$.*

1. $|sd(C_{3n}, n)| = 3$, $\forall\ n \in N$.

2. $|sd(C_n, m)| = |sd(C_{n-1}, m-1)| + |sd(C_{n-2}, m-1)| + |sd(C_{n-3}, m-1)|$, $\forall\ n \geq 4$, $m \geq \lceil \frac{n}{3} \rceil$,

3. *$|sd(C_{3n+2}, n+1)| = 3n+2$, $\forall\ n \in N$.*
4. *$|sd(C_{3n+1}, n+1)| = \frac{n(3n+7)+2}{2}$, $\forall\ n \in N$.*
5. *$|sd(C_n, m)| = 1$, $\forall\ n \geq 3$.*
6. *$|sd(C_n, n-1)| = n$, $\forall\ n \geq 3$.*
7. *$|sd(C_n, n-2)| = \frac{(n-1)n}{2}$, $\forall\ n \geq 3$.*
8. *$|sd(C_n, n-3)| = \frac{(n-4)(n)(n+1)}{6}$, $\forall\ n \geq 4$.*
9. *$\sum_{n=m}^{3m} |sd(C_n, m)| = 3\sum_{n=m-1}^{3m-3} |sd(C_n, m-1)|$, $\forall\ m \geq 4$.*
10. *$1 = |sd(C_n, n)| < |sd(C_{n+1}, n)| < |sd(C_{n+2}, n)| < \cdots < |sd(C_{2n-1}, n)| < |sd(C_{2n}, n)| > |sd(C_{2n+1}, n)| > \cdots > |sd(C_{3n-1}, n)| > |sd(C_{3n}, n)| = 3$, $\forall\ n \geq 3$.*
11. *If $A_n = \sum_{m=\lceil \frac{n}{3} \rceil}^{n} |sd(C_n, m)|$ then for every $n \geq 4$, $A_n = A_{n-1} + A_{n-2} + A_{n-3}$ with initial values $A_1 = 1$, $A_2 = 3$ and $A_3 = 7$.*

Proof.

1. Since $|sd(C_n, 3n)| = \{\{1, 4, 7, \ldots 3n-2\}, \{2, 5, 8, \ldots 3n-1\}, \{3, 6, 9, \ldots 3n\}\}$, so $|sd(C_{3n}, n)| = 3$.
2. It follows from theorem-4.2.
3. By induction on n, the result is true for $n = 1$, because $|sd(C_2, 5)| = \{\{1,3\}, \{1,4\}, \{2,4\}, \{2,5\}, \{3,5\}\}$. Suppose result is true for all $n-1$ then we prove for n. By (1),(2) and induction we have

$$\begin{aligned} |sd(C_{3n+2}, n+1)| &= |sd(C_{3n+1}, n)| + |sd(C_{3n}, n)| + |sd(C_{3n-1}, n)| \\ &= 3n+2 \end{aligned}$$

4. By mathematical induction $|sd(C_2, 4)| = \{\{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}\}$. So $|sd(C_4, 2)| = 6$, the result is true for $n = 1$. Now suppose that the result is true for all natural numbers less than n and we prove it for n. By (1),(2),(3) and induction we have

$$\begin{aligned} |sd(C_{3n+1}, n+1)| &= |sd(C_{3n}, n)| + |sd(C_{3n-1}, n)| + |sd(C_{3n-2}, n)| \\ &= 3 + 3(n-1) + 2 + (n-1)\frac{3(n-1)+7)+2}{2} \\ &= \frac{n(3n+7)+2}{10} \end{aligned}$$

5. It is obvious that for a graph with n vertices $|sd(G,n)| = 1$.

6. It is obvious that for a graph G with n vertices $|sd(G,n-1)| = n$.

7. By induction on n. The result is true for $n = 3$. Since $|sd(C_3,1)| = 3$. Assume it is true for all $n-1$. We prove for n. By parts (1), (2), (4), (5) and induction

$$\begin{aligned} |sd(C_n,n-2)| &= |sd(C_{n-1},n-3)| + |sd(C_{n-2},n-3)| + |sd(C_{n-3},n-3)| \\ &= \frac{(n-2)(n-1)}{2} + n - 2 + 1 \\ &= \frac{(n-1)n}{2} \end{aligned}$$

8. By induction on n. The result is true for $n = 4$. Since $|sd(C_4,1)| = 0$. Assume the result is true for all $n-1$. We prove for n, by parts (2), (6), (7) and induction we have

$$\begin{aligned} |sd(C_n,n-3)| &= |sd(C_{n-1},n-4)| + |sd(C_{n-2},n-4)| + |sd(C_{n-3},n-4)| \\ &= \frac{(n-5)(n-1)n}{6} + \frac{(n-2)(n-3)}{2} + n - 3 \\ &= \frac{(n-4)n(n+1)}{6} \end{aligned}$$

9. Proof by induction on m. Suppose $m = 3$ then $\sum_{n=3}^{9} sd(C_n,3) = 54 = 3\sum_{n=2}^{6} |sd(C_n,2)|$. Now suppose the result is true for every $m < t$ and we prove for $m = t$

$$\begin{aligned} \sum_{n=t}^{3t} |sd(C_n,t)| &= \sum_{n=t}^{3t} |sd(C_{n-1},t-1)| + \sum_{n=t}^{3t} |sd(C_{n-2},t-1)| + \sum_{n=t}^{3t} |sd(C_{n-3},t-1)| \\ &= 3\sum_{n=t-1}^{3(t-1)} |sd(C_{n-1},t-2)| + 3\sum_{n=t-1}^{3(t-1)} |sd(C_{n-2},t-2)| + 3\sum_{n=t-1}^{3(t-1)} |sd(C_{n-3},t-2)| \\ &= 3\sum_{n=t-1}^{3t-3} |sd(C_n,t-1)| \end{aligned}$$

10. We plan for every m, $|sd(C_n,m)| < |sd(C_{n+1},m)|$ for $m \le n \le 2m-1$ and $|sd(C_n,m)| > |sd(C_{n+1},m)|$ for $2m \le n \le 3m-1$. We prove first inequality by induction on m. The result hold for $m = 3$. Suppose that result is true for all $m \le t$. Now we prove it for $m = t+1$. That is $|sd(C_n,t+1)| < |sd(C_{n+1},t+1)|$ for $t+1 \le n \le 2t+1$.

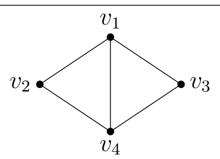
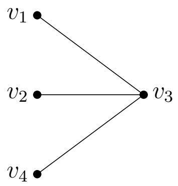
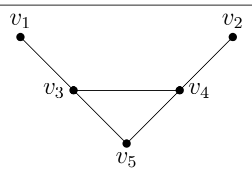
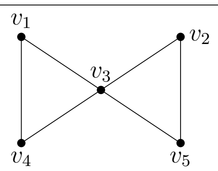
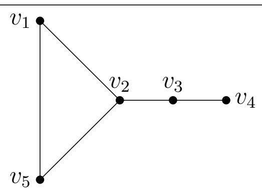
$$\begin{aligned} |sd(C_n,t+1)| &= |sd(C_{n-1},t)| + |sd(C_{n-2},t)| + |sd(C_{n-3},t)| \\ &< |sd(C_n,t)| + |sd(C_{n-1},t)| + |sd(C_{n-2},t)| \\ &= |sd(C_{n+1},t+1)| \end{aligned}$$

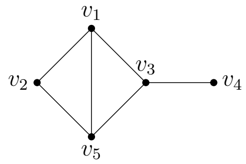
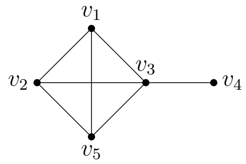
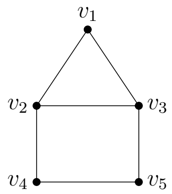
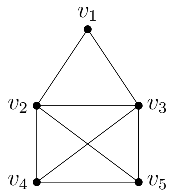
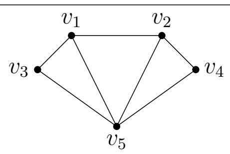
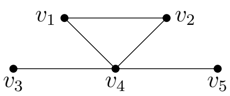
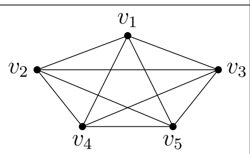
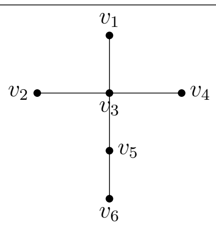
The other inequality follows in same way.

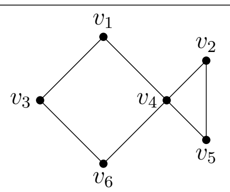
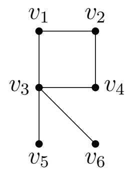
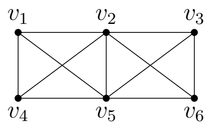
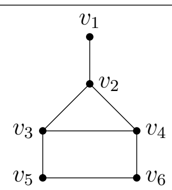
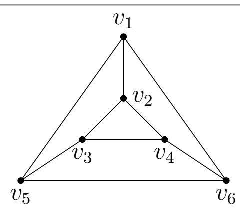
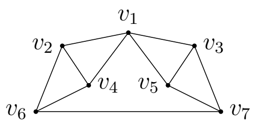
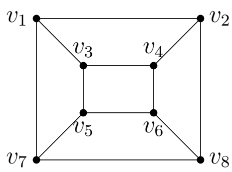
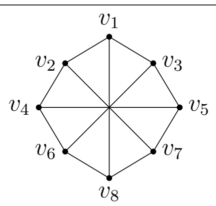
11. By theorem-4.2, we have

$$\begin{aligned}
A_n &= \sum_{m=\lceil \frac{n}{3} \rceil}^{n} |sd(C_n, m)| \\
&= \sum_{m=\lceil \frac{n}{3} \rceil}^{n} |sd(C_{n-1}, m-1)| + |sd(C_{n-2}, m-1)| + |sd(C_{n-3}, m-1)| \\
&= \sum_{m=\lceil \frac{n}{3} \rceil -1}^{n-1} |sd(C_{n-1}, m)| + \sum_{m=\lceil \frac{n}{3} \rceil -1}^{n-2} |sd(C_{n-2}, m)| + \sum_{m=\lceil \frac{n}{3} \rceil -1}^{n-3} |sd(C_{n-3}, m-1)| \\
&= A_{n-1} + A_{n-2} + A_{n-3}
\end{aligned}$$

Table: $SD(G,x)$ of different standard graphs and their roots are tabulated in the table below

Graph	Figure	Superior domination polynomial $SD(G,x)$	Roots
Diamond graph		$x^4 + 4x^3 + 6x^2 + 2x$	$x_1 = 0$, $x_2 = -0.4563$, $x_3 = -1.7718 + 1.1151i$, $x_4 = -1.7718 - 1.1151i$.
Claw graph		$x^4 + 3x^3 + 3x^2 + x$	$x_1 = 0$, $x_2 = -1$, $x_3 = -1$, $x_4 = -1$.
Bull graph		$x^5 + 3x^4 + 3x^3 + x^2$	$x_1 = 0$, $x_2 = -1$, $x_3 = -1$, $x_4 = -1$.
Butterfly graph		$x^5 + 4x^4 + 6x^3 + 4x^2 + x$	$x_1 = 0$, $x_2 = -1$, $x_3 = -1$, $x_4 = -1$, $x_5 = -1$.
(3,2)-Tadpole graph		$x^5 + 4x^4 + 5x^3 + 2x^2$	$x_1 = 0$, $x_2 = -1$, $x_3 = -2$, $x_4 = -1$.

Graph	Figure	Superior domination polynomial $SD(G,x)$	Roots
Kite graph		$x^5+5x^4+9x^3+5x^2+x$	$x_1=0$, $x_2=-0.378+0.1877i$, $x_3=-0.378-0.1877i$, $x_4=-2.122+1.0538i$, $x_5=-2.122-1.0538i$.
(4,1)-Lollipop graph		$x^5+4x^4+6x^3+4x^2+x$	$x_1=0$, $x_2=-1$, $x_3=-1$, $x_4=-1$, $x_5=-1$.
House graph		$x^5+5x^4+8x^3+4x^2+x$	$x_1=0$, $x_2=-0.3076+0.3182i$, $x_3=-0.3076-0.3182i$, $x_4=-2.1924+0.5479i$, $x_5=-2.1924-0.5479i$.
House X graph		$x^5+5x^4+8x^3+5x^2+x$	$x_1=0$, $x_2=-1$, $x_3=-1$, $x_4=-0.382$, $x_5=-2.618$.
Gem graph		$x^5+4x^4+4x^3$	$x_1=0$, $x_2=-2$.
Cricket graph		$x^5+4x^4+6x^3+4x^2+x$	$x_1=0$, $x_2=-1$, $x_3=-1$, $x_4=-1, x_5=-1$.
Pentatope graph		$x^5+5x^4+10x^3+10x^2+5x$	$x_1=0$, $x_2=-0.691=0.9511i$, $x_3=-0.691-0.9511$, $x_4=-1.809+0.5878i$, $x_5=-1.809-0.5878i$.
Cross graph		$x^6+5x^5+9x^4+7x^3+2x^2$	$x_1=0$, $x_2=-1$, $x_3=-2$, $x_4=-1$, $x_5=-1$.

Graph	Figure	Superior domination polynomial $SD(G,x)$	Roots
Fish graph		$x^6+5x^5+10x^4+9x^3+3x^2$	$x_1=0,$ $x_2=-1,$ $x_3=-1,$ $x_4=-\frac{3}{2}+\frac{\sqrt{3}}{2}i,$ $x_5=-\frac{3}{2}-\frac{\sqrt{3}}{2}i.$
R graph		$x^6+5x^5+10x^4+9x^3+3x^2$	$x_1=0,$ $x_2=-1,$ $x_3=-1,$ $x_4=-\frac{3}{2}+\frac{\sqrt{3}}{2}i,$ $x_5=-\frac{3}{2}-\frac{\sqrt{3}}{2}i.$
(2,3)-King graph		$x^6+4x^5+6x^4+4x^3+x^2$	$x_1=0,$ $x_2=-1,$ $x_3=-1,$ $x_4=-1,$ $x_5=-1.$
Antenna graph		$x^6+4x^5+5x^4+2x^3$	$x_1=0,$ $x_2=-1,$ $x_3=-1,$ $x_4=-1.$
3-prism graph		$x^6+6x^5+15x^4+20x^3$ $+15x^2+6x$	$x_1=0,$ $x_2=-2,$ $x_3=-0.5+0.866i,$ $x_4=-0.5-0.866i,$ $x_5=-1.5+0.866i,$ $x_6=-1.5-0.866i.$
Moser Spindle graph		$x^7+4x^6+6x^5$ $+4x^4+x^3$	$x_1=0,$ $x_2=-1,$ $x_3=-1,$ $x_4=-1,$ $x_5=-1.$
Cubical graph		$x^8+8x^7+28x^6+56x^5$ $+70x^4+48x^3+16x^2$	$x_1=0,$ $x_2=-0.6714+0.5756i,$ $x_3=-0.6714-0.5756i,$ $x_4=-0.8352+1.4854i,$ $x_5=-0.8352-1.4854i,$ $x_6=-2.4934+0.9097i,$ $x_7=-2.4934-0.9097i.$
Wagner graph		$x^8+8x^7+24x^6$ $+32x^5+16x^4$	$x_1=0,$ $x_2=-2,$ $x_3=-2,$ $x_4=-2,$ $x_5=-2.$

5 Conclusions

In this paper we introduced superior domination polynomial, this is a distance based domination polynomial. Emphasis was given to the family of stars and cycles. Formulas to find the coeffcients of the superior domination polynomials of cycles and stars were stated and proved. These formulas helps us to calculate the number of superior dominating sets of a specific desired cardinality for any given value of n. The superior domination polynomial of different standard graphs and their roots are calculated.

6 Acknowledgements

The authors express their gratitude to the managemanet- Ratio Mathematica for their constant support towards the successful completion of this work. We wish to thank the anonymous reviewers for the valuable suggestions and comments.

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