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D_4-Magic Graphs

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Abstract

Consider the set $X = \{1, 2, 3, 4\}$ with 4 elements. A permutation of X is a function from X to itself that is both one one and on to. The permutations of X with the composition of functions as a binary operation is a nonabelian group, called the symmetric group S_4. Now consider the collection of all permutations corresponding to the ways that two copies of a square with vertices $1, 2, 3$ and 4 can be placed one covering the other with vertices on the top of vertices. This collection form a nonabelian subgroup of S_4, called the dihedral group D_4. In this paper, we introduce A-magic labelings of graphs, where A is a finite nonabelian group and investigate graphs that are D_4-magic. This did not attract much attention in the literature.

Keywords: A-magic labeling; Dihedral group D_4 ; D_4-magic.
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1 Introduction

A graph G is an ordered pair $(V(G), E(G))$, where $V(G)$ is a finite nonempty set whose elements are called vertices and $E(G)$ is a binary irreflexive and symmetric relation on $V(G)$ whose elements are called edges. For any abelian group A, written additively, any mapping $\ell : E(G) \to A \setminus \{0\}$ is called a labeling. Given a labeling on the edge set $E(G)$, one can introduce a vertex set labeling $\ell^+ : V(G) \to A$ as follows:

$$\ell^+(v) = \sum_{uv \in E(G)} l(uv)$$

A graph G is said to be A-magic if there is a labeling $\ell : E(G) \to A\setminus\{0\}$ such that for each vertex $v,$ the sum of the labels of the edges incident with v are all equal to the same constant, that is, $\ell^+(v) = a$ for some fixed $a \in A$. The original concept of A-magic graph was introduced by Sedláček[1]. According to him, a graph G is A-magic if there exists an edge labeling on G such that (i) distinct edges have distinct non-negative labels; and (ii) the sum of the labels of the edges incident to a particular vertex is same for all vertices. When $A = \mathbb{Z}$, the $\mathbb{Z}$-magic graphs are considered in Stanley[7]. Doob [5, 4] also considered A-magic graphs where A is an abelian group. Also he determined which wheels are $\mathbb{Z}$-magic. Observe that several authors studied V_4-magic graphs[8, 6]. It is natural to ask does there exist graphs which admits A-magic labeling, when A is nonabelian? In this paper, we address this question and investigate graphs that are D_4-magic.

2 Main results

Let $G = (V(G), E(G))$ be a finite (p, q) graph and let $(A, *)$ be a finite nonabelain group with identity element 1. Let $f : E(G) \to N_q = \{1, 2, \ldots, q\}$ and let $g : E(G) \to A \setminus \{1\}$ be two edge labelings of G such that f is bijective. Define an edge labeling $\ell : E(G) \to N_q \times A \setminus \{1\}$ by

$$l(e) := (f(e), g(e)),\ e \in E(G).$$

Define a relation $\leq$ on the range of ℓ by:

$$(f(e), g(e)) \leq (f(e^{'}), g(e')) \text{ if and only if } \quad f(e) \leq f(e').$$

Then obviously the relation $\leq$ is a partial order on the range of ℓ.
Let $\{(f(e_1), g(e_1)), (f(e_2), g(e_2)), \ldots, (f(e_k), g(e_k))\}$ be a chain in the range of ℓ. We define the product of elements of this chain as follows:

$$\prod_{i=1}^{k}(f(e_i), g(e_i)) := ((((g(e_1) * g(e_2)) * g(e_3)) * g(e_4)) * \ldots) * g(e_k).$$

Let $u \in V$ and let $N^*(u)$ be the set of all edges incident with u. Note that the range of $\ell|_{N^*(u)}$ is a chain, say $(f(e_1), g(e_1)) \leq (f(e_2), g(e_2)) \leq \cdots \leq (f(e_n), g(e_n))$. We define,

$$\ell^*(u) = \prod_{i=1}^{n} (f(e_i), g(e_i)). \tag{1}$$

If $\ell^*(u)$ is a constant, say a for all $u \in V(G)$, we say that the graph G is A-magic. The map ℓ^* is called an A-magic labeling of G and the corresponding constant a is called the magic constant. For example, consider the cycle graph $C_4 = (uv, vw, wx, xu)$ and the permutation group D_4. Note that the group D_4 is a non abelian group of order 8 and its elements are given by

$$\rho_0 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}, \quad \mu_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix},$$
$$\rho_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}, \quad \mu_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix},$$
$$\rho_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}, \quad \delta_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix},$$
$$\rho_3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}, \quad \delta_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix}.$$

Define $f : E(G) \to N_4 = \{1, 2, 3, 4\}$ as $f(uv) = 1, f(wx) = 2, f(vw) = 3, f(xu) = 4$ and $g : E(G) \to D_4 \setminus \{\rho_0\}$ as $g(uv) = g(wx) = \rho_1,\ g(vw) = g(xu) = \delta_1$. Thus

$$\ell^*(u) = (1, \rho_1)(4, \delta_1) = \rho_1\delta_1 = \mu_2,$$

$\ell^*(v) = (1, \rho_1)(3, \delta_1) = \rho_1\delta_1 = \mu_2$. Similarly, $\ell^*(w) = \mu_2$ and $\ell^*(x) = \mu_2$. Thus C_4 is D_4-magic with magic constant μ_2.

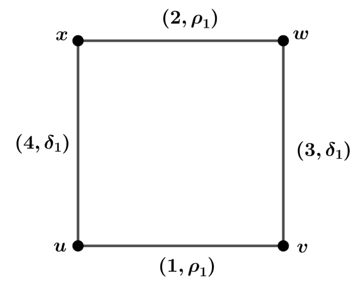

Figure 1: D_4-magic labeling of C_4.

In this paper, we will consider the symmetric group D_4 and investigate graphs that are D_4-magic.

Theorem 2.1. *Let A be a non abelian group having an element of order 2 and let G be a graph. If either the degree of the vertices of G are all even or odd. Then G is A-magic.*

Proof. Let G be a (p, q) graph and A be a nonabelian group having an element of order 2. Let $a \in G$ is of order 2. Let $g : E(G) \to A \setminus \{1\}$ be the constant map $g(e) = a, \ \forall e \in E(G)$ and let f be any bijection from $E(G) \to N_q$. First assume that all the vertices of G are of even degree then $l^*(u) = 1, \ \forall u \in V(G)$. Similarly, if all the vertices of G are of odd degree then $l^*(u) = a, \ \forall u \in V(G)$. Hence the proof.

Corolary 2.1. *All Eulerian graphs are D_4-magic.*

Theorem 2.2. *Any regular graph is D_4-magic.*

Proof. Let $G = (V(G), E(G))$ be a regular graph with $|E(G)| = q$. Let $f : E(G) \to N_q$ be any bijection and g be any constant map from $E(G) \to D_4 \setminus \{\rho_0\}$. Obviously, f and g will determine a D_4-magic labeling of G. This completes the proof of the theorem.

Corolary 2.2. *For any $n \geq 3$, the cycle graph C_n is D_4-magic.*

Corolary 2.3. *For any $n \geq 2$, the complete graph K_n is D_4-magic.*

Corolary 2.4. *The Peterson graph is D_4-magic.*

Theorem 2.3. *The star graph $K_{1,n}, n \geq 2$ is D_4-magic iff n is odd.*

Proof. Let $G = K_{1,n}$. Suppose that n is odd. Let $f : E(G) \to N_{n+1}$ be a bijection. Define $g : E(G) \to D_4 \setminus \{\rho_0\}$ by $g(e) = \mu_1$. Then clearly it is D_4-magic with magic constant μ_1.
Conversely, suppose $K_{1,n}$ is D_4-magic with magic constant, say 'a'. So every pendent edge of $K_{1,n}$ should be mapped to a under g. Let u be the vertex of $K_{1,n}$ with degree n. Then

$$\ell^*(u) = \underbrace{aa \cdots a}_{n \text{ times}} = a.$$

This implies that $a^{n-1} = \rho_0$. If n is odd, the equation $a^{n-1} = \rho_0$ has five non trivial solutions in D_4 viz. $\mu_1, \mu_2, \delta_1, \delta_2$ and ρ_2. On the other hand, if n is even there are no element in D_4 such that $a^{n-1} = \rho_0$. This completes the proof.

A bistar graph B_n is the graph obtained by connecting the apex vertices of two copies of star $K_{1,n}$ by a bridge.

Theorem 2.4. *The bistar graph $B_n, \ n > 1$ is D_4-magic when $n \not\equiv 1 (mod\, 4)$.*

Proof. First, observe that there are $2n$ pendant edges and one bridge in B_n. Here we consider the following cases:

Case (i): n is even ($n \equiv 2(\text{mod } 4)$ or $n \equiv 0(\text{mod } 4)$).
If n is even, define $g : E(B_n) \to D_4 \setminus \{\rho_0\}$ by $g(e) = \mu_1$, $\forall e \in E(B_n)$. Let f be any bijective map from $E(B_n) \to N_{2n+1}$. Then obviously, B_n is D_4-magic with magic constant μ_1.

Case (ii): $n \equiv 3(\text{mod } 4)$.
In this case we define $g : E(B_n) \to D_4 \setminus \{\rho_0\}$ by

$$g(e) = \begin{cases} \rho_1, & \text{if e is a pendant edge,} \\ \rho_2, & \text{if e is the bridge.} \end{cases}$$

Let f be any bijective map from $E(B_n)$ to N_{2n+1}. Then obviously B_n is D_4-magic with the magic constant ρ_1.

Case (iii): $n \equiv 1(\text{mod } 4)$.
Suppose that $n \equiv 1(\text{mod } 4)$. Let k_1 and k_2 be the apex vertices of the bistar graph. Assume that B_n is D_4-magic with magic constant μ_1. Therefore, $g(e) = \mu_1$ for all pendant edges e. Assume that $g(k_1k_2) = a$, where $a \in D_4 \setminus \{\rho_0\}$. Without loss of generality assume that $f(k_1k_2) > f(b)$, $\forall b \in E(G)$, where b denotes the pendant edge with one end point k_1. Then

$$\ell^*(k_1) = \underbrace{\mu_1\mu_1 \dots \mu_1}_{(n\ times)} a = \mu_1.$$

The above equation tells us that $a = \rho_0$, which is a contradiction. This contradiction shows that B_n is not D_4-magic with magic constant μ_1. In a similar manner, we can prove that B_n is not D_4-magic with magic constants $\mu_2,\ \rho_1,\ \rho_2,\ \rho_3,\ \delta_1$ or δ_2. Thus the bistar graph B_n is not D_4-magic when $n \equiv 1(\text{mod } 4)$. This completes the proof of the theorem.

Theorem 2.5. *The complete bipartite graph $K_{m,n}$ is D_4-magic, $m, n > 1$.*

Proof. Let $G = K_{m,n}$. Suppose $U = \{u_1, u_2, \dots, u_n\}$, and $V = \{v_1, v_2, \dots, v_m\}$. be the two partite sets of $K_{m,n}$. If m and n are both even or odd then the theorem is obvious by taking any constant map $g : E(G) \to \{\rho_2, \mu_1, \mu_2, \delta_1, \delta_2\}$.

Case (i): $n \equiv 0(\text{mod } 2)$ and $m \equiv 1(\text{mod } 4)$.
Let $U = \{u_1, u_2, \dots, u_{2l}\}$ and $V = \{v_1, v_2, \dots, v_{4r+1}\}$ where $n = 2l$, $m = 4r + 1$, and $l, r \in \mathbb{N}$. For $1 \le i \le n$ and $1 \le j \le m$ define

$$\begin{aligned} g(u_iv_{5k+1}) &= \mu_1, \text{ where } k < m,\ k = 0, 1, 2, 3, \dots \\ g(u_iv_{5k+2}) &= \mu_2, k < m,\ k = 0, 1, 2, 3, \dots \\ g(u_iv_j) &= \rho_2, j \neq 5k + 1,\ 5k + 2 \text{ where } k = 0, 1, 2, \dots \\ f(u_iv_j) &= (i - 1)m + j,\ 1 \le i \le n,\ 1 \le j \le m. \end{aligned}$$

The maps f and g will determine a D_4-magic labeling for $K_{m,n}$ with magic constant ρ_0.

Case (ii): $n \equiv 0(\text{mod } 2)$ and $m \equiv 3(\text{mod } 4)$. Define g as follows:

$$g(u_iv_j) = \begin{cases} \rho_1, & \text{if } i \text{ is odd } 1 \le j < m, \\ \rho_3, & \text{if } i \text{ is even } 1 \le j < m, \end{cases} \quad \text{and}$$
$$g(u_iv_m) = \rho_2, \ \forall i, \ 1 \le i \le n,$$

and let f be any bijection from $E(G)$ to $\{1, 2, \ldots, mn\}$. Then clearly f and g will determine a D_4-magic labeling of $K_{m,n}$ with magic constant ρ_0.

This completes the proof of the theorem.

3 Cycle Generated Graphs

In this section, we consider certain graphs which are constructed from cycles.

A wheel W_n of order $n + 1$, sometimes simply called an n wheel is a graph that contains a cycle of order n and for which every graph vertex in the cycle is connected to one other graph vertex (which is known as the hub). The edges of a wheel which include the hub are called spokes. The wheel W_n can be defined as the graph join $K_1 + C_n$, where K_1 is the singleton graph and C_n is the cycle graph.

Theorem 3.1. *If $n \ge 3$, the wheel W_n is D_4-magic.*

Proof. Let the vertices of C_n be $u_1, u_2, \ldots, u_n$ such that $u_iu_{i+1} \in E(C_n)$, $i = 1, 2, \ldots, n$ and $u_{n+1} = u_1$. Denote the vertex of K_1 by k. Now we consider the following cases:

Case (i): n is odd.

If n is odd then every vertex of W_n is of odd degree. Thus we can take $g : E(W_n) \to D_4 \backslash \{\rho_0\}$ as any constant map from $E(W_n)$ to $\{\rho_2, \mu_1, \mu_2, \delta_1, \delta_2\}$. Since g is constant we can take f as any bijection from $E(W_n)$ to N_{2n}. Clearly this f and g will constitute a D_4-magic labeling for W_n.

Case (ii): n is even.

Suppose n is even define $f : E(W_n) \to N_{2n}$ as

$$f(ku_i) = i, \ i = 1, 2, \ldots, n,$$
$$f(u_iu_{i+1}) = n + i, \ 1 \le i \le n - 1,$$
$$f(u_1u_n) = 2n.$$

Now we can define $g : E(W_n) \to D_4 \setminus \{\rho_0\}$ by labeling each spokes by μ_1 and all the outer edges by μ_2 and ρ_2 alternatively. Then W_n becomes D_4-magic with magic constant ρ_0.

This completes the proof of the theorem.

The helm H_n is a graph obtained from a wheel W_n by attaching a pendant edge at each vertex of the n cycle.

Theorem 3.2. *The Helm graph H_n is D_4-magic.*

Proof. Let $\{k, u_i, v_i : i = 1, 2, \ldots, n\}$ be the vertex set of H_n, where k be the central vertex, $u_1, u_2, \ldots, u_n$ are the vertices of the cycle, $v_1, v_2, \ldots, v_n$ are the pendant vertices adjacent to $u_1, u_2, \ldots, u_n$. The edge set of H_n is $E(H_n) = \{u_iu_{i+1}, ku_i, u_iv_i : i = 1, 2, \ldots, n,\ u_{n+1} = u_1\}$. Now consider the following two cases:

Case(i): n is odd.

Suppose that n is odd. Define f and g as follows: Let $g : E(G) \to D_4\setminus\{\rho_0\}$ be defined as $g(ku_i) = \rho_2,\ 1 \le i \le n,\ g(u_ju_{j+1}) = \rho_1,\ 1 \le j \le n-1,\ g(u_1u_n) = \rho_1,\ g(u_kv_k) = \rho_2,\ 1 \le k \le n$. Now let $f : E(G) \to N_{2n+1}$ be any bijection. Then clearly f and g will give a D_4-magic labeling of $H_n,$ where n is odd.

Case(ii): n is even.

Let f be defined as above and define $g : E(G) \to D_4 \setminus \{\rho_0\}$ by

$$g(u_iv_i) = \rho_2,\ 1 \le i \le n,\ g(v_1v_n) = \rho_1$$

$$g(ku_j) = \begin{cases} \rho_2, & \text{if } 1 \le j \le n-2, \\ \rho_1, & \text{if } j = n-1, n. \end{cases},$$

$$g(u_ku_{k+1}) = \begin{cases} \rho_1, & \text{if } 1 \le k \le n-2, \\ \rho_2, & \text{if } k = n-1. \end{cases}$$

It follows that $l^*(u) = \rho_2, \forall u \in V(G)$. Hence H_n is D_4-magic when n is even.

This completes the proof of the theorem.

The web graph $W(2, n)$ is a graph obtained joining the pendant points of a helm to form a cycle and adding a single pendant edge to each vertex of this outer graph.

Theorem 3.3. *The web graph $W(2, n), n \ge 3$ is D_4-magic.*

Proof. Let $\{k, u_i, v_i, w_i : i = 1, 2, 3, \ldots, n\}$ be the vertex set of $W(2, n)$, where k be the central vertex, $u_1, u_2, u_3, \ldots, u_n$ are the vertices of inner cycle, $v_1, v_2, v_3, \ldots, v_n$ are the vertices of outer cycle and $w_1, w_2, w_3, \ldots, w_n$ are the pendant vertices adjacent to $v_1, v_2, v_3, \ldots, v_n$ of $W(2, n)$. Let $E(W(2, n)) = \{u_iu_{i+1}, v_iv_{i+1}, u_iv_i, v_iw_i : i = 1, 2, \ldots, n$ and $u_{n+1} = u_1, v_{n+1} = v_1\}$. We define a D_4-magic labeling for $W(2, n)$ with magic constant ρ_2 as follows:

Case (i): n is odd.
Let $f : E(G) \to N_{3n+1}$ be any bijection.
Define $g : E(G) \to D_4 \setminus \{\rho_0\}$ as

$$\begin{aligned} g(ku_i) &= \rho_2 = g(u_iv_i) = g(v_iw_i),\ 1 \leq i \leq n, \\ g(u_iu_{i+1}) &= \rho_1 = g(v_iv_{i+1}),\ 1 \leq i \leq n-1, \\ g(u_1u_n) &= \rho_1 = g(v_1v_n). \end{aligned}$$

Case (ii): n is even.
Let $f : E(G) \to N_{3n+1}$ be any bijection.
Define $g : E(G) \to D_4 \setminus \{\rho_0\}$ as
$g(ku_i) = \rho_2,\ 1 \leq i < n-1,\ g(ku_n) = g(ku_{n-1}) = \rho_1,\ g(v_iv_{i+1}) = \rho_1 = g(u_iu_{i+1}) = \rho_1,\ 1 \leq i \leq n-1,\ g(v_iw_i) = \rho_2 = g(u_iv_i),\ 1 \leq i \leq n,\ g(v_1v_n) = \rho_1, g(u_1u_n) = \rho_2$.

This completes the proof of the theorem.

A shell graph $S_{n,n-3}$ of width n is a graph obtained by taking $n-3$ concurrent chords in a cycle C_n of n vertices. The vertex at which all chords are concurrent is called is called the apex. The two vertices adjacent to the apex have degree 2, apex has degree $n-1$ and all other vertices have degree 3.

Theorem 3.4. *Shell graphs $S_{n,n-3}$ are D_4-magic.*

Proof. Let us denote the vertices of the shell graph $S_{n,n-3}$ by $u_1, u_2, \ldots, u_n$ such that u_i is adjacent to u_{i+1}, where $i = 1, 2, \ldots, n$ and $u_{n+1} = u_1$. Without loss of generality let the apex be u_1. Now consider the following cases:

Case (i): n is even.
We will define the map $f : E(S_{n,n-3}) \to N_{2n-3}$ as

$$\begin{aligned} f(u_iu_{i+1}) &= i,\ 1 \leq i \leq n-1, \\ f(u_nu_1) &= n, \\ f(u_1u_j) &= n + (j-2),\ 3 \leq j \leq n-1 \end{aligned}$$

and we define $g : E(S_{n,n-3}) \to D_4 \setminus \{\rho_0\}$ as

$$\begin{aligned} g(u_1u_2) &= g(u_nu_1) = \rho_2, \\ g(u_1u_i) &= \mu_1, 3 \le i \le n-1, \\ g(u_iu_{i+1}) &= \mu_2,\ 2 \le i \le n-1. \end{aligned}$$

Clearly f and g define a D_4-magic labeling with magic constant μ_1.

Case (ii): n is odd.

Define f as

$$\begin{aligned} f(u_iu_{i+1}) &= i,\ 1 \le i \le n-1, \\ f(u_1u_n) &= n, \\ f(u_1u_j) &= n + (j-2),\ 3 \le j \le n-1 \end{aligned}$$

and define g as

$$\begin{aligned} g(u_1u_2) &= g(u_1u_n) = \rho_2, \\ g(u_1u_j) &= \mu_1,\ 3 \le j \le n-1, \\ g(u_iu_{i+1}) &= \begin{cases} \rho_2, & \text{if } i \text{ is even}, \\ \mu_2, & \text{if } i \text{ is odd}, 1 < i \le n-1. \end{cases} \end{aligned}$$

Obviously the functions f and g define a D_4-magic labeling of $S_{n,n-3}$ with magic constant ρ_0.

This completes the proof of the theorem.

When k copies of C_n share a common edge it will form the n-gon book of k pages and is denoted by $B(n,k)$.

Theorem 3.5. *The graph n-gon book of k pages $B(n,k)$ is D_4-magic.*

Proof. Let G be the graph $B(n,k)$. Denote the vertices of common edge by k_1 and k_n and the edges of i^{th} page other than k_1 and k_n by $u_{i2}, u_{i3}, \ldots, u_{in-1}$ such that u_{i2} is adjacent to k_1 and u_{in-1} adjacent to k_n and u_{ij} adjacent to u_{ij+1} for all $2 \le j < n-1$. Consider the following cases:

Case (i): k is even.

Define $g : E(G) \to D_4 \setminus \{\rho_0\}$ as

$$\begin{aligned} g(k_1k_n) &= \rho_2, \\ g(u_{1j}u_{1j+1}) &= \mu_1,\ \ 2 \le j \le n-2, \\ g(u_{1n-1}k_n) &= \mu_1 = g(k_1u_{12}), \\ g(u_{ij}u_{ij+1}) &= \mu_2,\ 2 \le i \le k,\ 2\ \le j \le n-1, \\ g(k_1u_{l2}) &= g(u_{ln-1}) = \mu_2,\ 2 \le l \le k. \end{aligned}$$

Now define f as

$$\begin{aligned}
f(k_1k_n) &= 1,\ f(k_1u_{12}) = 2, f(u_{1n-1}k_n) = n,\\
f(u_{1j}u_{1j+1}) &= j+1,\ \forall\, 2 \leq j \leq n-2,\\
f(k_1u_{i2}) &= n + (i-2)(n-1) + 1,\ i \geq 2,\\
f(u_{ij}u_{ij+1}) &= n + (i-2)(n-1) + j,\ 2 \leq j \leq n-2,\ 2 \leq i \leq k,\\
f(u_{i(n-1)}k_n) &= n + (i-2)(n-1) + (n-1),\ 2 \leq i \leq k.
\end{aligned}$$

The functions f and g determine a D_4-magic labeling with magic constant ρ_0.

Case (ii): k is odd.
Here define g as $g(e) = \rho_2,\ \forall e \in E(G)$ then g together with any bijection $f : E(G) \to N_{kn-1}$ will define a D_4-magic labeling of $B(n,k)$ with magic constant ρ_0.

This completes the proof of the theorem.

Note that, for any $n \geq 3$ the path graph of order n is not D_4-magic.

4 Path Generated Graphs

In this section we will consider some graphs which are constructed from Paths. We start with the Splitting graph of Path.

A splitting graph $S(G)$ of a graph G is the graph obtained from G by adding to G a new vertex z' for each vertex z of G and joining z' to the neighbors of z in G.

Theorem 4.1. *Splitting graph of the path graph $P_n, n \geq 3$ is D_4-magic.*

Proof. Let P_n be a path graph of order n, where $n \geq 3$. Let $u_1, u_2, \ldots, u_n$ be the vertices of P_n, where $u_iu_{i+1} \in E(P_n), i = 1, 2, \ldots, n-1$. There are $2n$ vertices and $3n-3$ edges in $S(P_n)$. Let u_{n+i} be the vertex corresponding to the i^{th} vertex in $S(P_n)$. Observe that there are two pendant edges in $S(P_n)$, one with end points u_2 and u_{n+1} and the other with end points u_{n-1} and u_{2n}.

Case (i): $n = 3$.
In this case, define $f : E(S(P_3)) \to N_6$ as
$f(u_1u_2) = 1,\ f(u_2u_3) = 3,\ f(u_3u_5) = 2,\ f(u_1u_5) = 4,\ f(u_2u_4) = 5,\ f(u_2u_6) = 6$. Now define $g : E(G) \to D_4 \setminus \{\rho_0\}$ as
$g(u_1u_2) = g(u_3u_5) = \rho_1,\ g(u_2u_3) = g(u_1u_5) = \delta_2,\ g(u_2u_4) = g(u_2u_6) = \mu_1$.

Case (ii): $n > 3$.

In this case, define f and g as follows:

$$
\begin{aligned}
f(u_iu_{i+1}) &= i,\ 1 \le i \le n-1,\\
f(u_iu_{n+(i-1)}) &= n + (i-2),\ 2 \le i \le n,\\
f(u_iu_{n+(i+1)}) &= (2n-2) + i,\ 1 \le i \le n-1 \text{ and}\\
g(u_1u_2) &= \rho_2,\ g(u_{n-1}u_n) = \mu_2,\\
g(u_2u_{n+1}) &= g(u_{n-1}u_{2n}) = \mu_1,\\
g(u_iu_{i+1}) &= \mu_1,\ 2 \le i < n-1,\\
g(u_iu_{n+(i-1)}) &= \rho_2,\ 3 \le i \le n,\\
g(u_iu_{n+(i+1)}) &= \mu_2,\ 1 \le i \le n-2.
\end{aligned}
$$

In all the above cases, we can prove that the functions f and g defines a D_4-magic labeling of $S(P_n)$ with magic constant μ_1.

This completes the proof of the theorem.

The middle graph of a connected graph G denoted by $M(G)$ is the graph whose vertex set is $V(G) \cup E(G)$ where two vertices are adjacent if

(i) They are adjacent edges of G or

(ii) One is a vertex of G and the other is an edge incident with it.

Theorem 4.2. *Middle graph of the path graph P_n is D_4-magic for $n \ge 3$.*

Proof. Let $M(P_n)$ be the middle graph of the path P_n. Denote the vertices of P_n by $u_1, u_2, \ldots, u_n$ and edges by $e_1, e_2, \ldots, e_{n-1}$ where e_i incident with u_i and u_{i+1}. There are $2n-1$ vertices and $3n-4$ edges in $M(P_n)$. Consider the following cases:

Case(i): $n = 3$.

Define $f : E(M(P_3)) \to N_{3n-4}$ as $f(e_1u_1) = 1,\ f(e_1u_2) = 2,\ f(e_1e_2) = 3,\ f(e_2u_2) = 4$ and $f(e_2u_3) = 5$ and define $g : E(M(P_3)) \to D_4 \setminus \{\rho_0\}$ as $g(e_1u_1) = \rho_2 = g(e_2u_3),\ \ g(e_1u_2) = \rho_1 = g(e_2u_2),\ \ g(e_1e_2) = \rho_3$. Then clearly the middle graph of the path P_3 is D_4-magic with magic constant ρ_2.

Case(ii): $n > 3$.

Define $f : E(M(P_n)) \to N_{3n-4}$ as follows:

For $1 \le i \le n-2,\ 1 \le j \le n,\ f(e_iu_j) = 2(i-1) + j$ and $f(e_ie_{i+1}) = 3i$.

Now define $g : E(M(P_n)) \to D_4 \setminus \{\rho_0$ by

$$
\begin{aligned}
g(e_1u_1) &= \rho_2 = g(e_{n-1}u_n),\\
g(e_1u_2) &= \mu_2,\ g(e_2u_2) = \mu_1,\ g(e_2e_3) = \rho_3\\
g(e_2u_3) &= \rho_1 = g(e_3u_3),\\
g(e_ie_{i+1}) &= \mu_2,\ \text{where } i \neq 2 \text{ and } 1 \leq i \leq n-1,\\
g(e_iu_j) &= \begin{cases} \mu_1,\ if\ j = i+1 \text{ and } 3 \leq i < n-1,\\ \mu_2,\ if\ i = j \text{ and } 4 \leq i \leq n-1. \end{cases}
\end{aligned}
$$

The above functions f and g will define a D_4-magic labeling of $M(P_n)$ with magic constant ρ_2.

This completes the proof of the theorem.

A triangular snake T_n is obtained from the path P_n by replacing each edge of the path by a triangle C_3.

Theorem 4.3. *The Triangular snake T_n is D_4-magic.*

Proof. Note that every vertex of T_n has even degree . So the proof is indisputable from Theorem 2.1.

The alternate triangular snake $A(T_n)$ is obtained from the path $u_1, u_2, \ldots, u_n$ by joining u_iu_{i+1}(alternatively) to a new vertex v_i.

Theorem 4.4. *The alternate triangular graph $A(T_n)$ is D_4-magic.*

Proof. Let us denote the vertices of the path P_n be $u_1, u_2, \ldots, u_n$ and the vertex which join u_i and u_{i+1} be denoted by v_i. Now consider the following cases:

Case (i): n is even and triangle starts from u_1.

Suppose that n is even and the triangle starts from the first vertex u_i, then there are $n + \frac{n}{2}$ vertices and $2n-1$ edges.

Suppose $n = 2$ then $A(T_n)$ is C_3 itself. So there is nothing to prove.

Suppose $n = 4$ then take f be any bijection from $E(A(T_n))$ to N_7 and define $g : E(A(T_n)) \to D_4 \setminus \{\rho_0\}$ by

$g(u_1v_1) = g(u_4v_3) = \rho_1$, $g(u_2u_3) = \rho_2$, $g(u_2v_1) = g(u_3v_3) = g(u_1u_2) = g(u_3u_4) = \rho_3$. Then $A(T_4)$ becomes D_4-magic with magic constant ρ_0.

Suppose $n > 4$, then let $f : E(A(T_n)) \to N_{2n-1}$ be any bijection and define

$g : E(A(T_n)) \to D_4 \setminus \{\rho_0\}$ as

$$g(u_1u_2) = g(u_{n-1}u_n) = g(u_2v_1) = g(u_{n-1}v_{n-1}) = \rho_3,$$
$$g(u_1v_1) = g(u_{n-1}v_{n-1}) = \rho_1. \text{ For } 2 \leq i < n-1, \text{ define}$$
$$g(u_iu_{i+1}) = \begin{cases} \rho_2, & \text{if } i \text{ is even,} \\ \mu_2, & \text{if } i \text{ is odd.} \end{cases}$$
$$g(u_{2k+1}v_{2k+1}) = g(u_{2k+2}v_{2k+1}) = \mu_1, k = 1, 2, \ldots, \frac{n-4}{2}$$

Obviously the functions f and g will constitute a D_4-magic labeling for $A(T_n)$ with $l^*(u) = \rho_0, \forall u \in V(A(T_n))$.

Case (ii): n is even and the triangle starts from the second vertex u_2.
We can define a magic labeling for $A(T_n)$, where n is even and cycle starts from u_2 as follows:
Let f be any bijection as above and define g as

$$g(u_iu_{i+1}) = \begin{cases} \rho_2, & \text{if } i \text{ is odd,} \\ \rho_3, & \text{if } i \text{ is even,} \end{cases} \quad , 1 \leq i \leq n-1,$$
$$g(u_{2k}v_{2k}) = g(u_{2k+1}v_{2k}) = \rho_1,\ k = 1, 2, 3, \ldots, \frac{n-2}{2}.$$

Clearly l^* is a constant map, i.e., $l^*(u) = \rho_2,\ \forall u \in V(A(T_n))$.

Case (iii): n is odd and the triangle starts from the first vertex.
Suppose $n = 3$ and the triangle starts from the first vertex u_1.
Let $f : E(A(T_n)) \to N_4$ be any bijection.
Now define $g : E(A(T_n)) \to D_4 \setminus \{\rho_0\}$ by
$g(u_1u_2) = g(u_2v_1) = \delta_2,\ g(u_1v_1) = \delta_1,\ g(u_2u_3) = \rho_2$. Using these maps we can show that the graph is D_4-magic with magic constant ρ_2.
When n is odd and $n > 3$, there are $n + \frac{(n-1)}{2}$ vertices and $2(n-1)$ edges in $A(T_n)$. Suppose that $n > 3,\ n$ is odd and the triangle of $A(T_n)$ starts from the first vertex u_1. Here we take f as any bijection and $g : E(A(T_n)) \to D_4 \setminus \{\rho_0\}$ be defined as follows:

$$g(u_1u_2) = g(u_2v_1) = \delta_2,\ g(u_1v_1) = \delta_1,$$
$$g(u_iu_{i+1}) = \begin{cases} \rho_2, & \text{if } i \text{ is even,} \\ \rho_3, & \text{if } i \text{ is odd,} \end{cases} \quad , 1 < i < n.$$
$$g(u_iv_i) = g(u_{i+1}v_i) = \rho_1, 1 < i < n \text{ and } i \text{ is odd.}$$

Then clearly $l^*(u) = \rho_2,\ \forall v \in V(A(T_n))$

Case (iv): n is odd and triangle starts from the second vertex.
$A(T_n)$ with n odd and the triangle starts from the first vertex is just the mirror image of the $A(T_n)$ in Case (iii). So we can define f and g similarly as in Case(iii) and obtain a D_4-magic labeling for $A(T_n)$ with magic constant ρ_2.

This completes the proof of the theorem.

A double triangular snake $D(T_n)$ consists of two triangular snakes that have a common path P_n.

Theorem 4.5. *The double triangular graph is D_4-magic.*

Proof. Let $u_1, u_2, \ldots, u_n$ be the vertices of the path P_n and let $v_1, v_2, \ldots, v_{n-1}$, $w_1, w_2, \ldots, w_{n-1}$ be the remaining vertices of $D(T_n)$ such that the vertex v_i is adjacent to u_i and u_{i+1}, where $1 \leq i < n$. Similarly the vertex w_i is adjacent to u_i and u_{i+1}. Without loss of generality let $v_1, v_2, \ldots, v_{n-1}$ and $w_1, w_2, \ldots, w_{n-1}$ be the vertices of upper triangles and lower triangles respectively.
Now we define a D_4-magic labeling for $D(T_n)$ as follows:
Let $f : E(D(T_n)) \to N_{5(n-1)}$ be any bijection and let $g : E(D(T_n)) \to D_4 \setminus \{\rho_0\}$ be defined by $g(u_iu_{i+1}) = \rho_2,\ g(u_iv_i) = g(u_iw_i) = \rho_1,$ and $g(u_{i+1}v_i) = g(u_{i+1}w_i) = \rho_3$ where $1 \leq i < n$. Thus we can see that $l^*(u) = \rho_0,\ \forall u \in V(D(T_n))$. This completes the proof.

5 Conclusions

In this paper, we introduced the concept of A-magic labeling of graphs, where A is a nonabelian group. Furthermore, we characterised graphs which are D_4 magic.

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