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A characterization of strong fuzzy diameter zero in intuitionistic fuzzy metric spaces

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Abstract

The idea of intuitionistic fuzzy metric space introduced by Park (2004). In this paper, we introduce the notion of strong intuitionistic fuzzy diameter zero for a family of subsets based on the intuitionistic fuzzy diameter for a subset of A. Then we introduce nested sequence of subsets having strong intuitionistic fuzzy diameter zero using their intuitionistic fuzzy diameter.

Keywords: strong fuzzy diameter; intuitionistic fuzzy metric space; strong completeness.

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1 Introduction

The theory of fuzzy sets was introduced by L.A. Zadeh [17] in 1965. Kramosil and Michalek [6] introduced the fuzzy metric spaces (FM-spaces) by generalizing the concept of probabilistic metric spaces to fuzzy situation. George and Veeramani [4] modified the concept of fuzzy metric space introduced by Kramosil and Michalek [6] with a view to obtain a Hausdorff topology on fuzzy metric spaces which have very important applications in quantum particle particularly in connection with both string and E-infinity theory. In 2004, Park [8] defined the concept of intuitionistic fuzzy metric space with the help of continuous t-norms and continuous t-conorms.

Several researchers have shown interest in the intuitionistic fuzzy metric space successfully applied in many fields, it can be found in [5, 10, 11, 13, 14, 15, 16]. Theory of fuzzy sets have been widely used and developed in different fields of sciences, including mathematical programing, theory of modeling, theory of optimal control, theory of neural network, engineering and medical sciences, coloured image processing, etc.

In this paper, the concept of characterization of strong fuzzy diameter zero in intuitionistic fuzzy metric spaces are introduced and also discuss some properties of strong fuzzy diameter zero in intuitionistic fuzzy metric spaces.

2 Preliminaries

Definition 2.1[17] Let X be a nonempty set. A fuzzy set A in X is characterized by its membership function $\mu_A : X \to [0, 1]$ and $\mu_A(x)$ is interpreted as the degree of membership of element x in fuzzy set A for each $x \in X$. It is clear that A is completely determined by the set of tuples $A = \{(x, \mu_A(x)) | x \in X\}$.

Definition 2.2[4] The 3-tuple $(A, M, *)$ is said to be a fuzzy metric space if A be a non-empty set and $*$ be a continuous t-norm. A fuzzy set $A^2 \times (0, \infty)$ is called a fuzzy metric on A if $a, b, c \in A$ and $s, t > 0$, the following condition holds:

1. $M\ (a, b, t) = 0$
2. $M\ (a, b, t) = 1$ if and only if $a = b$

3. $M\ (a, b, t)\ =\ M(b, a, t\ )$
4. $M\ (a, b, t\ +\ s)\ \geq\ M(a, b, t) * M(a, b, s)$
5. $M\ (a, b,\ \bullet)\colon (0,\ \infty) \to [0, 1]$ is left continuous

The function $M(a, b, t)$ denote the degree of nearness between a and b with respect to t respectively.

Definition 2.3[1, 2] Let a set E be fixed. An intuitionistic fuzzy set A in E is an object of the following $A = \{(x, \mu_A(x), \upsilon_A(x)),\ x \in E\ \}$ where the functions $\mu_A(x)\colon E \to [0, 1]$ and $\upsilon_A\ (x\ )\colon E \to [0, 1]$ determine the degree of membership and the degree of non-membership of the element $x \in E$, respectively, and for every $x \in E\colon 0 \leq \mu_A(x) + \upsilon_A(x) \leq 1$, when $\upsilon_A(x) = 1 - \mu_A(x)$ for all $x \in E$ is an ordinary fuzzy set. In addition, for each IFS A in E, if $\pi_A(x) = 1 -\ \mu_A(x) - \upsilon_A(x)$. Then $\mu_A(x)$ is called the degree of indeterminacy of x to A or called the degree of hesitancy of x to A. It is obvious that $0 \leq \pi_A(x) \leq 1$, for each $x \in E$.

Definition 2.4 [7] A 5-tuple $(A, M, N, *, \circ)$ is said to be an intuitionistic fuzzy metric space if A is an arbitrary set, $*$ is a continuous t-norm, $\circ$ is a continuous t- conorm and, M, N are fuzzy sets on $A^2\ \times\ [0, \infty)$ satisfying the conditions:

1. $M(a, b, t) +\ N(a, b, t) \leq\ 1,$ for all $a, b \in\ A$ and $t > 0$
2. $M(a, b, 0) =\ 0,$ for all $a, b \in A$
3. $M(a, b, t) =\ 1$, for all $a, b \in A$ and $t > 0$ if and only if $a\ =\ b$
4. $M(a, b, t) =\ M(b, a, t)$, for all $a, b \in A$ and $t > 0$
5. $M(a, b, t) * M(b, c, s) \leq\ M(a, c, t + s)$, for all $a, b, c \in A$ and $s, t > 0$
6. $M(a, b, \bullet)\colon [0, \infty) \to [0, \infty]$ is left continuous for all $a, b \in A$
7. $\lim\limits_{t\to\infty} M(a, b, t) = 1$, for all $a, b \in A$ and $t > 0$
8. $N(a, b, 0) =\ 1$, for all a, $b \in A$
9. $N(a, b, t) =\ 0$, for all $a, b \in A$ and $t > 0$ if and only if $a = b$
10. $N(a, b, t) =\ N(b, a, t)$, for all $a, b \in A$ and $t > 0$
11. $N(a, b, t) \circ\ N(b, c, s) \geq\ N(a, c, t + s)$, for all $a, b, c \in A$ and $s, t > 0$
12. $N(a, b, \bullet)\colon [0, \infty) \to [0,1]$ is right continuous for all $a, b \in A$
13. $\lim\limits_{t\to\infty} N(a, b, t) = 0$, for all $a, b \in A$.

The functions $M(a, b, t)$ and $N(a, b, t)$ denote the degree of nearness and the degree of non-nearness between a and b w.r.t t respectively.

Definition 2.5 [9] The **fuzzy diameter** of a non-empty set B of a fuzzy metric space A, with respect to t, is the function $\varphi_B\colon (0, +\infty) \to\ [0, 1]$ given by $\varphi_B(t) = inf\{M(a, b, t)\colon a, b\ \in B\}$ for each $t \in R^+$.

Definition 2.6 [9] A collection of sets $\{B_i\}_{i\in I}$ is said to have fuzzy diameter zero if given $r \in (0,1)$ and $t \in R^+$ there exists $i \in I$ such that $\mathrm{M(a,b,t)} > 1 - \mathrm{r}$ for all $a,b \in B_i$.

3 Strong fuzzy diameter zero in intuitionistic fuzzy metric spaces

Definition 3.1 The fuzzy diameter of a non-empty set B of a intuitionistic fuzzy metric space $(A,M,N,*,\circ)$, with respect to t, is the function $\varphi_B:(0,+\infty) \to [0,1]$ given by $\varphi_B(t) = inf\{M(a,b,t): a,b \in B\}$ and $\psi_B:(0,+\infty) \to [0,1]$ given by $\psi_B(t) = sup\{N(a,b,t): a,b \in B\}$ for each $t \in R^+$

Definition 3.2 A collection of sets $\{B_i\}_{i\in I}$ of a intuitionistic fuzzy metric space $(A,M,N,*,\circ)$ is said to have fuzzy diameter zero if given $r \in (0,1)$ and $t \in R^+$ there exists $i \in I$ such that $M(a,b,t) > 1 - r \;\; N(a,b,t) < r$ for all $a,b \in B_i$.

Theorem 3.3 Let $\{B_n\}_{n\in\mathbb{N}}$ be a nested sequence of sets of the intuitionistic fuzzy metric space $(A,M,N,*,\circ)$. Then the following statements are equivalent:

(i) $\{B_n\}_{n\in\mathbb{N}}$ has fuzzy diameter zero.

(ii) $\lim_{n\to\infty} \varphi_{B_n}(t) = 1$, $\lim_{n\to\infty} \psi_{B_n}(t) = 0$ for all $t \in R^+$.

Proof:

(i)→(ii):

Let $t \in R^+$. Given $r \in (0,1)$ exists $n_{r,t} \in \mathbb{N}$ such that $M(a,b,t) > 1 - r, N(a,b,t) < r$ for each $a,b \in B_n$ with $n \geq n_{r,t}$.

Then, $\varphi_{B_n}(t) = inf\{M(a,b,t): a,b \in B_n\} \geq 1 - r$ and

$\psi_{B_n}(t) = sup\{N(a,b,t): a,b \in B_n\} \leq 1 - r$ for all $\mathrm{n} \geq \mathrm{n_{r,t}}$.

Hence,

$\lim_{n\to\infty} \varphi_{B_n}(t) = 1$ and $\lim_{n\to\infty} \psi_{B_n}(t) = 0$, since r is arbitrary in (0,1).

(ii)→(i):

Suppose $\lim_{n\to\infty} \varphi_{B_n}(t) = 1$ and $\lim_{n\to\infty} \psi_{B_n}(t) = 0$, for all $t \in R^+$.

Let $t \in R^+$ and let $r \in (0,1)$.

We can find $n_{r,t} \in \mathbb{N}$ such that $\varphi_{B_n}(t) > 1 - r$ and $\psi_{B_n}(t) < r$ for all $\geq n_{r,t}$. Thus, $M(a,b,t) > 1 - r$ and $N(a,b,t) < r$ for each $a,b \in B_n$ with $n \geq n_{r,t}$ i.e., $\{B_n\}_{n\in\mathbb{N}}$ has fuzzy diameter zero.

Definition 3.4 A family of non-empty sets $\{B_i\}_{i\in I}$ of a intuitionistic fuzzy metric space $(A, M, N,*,\circ)$ has strong fuzzy diameter zero if for $r \in (0,1)$ there exists $i \in I$ such that $M(a,b,t) > 1-r$ and $N(a,b,t) < r$ for each $a,b \in B_n$ and all $t \in R^+$.

Theorem 3.5 Let $(A, M, N,*,\circ)$ be an intuitionistic fuzzy metric space and let $\{B_n\}_{n\in\mathbb{N}}$ be a nested sequence of sets of A. Then the following statements are equivalent.

(i) $\{B_n\}_{n\in\mathbb{N}}$ has strong fuzzy diameter zero.

(ii) $\lim_{n\to\infty} \varphi_{B_n}(t_n) = 1$, $\lim_{n\to\infty} \psi_{B_n}(t_n) = 0$ for every decreasing and increasing sequence of positive real numbers $\{t_n\}_{n\in\mathbb{N}}$ that converges and diverges respectively.

Proof:

(i) → (ii): Let $\{t_n\}_{n\in\mathbb{N}}$ be a decreasing increasing sequence of positive real numbers that converges and diverges respectively. Given $r \in (0,1)$, we can find $n_r \in \mathbb{N}$ such that $M(a,b,t) > 1-r$ and $N(a,b,t) < r$ for each $\mathrm{a,b} \in \mathrm{B_n}$ with $n \geq n_r$ and all $t \in R^+$. In particular, $M(a,b,t_n) > 1-r$ and $N(a,b,t_n) < r$ for all $\mathrm{a,b} \in \mathrm{B_n}$ with $n \geq n_r$, i.e., $\varphi_{B_n}(t_n) \geq 1-r$, $\psi_{B_n}(t_n) \leq r$ for all $n \geq n_r$, i.e., $\lim_{n\to\infty} \varphi_{B_n}(t_n) = 1, \lim_{n\to\infty} \psi_{B_n}(t_n) = 0$.

(ii) → (i): Suppose that $\{B_n\}_{n\in\mathbb{N}}$ has not strong fuzzy diameter zero. Let $\mathrm{r} \in (0,1)$ such that $I = \{n \in \mathbb{N}: M(a,b,t) \leq 1-r,\ N(a,b,t) \geq r$ for some $a,b \in B_n$ and some $t \in R^+\}$, is infinite. Take $n_1 = \min I$. Then, there exist $a_{n_1}, b_{n_1} \in B_{n_1}$ such that $M(a_{n_1}, b_{n_1}, t_{n_1}) \leq 1-r, N(a_{n_1}, b_{n_1}, t_{n_1}) \geq r$ with $0 < t_{n_1} < 1$.

Take $n_2 > n_1$, with $n_2 \in \mathbb{N}$, such that $M(a_{n_1}, b_{n_1}, t_{n_1}) \leq 1-r, N(a_{n_1}, b_{n_1}, t_{n_1}) \geq r$ for some $a_{n_2}, b_{n_2} \in B_{n_2}$ and $0 < t_{n_2} < \min\{t_{n_1}, \frac{1}{2}\}$. In this way, we construct, by induction, a sequence $\{t_{n_i}\}_{i\in\mathbb{N}}$ such that $M(a_{n_i}, b_{n_i}, t_{n_i}) \leq 1-r, N(a_{n_i}, b_{n_i}, t_{n_i}) \geq r$ for some $a_{n_i}, b_{n_i} \in B_{n_i}, n_i \in \mathbb{N}$ with $n_i > n_{i-1}$ and $0 < t_{n_i} < \min\{t_{n_{i-1}}, \frac{1}{i}\}$.
Then,

$$\varphi_{B_{n_i}}(t_{n_i}) = \inf\{M(a,b,t_{n_i}): a,b \in B_{n_i}\} \leq 1-r,$$

$$\psi_{B_{n_i}}(t_{n_i}) = \sup\{N(a,b,t_{n_i}): a,b \in B_{n_i}\} \geq r \text{ for all } i \in \mathbb{N}.$$

Hence $\{\varphi_{B_{n_i}}(t_{n_i})\}_{i\in\mathbb{N}}$, $\{\psi_{B_{n_i}}(t_{n_i})\}_{i\in\mathbb{N}}$ does not converge and diverge respectively. Now, $\{t_{n_i}\}_{i\in\mathbb{N}}$ is a subsequence of the decreasing and increasing sequence $\{\mathrm{t_n}\}_{\mathrm{n}\in\mathbb{N}}$ that converges and diverges respectively, given by

$$t_n = \begin{cases} t_{n_1} & n \le n_1 \\ t_{n_{i+1}} & n_i \le n \le n_{i+1} \end{cases}$$

and the sequence $\{\varphi_{B_n}(t_n)\}_{n\in\mathbb{N}}$, $\{\psi_{B_n}(t_n)\}_{n\in\mathbb{N}}$ does not converge and diverge respectively. Thus, we get the contradiction.

Theorem 3.6 Let $\{B_n\}_{n\in\mathbb{N}}$ be a nested sequence of sets with fuzzy diameter zero in a intuitionistic fuzzy metric space $(A, M, N,*,\circ)$. $\{B_n\}_{n\in\mathbb{N}}$ has strong fuzzy diameter zero if and only if $\{B_n\}$ is a singleton set after a certain stage.

Proof:

Suppose $\{B_n\}_{n\in\mathbb{N}}$ is not eventually constant. Put $p_n = sup\{d(a,b): a,b \in B_n\}$, $q_n = inf\{d(a,b): a,b \in B_n\}$ and take $t_n = p_n$ and $t_n = q_n$ for all $n \in \mathbb{N}$. Then, $\{t_n\}_{n\in\mathbb{N}}$ is a decreasing and increasing sequence of positive real numbers converges and diverges respectively.

Then, $\lim_{n\to\infty} \varphi_{B_n}(t) = \lim_{n\to\infty} inf\{M_d(a,b,t_n): a,b \in B_n\}$

$$= \lim_{n\to\infty} \frac{t_n}{t_n + diam(B_n)}$$

$$= \lim_{n\to\infty} \frac{p_n}{p_n + p_n} = \frac{1}{2}$$

and

$$\lim_{n\to\infty} \psi_{B_n}(t) = \lim_{n\to\infty} sup\{N_d(a,b,t_n): a,b \in B_n\}$$

$$= \lim_{n\to\infty} \frac{t_n}{t_n + diam(B_n)}$$

$$= \lim_{n\to\infty} \frac{q_n}{q_n + q_n} = \frac{1}{2}$$

Hence $\{B_n\}_{n\in\mathbb{N}}$ has not strong fuzzy diameter zero.

Theorem 3.7 Let $(A, M, N,*,\circ)$ be a intuitionistic fuzzy metric space. If $\{B_n\}_{n\in\mathbb{N}}$ is a nested sequence of sets of A which has strong fuzzy diameter zero then $\{B_n\}_{n\in\mathbb{N}}$ has strong fuzzy diameter zero.

Proof:

First, we prove that $\varphi_{\bar{B}}(t) = \varphi_B(t)$, $\psi_{\bar{B}}(t) = \psi_B(t)$ for every subset B of A . Indeed, take $a, b \in B$. Then, we can find two sequences $\{a_n\}_{n\in\mathbb{N}}$ and

$\{b_n\}_{n\in\mathbb{N}}$ in B that converge to a and b, respectively. Let $t \in R^+$ and an arbitrary ε ∈ (0, 1). We have that

$$M(a,b,t+2\varepsilon) \geq M(a,b_n,\varepsilon) * M(a_n,b_n,t) * M(b_n,b,\varepsilon)$$
$$\geq M(a,a_n,\varepsilon) * \varphi_B(t) * M(b_n,b,\varepsilon),$$
$$N(a,b,t+2\varepsilon) \leq N(a,b_n,\varepsilon) \circ N(a_n,b_n,t) \circ N(b_n,b,\varepsilon)$$
$$\leq N(a,a_n,\varepsilon) \circ \psi_B(t) \circ N(b_n,b,\varepsilon)$$

and taking limit on the inequality when n tends to ∞, we obtain

$$M(a,b,t+2\varepsilon) \geq 1 * \varphi_B(t) * 1 = \varphi_B(t),$$
$$N(a,b,t+2\varepsilon) \leq 1 \circ \varphi_B(t) \circ 1 = \varphi_B(t).$$

Since ε is arbitrary, due to the continuity of $M(a,b,t), N(a,b,t)$ we obtain $M(a,b,t) \geq \varphi_B(t), N(a,b,t) \leq \psi_B(t)$ and then $\varphi_{\bar{B}}(t) \geq \varphi_B(t)$, $\psi_{\bar{B}}(t) \geq \psi_B(t)$. On the other hand, we have $\varphi_{\bar{B}}(t) \leq \varphi_B(t)$, $\psi_{\bar{B}}(t) \leq \psi_B(t)$ and hence $\varphi_{\bar{B}}(t) = \varphi_B(t)$, $\psi_{\bar{B}}(t) = \psi_B(t)$. Let $\{t_n\}_{n\in\mathbb{N}}$ be a decreasing and increasing sequence of positive real numbers convergent and divergent respectively. By theorem 3.5, we have that $\lim_{n\to\infty} \varphi_{B_n}(t_n) = 1$, $\lim_{n\to\infty} \psi_{B_n}(t_n) = 0$.

Then, by our last argument, we have that,

$$\lim_{n\to\infty} \varphi_{B_n}(t_n) = \lim_{n\to\infty} \varphi_{\bar{B}_n}(t_n) = 1,$$
$$\lim_{n\to\infty} \psi_{B_n}(t_n) = \lim_{n\to\infty} \psi_{\bar{B}_n}(t_n) = 0,$$

and consequently, by theorem 3.5, $\{B_n\}_{n\in\mathbb{N}}$ has strong fuzzy diameter zero.

4 Conclusion

Intuitionistic fuzzy set theory plays a vital role in uncertain situations in all aspects. In this paper, the characterizations of strong fuzzy diameter zero in intuitionistic fuzzy metric spaces are discussed and proved that the nested sequences having the strong fuzzy diameter zero in intuitionistic fuzzy metric space. We have also provided that nested sequences of subsets has strong fuzzy diameter zero if and only if singleton set after a certain stage in a intuitionistic fuzzy metric spaces.

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