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Insertion of terms satisfying the recurrence relations of Horadam sequence and Bifurcating Fibonacci sequences

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Abstract

In this article, we consider the problem of finding general formula for the terms introduced between two given positive integers 'a' and 'b' in such a way that the terms of newly formed finite sequence satisfy the recurrence relations of Horadam sequence and some bifurcating Fibonacci sequences.

Keywords: Generalized Fibonacci sequences, Horadam sequence, Bifurcating sequence, Inserted terms, Missing terms.
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1 Introduction

Consider any two fixed given positive integers a and b such that $a < b$. We insert k number of terms between a and b so that every term inserted between follows the given recurrence relation of any generalized Fibonacci sequence. Then we say x_n $(1 \le n \le k)$ to be the *inserted term* (or *missing term*, as defined by some of the authors) in the finite (generalized Fibonacci-like) sequence $a, x_1, x_2, x_3, \ldots., x_k, b$.

The problem of insertion of terms in arithmetic, harmonic, geometric sequence is considered to be elementary which occurs in the high school mathematics. However, not much work is done regarding the general formula for the inserted terms in a Fibonacci-like sequences. Howell [8] presented a proof for finding the n^{th} term of Fibonacci sequence using vectors and eigenvalues. But he said nothing for the Fibonacci-like sequences. Agnes, et al. [1] provided a formula for inclusion of three consecutive missing terms in Fibonacci-like sequence.

Horadam [7] defined a linear recurrence sequence of second order $F_n(p,q)$, acknowledged as *Horadam sequence*, by the recurrence relation $F_n^{(p,q)} = pF_{n-1}^{(p,q)} + qF_{n-2}^{(p,q)}$ with the initial conditions $F_0^{(p,q)} = a, F_1^{(p,q)} = c$, where a, c and p, q are arbitrary positive integers. First few terms of this sequence are shown in the Table 1.

$\boldsymbol{n}$	$\boldsymbol{F_n^{(p,q)}}$
0	a
1	c
2	$pc + qa$
3	$p^2c + pqa + qc$
4	$p^3c + p^2qa + 2pqc + q^2a$
5	$p^4c + p^3qa + 3p^2qc + 2pq^2a + q^2c$
6	$p^5c + p^4qa + 4p^3qc + 3p^2q^2a + 3pq^2c + q^3a$

Table 1

The Binet-type explicit formula for the terms of this sequence is given by $F_k^{(p,q)} = \frac{(c-a\beta)\alpha^k-(c-a\alpha)\beta^k}{\alpha-\beta}$, where $\alpha = \frac{p+\sqrt{p^2+4q}}{2}, \beta = \frac{p-\sqrt{p^2+4q}}{2}$. Here we note that $\alpha - \beta = \sqrt{p^2+4q},\ \alpha+\beta = p$ and $\alpha\beta = -q$.

Diwan, Shah [4,5] considered the sequence $\left\{F_n^{(p,q,r,s)}\right\}$ defined by the recurrence relation

$$F_n^{(p,q,r,s)} = p^{\chi(n)}q^{1-\chi(n)}F_{n-1}^{(p,q,r,s)} + r^{\chi(n)}s^{1-\chi(n)}F_{n-2}^{(p,q,r,s)}, \qquad (1.1)$$

where p,q,r,s are any fixed positive integers and $\chi(n) = \begin{cases} 1; \text{if } n \text{ is odd} \\ 0; \text{if } n \text{ is even} \end{cases}$. They studied this sequence extensively for some explicit values of p,q,r,s. Verma and Bala, Bilgici, Yayenie [2,9,10] also studied this sequence for some specific values of p,q,r,s. It is easy to observe that this sequence is bifurcating sequence depending on the parity of n.

In this paper we derive the general formula which gives the value of any inserted term $x_n\ (1 \le n \le k)$ for the sequence $\{F_n(p,q)\}$ and various bifurcating subsequence of $\left\{F_n^{(p,q,r,s)}\right\}$ by considering some fixed values of p,q,r,s. Throughout we assume that the positive integers a,b are the given first and last term respectively in the sequence to be considered.

2 Insertion of terms satisfying the recurrence relation of Horadam sequence

In this section, we find the formula for the inserted terms between the given fixed positive integers a,b so that terms of the sequence $a,x_1,x_2,x_3,\dots,x_k,b$ satisfies the recurrence relation of $F_n(p,q)$. Before finding the general formula, we find the formula for the first term x_1 in the finite sequence $a,x_1,x_2,x_3,\dots.,x_k,b$. This value of x_1 will be further used to find the general formula for any $x_n\ (1 \le n \le k)$.

In [7], Horadam obtained only the formula for the first missing term (inserted term) x_1 when k terms are inserted between given two positive integers a and b, so that all x_i's satisfy the recurrence relation of $F_n^{(p,q)}$. In fact, he proved that

$$x_1 = \frac{b - aqF_k^{(p,q)}}{F_{k+1}^{(p,q)}}. \tag{2.1}$$

In the following theorem, we obtain the general formula for any arbitrary term x_n $(1 \le n \le k)$ when k terms $x_1, x_2, x_3, \ldots, x_k$ are inserted between positive integers a and b, so that all x_i's satisfy the recurrence relation of $F_n^{(p,q)}$.

Theorem 2.1. When the terms $x_1, x_2, x_3, \ldots, x_k$ are inserted between given integers a and b in such a way that the finite sequence $a, x_1, x_2, x_3, \ldots, x_k, b$ satisfy the recurrence relation of $F_n^{(p,q)}$, then the general formula for any arbitrary term x_n $(1 \le n \le k)$ is given by

$$x_n = \frac{F_n^{(p,q)} b(\alpha-\beta) - aqF_n^{(p,q)}\left\{(c-a\beta)\alpha^k - (c-a\alpha)\beta^k\right\} + aqF_{n-1}^{(p,q)}\left\{(c-a\beta)\alpha^{k+1} - (c-a\alpha)\beta^{k+1}\right\}}{(c-a\beta)\alpha^{k+1} - (c-a\alpha)\beta^{k+1}}. \tag{2.2}$$

Proof. We prove the result by the principle of mathematical induction. For we $k = 1$, we get $n = 1$ and

$$x_1 = \frac{F_1^{(p,q)} b(\alpha-\beta) - aqF_1^{(p,q)}\{(c-a\beta)\alpha - (c-a\alpha)\beta\} + aqF_0^{(p,q)}\left\{(c-a\beta)\alpha^2 - (c-a\alpha)\beta^2\right\}}{(c-a\beta)\alpha^2 - (c-a\alpha)\beta^2} = \frac{b-aqc}{cp+aq},$$

which is same as (2.1).

Next, we assume that (2.2) holds for some positive integer not exceeding n. Then both the following results hold:

$$x_{n-1} = \frac{F_{n-1}^{(p,q)} b(\alpha-\beta) - aqF_{n-1}^{(p,q)}\left\{(c-a\beta)\alpha^k - (c-a\alpha)\beta^k\right\} + aqF_{n-2}^{(p,q)}\left\{(c-a\beta)\alpha^{k+1} - (c-a\alpha)\beta^{k+1}\right\}}{(c-a\beta)\alpha^{k+1} - (c-a\alpha)\beta^{k+1}}$$

and

$$x_n = \frac{F_n^{(p,q)} b(\alpha-\beta) - aqF_n^{(p,q)}\left\{(c-a\beta)\alpha^k - (c-a\alpha)\beta^k\right\} + aqF_{n-1}^{(p,q)}\left\{(c-a\beta)\alpha^{k+1} - (c-a\alpha)\beta^{k+1}\right\}}{(c-a\beta)\alpha^{k+1} - (c-a\alpha)\beta^{k+1}}$$

Now consider

$$px_n + qx_{n-1} = \frac{\left(pF_n^{(p,q)} + qF_{n-1}^{(p,q)}\right) b(\alpha-\beta) - aq\left(pF_n^{(p,q)} + qF_{n-1}^{(p,q)}\right)\{(c-a\beta)\alpha^k - (c-a\alpha)\beta^k\} + aq\left(pF_{n-1}^{(p,q)} + qF_{n-2}^{(p,q)}\right)\{(c-a\beta)\alpha^{k+1} - (c-a\alpha)\beta^{k+1}\}}{(c-a\beta)\alpha^{k+1} - (c-a\alpha)\beta^{k+1}}$$

$$= \frac{F_{n+1}^{(p,q)} b(\alpha-\beta) - aqF_{n+1}^{(p,q)}\{(c-a\beta)\alpha^k - (c-a\alpha)\beta^k\} + aqF_n^{(p,q)}\{(c-a\beta)\alpha^{k+1} - (c-a\alpha)\beta^{k+1}\}}{(c-a\beta)\alpha^{k+1} - (c-a\alpha)\beta^{k+1}}.$$

It can be observed that the right side of this result simplifies to x_{n+1}. Thus, by (2.2) is true for every positive integer n.

3 Insertion of terms satisfying the recurrence relation of bifurcating sequence $\boldsymbol{F_n^{(p,q,1,1)}}$

In this section, we find the formula for the inserted terms between the numbers a and b so that terms of the sequence $a, x_1, x_2, x_3, \dots, x_k, b$ satisfies the recurrence relation (1.2).

If we let $p = q = r = s = 1$, the sequence $\left\{F_n^{(p,q,r,s)}\right\}$ is the sequence of usual Fibonacci numbers. If we define $F_0^{(p,q,r,s)} = 0, F_1^{(p,q,r,s)} = 1$, then first few terms of this sequence are shown in Table 2.

$\boldsymbol{n}$	$\boldsymbol{F_n^{(p,q,r,s)}}$
0	0
1	1
2	q
3	$pq + r$
4	$pq^2 + qr + qs$
5	$p^2q^2 + 2pqr + pqs + r^2$
6	$p^2q^3 + 2pq^2r + 2pq^2s + r^2q + sqr + qs^2$

Table 2

If we consider $r = s = 1$ in (1.1), we get the sequence $\left\{F_n^{(p,q,1,1)}\right\}$ whose terms are defined by the recurrence relation

$$F_n^{(p,q,1,1)} = p^{\chi(n)} q^{1-\chi(n)} F_{n-1}^{(p,q,1,1)} + F_{n-2}^{(p,q,1,1)},$$

where $F_0^{(p,q,1,1)} = 0, F_1^{(p,q,1,1)} = 1$. This can be written in the form

$$F_n^{(p,q,1,1)} = \begin{cases} pF_{n-1}^{(p,q,1,1)} + F_{n-2}^{(p,q,1,1)} \text{ ; when } n \text{ is odd} \\ qF_{n-1}^{(p,q,1,1)} + F_{n-2}^{(p,q,1,1)} \text{ ; when } n \text{ is even} \end{cases} (n \geq 2) \tag{3.1}$$

with the initial conditions $F_0^{(p,q,1,1)} = 0, F_1^{(p,q,1,1)} = 1$. First few terms of this sequence are shown in Table 3.

This sequence was studied in detail by Diwan, Shah [4] as well as Edson, Yayenie [6]. They derived the Binet-type explicit formula for the terms of this sequence as

$$F_k^{(p,q,1,1)} = \frac{q^{1-\chi(k)}}{(pq)^{\left\lfloor \frac{k}{2} \right\rfloor}} \left(\frac{\alpha^k - \beta^k}{\alpha - \beta}\right), \text{ where } \alpha = \frac{pq + \sqrt{p^2q^2 + 4pq}}{2}, \beta = \frac{pq - \sqrt{p^2q^2 + 4pq}}{2}.$$

Here we note that $\alpha - \beta = \sqrt{p^2q^2 + 4pq},\ \alpha + \beta = pq$ and $\alpha\beta = -pq$.

n	$F_n^{(p,q,1,1)}$
0	0
1	1
2	q
3	$pq+1$
4	pq^2+2q
5	$p^2q^2+3pq+1$
6	$p^2q^3+4pq^2+3q$

Table 3

When we consider the sequence a, x_1, b, where x_1 is the only inserted term between a and b, then using $F_2^{(p,q,1,1)} = qF_1^{(p,q,1,1)} + F_0^{(p,q,1,1)}$, we get $b = qx_1 + a$. Thus $x_1 = \frac{b-a}{q}$.

When we consider the finite sequence a, x_1, x_2, b, so that it satisfies (3.1), we observe that $x_1 = \frac{x_2-a}{q}$ and $x_2 = \frac{b-x_1}{p}$. This gives $x_2 = \frac{bq+a}{pq+1}$ and $x_1 = \frac{b-ap}{pq+1}$.

Further, considering the sequence a, x_1, x_2, x_3, b, it is now easy to observe that $x_1 = \frac{x_2-a}{q}, x_2 = \frac{x_3-x_1}{p}$ and $x_3 = \frac{\mathrm{b}-x_2}{q}$. Solving these three equations in three variables x_1, x_2, x_3 we get $x_1 = \frac{b-a(pq+1)}{pq^2+2q}, x_2 = \frac{bq+aq}{pq^2+2q}, x_3 = \frac{b-a+bpq}{pq^2+2q}$.

If we continue extending the above finite sequence one more time, then we get the sequence a, x_1, x_2, x_3, x_4, b. Using the similar approach as above, we obtain

$x_1 = \frac{b-a(p^2q+2p)}{p^2q^2+3pq+1}, x_2 = \frac{bq+apq+a}{p^2q^2+3pq+1}, x_3 = \frac{b-ap+bpq}{p^2q^2+3pq+1}$ and $x_4 = \frac{a+2bq+bpq^2}{p^2q^2+3pq+1}$.

We mention these results in the Table 4.

Number of inserted terms	Formula			
	x_1	x_2	x_3	x_4
1	$\frac{b-a}{q}$	---	---	---
2	$\frac{b-ap}{pq+1}$	$\frac{bq+a}{pq+1}$	---	---
3	$\frac{b-a(pq+1)}{pq^2+2q}$	$\frac{bq+aq}{pq^2+2q}$	$\frac{b-a+bpq}{pq^2+2q}$	---
4	$\frac{b-a(p^2q+2p)}{p^2q^2+3pq+1}$	$\frac{bq+apq+a}{p^2q^2+3pq+1}$	$\frac{b-ap+bpq}{p^2q^2+3pq+1}$	$\frac{a+2bq+bpq^2}{p^2q^2+3pq+1}$

Table 4

From this table, we observe that there is a similar pattern for the first inserted term in case of any number of inserted terms. We thus generalize it for the case of any number of inserted terms and when the terms $x_1, x_2, x_3, \ldots, x_k$ are inserted between given two positive integers a and b, we can now write the general formula for x_1 as

$$x_1 = \frac{b - ap^{1-\chi(k)} q^{\chi(k)-1} F_k^{(p,q,1,1)}}{F_{k+1}^{(p,q,1,1)}} \,. \tag{3.2}$$

Next, when the terms $x_1, x_2, x_3, \ldots, x_k$ are inserted between given two positive integers a and b, we obtain the general formula for n^{th} inserted term $(1 \leq n \leq k)$ using the recurrence relation (3.1).

Theorem 3.1. When the terms $x_1, x_2, x_3, \ldots, x_k$ are inserted between given integers a and b, so that all x_i's satisfy the recurrence relation of $F_n^{(p,q,1,1)}$, the general formula for any arbitrary term x_n $(1 \leq n \leq k)$ is given by

$$x_n = \frac{F_n^{(p,q,1,1)} b - aF_n^{(p,q,1,1)} p^{1-\chi(k)} q^{\chi(k)-1} F_k^{(p,q,1,1)} + ap^{\chi(n)} q^{-\chi(n)} F_{n-1}^{(p,q,1,1)} F_{k+1}^{(p,q,1,1)}}{F_{k+1}^{(p,q,1,1)}}.$$

Proof. We use the principle of mathematical induction to prove the result. Since by (3.2), we have $x_1 = \frac{b - ap^{1-\chi(k)} q^{\chi(k)-1} F_k^{(p,q,1,1)}}{F_{k+1}^{(p,q,1,1)}}$, which proves the result for $n = 1$. We next assume that it is true for all positive integers not exceeding n. Then the following holds:

$$x_{n-1} = \frac{F_{n-1}^{(p,q,1,1)} b - aF_{n-1}^{(p,q,1,1)} p^{1-\chi(k)} q^{\chi(k)-1} F_k^{(p,q,1,1)} + ap^{\chi(n-1)} q^{-\chi(n-1)} F_{n-2}^{(p,q,1,1)} F_{k+1}^{(p,q,1,1)}}{F_{k+1}^{(p,q,1,1)}}$$

$$x_{n-2} = \frac{F_{n-2}^{(p,q,1,1)} b - aF_{n-2}^{(p,q,1,1)} p^{1-\chi(k)} q^{\chi(k)-1} F_k^{(p,q,1,1)} + ap^{\chi(n-2)} q^{-\chi(n-2)} F_{n-3}^{(p,q,1,1)} F_{k+1}^{(p,q,1,1)}}{F_{k+1}^{(p,q,1,1)}}.$$

Now, $p^{\chi(n)} q^{1-\chi(n)} x_{n-1} + x_{n-2}$

$$= \frac{\left\{ \begin{array}{c} \left(p^{\chi(n)} q^{1-\chi(n)} F_{n-1}^{(p,q,1,1)} + F_{n-2}^{(p,q,1,1)}\right) b - a\left(p^{\chi(n)} q^{1-\chi(n)} F_{n-1}^{(p,q,1,1)} + F_{n-2}^{(p,q,1,1)}\right) \\ \times p^{1-\chi(k)} q^{\chi(k)-1} F_k^{(p,q,1,1)} \\ + a\left(p^{\chi(n)} q^{1-\chi(n)} p^{\chi(n-1)} q^{-\chi(n-1)} F_{n-2}^{(p,q,1,1)} + p^{\chi(n-2)} q^{-\chi(n-2)} F_{n-3}^{(p,q,1,1)}\right) F_{k+1}^{(p,q,1,1)} \end{array} \right\}}{F_{k+1}^{(p,q,1,1)}}$$

$$= \frac{\left\{ \begin{array}{c} F_n^{(p,q,1,1)} b - aF_n^{(p,q,1,1)} p^{1-\chi(k)} q^{\chi(k)-1} F_k^{(p,q,1,1)} \\ + a\left(p^{\chi(n)+\chi(n-1)} q^{1-(\chi(n)+\chi(n-1))} F_{n-2}^{(p,q,1,1)} + p^{\chi(n)} q^{-\chi(n)} F_{n-3}^{(p,q,1,1)}\right) F_{k+1}^{(p,q,1,1)} \end{array} \right\}}{F_{k+1}^{(p,q,1,1)}}$$

$$= \frac{\left\{ \begin{array}{c} F_n^{(p,q,1,1)} b - aF_n^{(p,q,1,1)} p^{1-\chi(k)} q^{\chi(k)-1} F_k^{(p,q,1,1)} \\ +a\left(pF_{n-2}^{(p,q,1,1)} + p^{\chi(n)} q^{-\chi(n)} F_{n-3}^{(p,q,1,1)}\right) F_{k+1}^{(p,q,1,1)} \end{array} \right\}}{F_{k+1}^{(p,q,1,1)}}.$$

Now, when n is odd, we get

$$\begin{aligned} & pF_{n-2}^{(p,q,1,1)} + p^{\chi(n)} q^{-\chi(n)} F_{n-3}^{(p,q,1,1)} \\ &= p^{\chi(n)} q^{1-\chi(n)} F_{n-2}^{(p,q,1,1)} + p^{\chi(n)} q^{-\chi(n)} F_{n-3}^{(p,q,1,1)} \\ &= p^{\chi(n)} q^{-\chi(n)} \left(qF_{n-2}^{(p,q,1,1)} + F_{n-3}^{(p,q,1,1)}\right) \\ &= p^{\chi(n)} q^{-\chi(n)} \left(p^{\chi(n-1)} q^{1-\chi(n-1)} F_{n-2}^{(p,q,1,1)} + F_{n-3}^{(p,q,1,1)}\right) \\ &= p^{\chi(n)} q^{-\chi(n)} F_{n-1}^{(p,q,1,1)}. \end{aligned}$$

Also, when n is even, we get

$$\begin{aligned} & pF_{n-2}^{(p,q,1,1)} + p^{\chi(n)} q^{-\chi(n)} F_{n-3}^{(p,q,1,1)} \\ &= p^{1-\chi(n)} q^{\chi(n)} F_{n-2}^{(p,q,1,1)} + p^{\chi(n)} q^{-\chi(n)} F_{n-3}^{(p,q,1,1)} \\ &= p^{\chi(n)} q^{-\chi(n)} \left(p^{1-2\chi(n)} q^{2\chi(n)} F_{n-2}^{(p,q,1,1)} + F_{n-3}^{(p,q,1,1)}\right) \\ &= p^{\chi(n)} q^{-\chi(n)} \left(p^{\chi(n-1)} q^{1-\chi(n-1)} F_{n-2}^{(p,q,1,1)} + F_{n-3}^{(p,q,1,1)}\right) \\ &= p^{\chi(n)} q^{-\chi(n)} F_{n-1}^{(p,q,1,1)} \end{aligned}$$

Therefore, we have

$$x_n = p^{\chi(n)} q^{1-\chi(n)} x_{n-1} + x_{n-2} = \frac{\left\{ \begin{array}{c} F_n^{(p,q,1,1)} b - aF_n^{(p,q,1,1)} p^{1-\chi(k)} q^{\chi(k)-1} F_k^{(p,q,1,1)} \\ +a\left(p^{\chi(n)} q^{-\chi(n)} F_{n-1}^{(p,q,1,1)}\right) F_{k+1}^{(p,q,1,1)} \end{array} \right\}}{F_{k+1}^{(p,q,1,1)}},$$

which proves the result for every positive integer n.

4 Insertion of terms satisfying the recurrence relation of bifurcating sequence $F_n^{(p,1,1,s)}$

If we consider $q = r = 1$ in (1.1), the sequence $\left\{F_n^{(p,1,1,s)}\right\}$ whose terms are defined by the recurrence relation

$$F_n^{(p,1,1,s)} = p^{\chi(n)} F_{n-1}^{(p,1,1,s)} + s^{1-\chi(n)} F_{n-2}^{(p,1,1,s)},$$

where $F_0^{(p,1,1,s)} = 0, F_1^{(p,1,1,s)} = 1$. This can be written in the form

$$F_n^{(p,1,1,s)} = \begin{cases} pF_{n-1}^{(p,1,1,s)} + F_{n-2}^{(p,1,1,s)} & ;\text{when } n \text{ is odd} \\ F_{n-1}^{(p,1,1,s)} + sF_{n-2}^{(p,1,1,s)} & ;\text{when } n \text{ is even} \end{cases} (n \geq 2) \qquad (4.1)$$

with the initial conditions $F_0^{(p,1,1,s)} = 0, F_1^{(p,1,1,s)} = 1$. First few terms of this sequence are shown in Table 5.

n	$F_n^{(p,1,1,s)}$
0	0
1	1
2	1
3	$1+p$
4	$1+p+s$
5	$1+2p+ps+p^2$
6	$1+s+2p+2ps+p^2+s^2$

Table 5

This sequence was studied by Diwan, Shah [4]. They obtained Binet-type explicit formula for the terms of this sequence as

$$F_n^{(p,1,1,s)} = \frac{(\alpha-s)^{\chi(n)}\alpha^{\left[\frac{n}{2}\right]}-(\beta-s)^{\chi(n)}\beta^{\left[\frac{n}{2}\right]}}{\alpha-\beta},$$

where $\alpha = \frac{(p+s+1)+\sqrt{(p+s+1)^2-4s}}{2}, \beta = \frac{(p+s+1)-\sqrt{(p+s+1)^2-4s}}{2}$. This gives $\alpha-\beta = \sqrt{(p+s+1)^2-4s}, \alpha+\beta = p+s+1$ and $\alpha\beta = s$.

In this section, we find the formula for the inserted terms between the numbers a and b so that terms of the sequence $a, x_1, x_2, x_3, \dots, x_k, b$ satisfies the recurrence relation (4.1).

When we consider the sequence a, x_1, b, by using $F_2^{(p,1,1,s)} = F_1^{(p,1,1,s)} + sF_0^{(p,1,1,s)}$, we get $b = x_1 + sa$. Thus $x_1 = b - sa$. This basic formula will be used to find the other terms.

When we consider the finite sequence as a, x_1, x_2, b, so that it satisfies (4.1), we observe that $x_1 = x_2 - s$ and $x_2 = \frac{b-x_1}{p}$. This gives $x_2 = \frac{b+sa}{1+p}$ and $x_1 = \frac{b-sap}{1+p}$.

Further, considering the sequence a, x_1, x_2, x_3, b, it is now easy to observe that $x_1 = x_2 - sa, x_2 = \frac{x_3-x_1}{p}$ and $x_3 = b - sx_2$. Solving these three equations in three variables x_1, x_2, x_3 we get $x_1 = \frac{b-as(p+s)}{1+p+s}, x_2 = \frac{b+sa}{1+p+s}, x_3 = \frac{b+bp-as^2}{1+p+s}$.

If we continue extending the above finite sequence one more time, then we get the sequence a, x_1, x_2, x_3, x_4, b. Using the similar approach as above, we obtain $x_1 = \frac{b-sa(p^2+p+ps)}{1+2p+ps+p^2}, x_2 = \frac{b+sap+sa}{1+2p+ps+p^2}, x_3 = \frac{b+pb-s^2pa}{1+2p+ps+p^2}$ and $x_4 = \frac{b+pb+sb+s^2a}{1+2p+ps+p^2}$.

We mention these results in the Table 6.

Number of inserted terms	Formula			
	x_1	x_2	x_3	x_4
1	$b - sa$	---	---	---
2	$\frac{b - sap}{1 + p}$	$\frac{b + sa}{1 + p}$	---	---
3	$\frac{b - sa(p + s)}{1 + p + s}$	$\frac{b + sa}{1 + p + s}$	$\frac{b + bp - as^2}{1 + p + s}$	---
4	$\frac{b - sa(p^2 + p + ps)}{1 + 2p + ps + p^2}$	$\frac{b + sap + sa}{1 + 2p + ps + p^2}$	$\frac{b + pb - s^2pa}{1 + 2p + ps + p^2}$	$\frac{b + pb + sb + s^2a}{1 + 2p + ps + p^2}$

Table 6

From this Table, we observe that there is a similar pattern for the first inserted term in case of any number of inserted terms. We thus generalize it for the case of k number of inserted terms and we can now write the general formula for x_1 as

$$x_1 = \frac{b-sa\left\{F_{k+1}^{(p,1,1,s)} - F_{k-1}^{(p,1,1,s)}\right\}}{F_{k+1}^{(p,1,1,s)}}. \tag{4.2}$$

Next, when the terms $x_1, x_2, x_3, \dots, x_k$ are inserted between given two positive integers a and b, we obtain the general formula for n^{th} inserted term $(1 \leq n \leq k)$ using the recurrence relation (4.1).

Theorem 4.1. When the terms $x_1, x_2, x_3, \dots, x_k$ are inserted between given integers a and b, so that all x_i's satisfy the recurrence relation of $F_n^{(p,1,1,s)}$, the general formula for any arbitrary term x_n $(1 \leq n \leq k)$ is given by

$$x_n = \frac{F_n^{(p,1,1,s)} b + saF_n^{(p,1,1,s)} F_{k-1}^{(p,1,1,s)} - saF_{n-2}^{(p,1,1,s)} F_{k+1}^{(p,1,1,s)}}{F_{k+1}^{(p,1,1,s)}}.$$

Proof. We use the principle of mathematical induction to prove the result. Since by (4.2), we have $x_1 = \frac{b-sa\left\{F_{k+1}^{(p,1,1,s)} - F_{k-1}^{(p,1,1,s)}\right\}}{F_{k+1}^{(p,1,1,s)}}$, which proves the result for $n = 1$. We next assume that it is true for all positive integers not exceeding n. Then the following holds:

$x_{n-1} = \frac{F_{n-1}^{(p,1,1,s)}b + saF_{n-1}^{(p,1,1,s)}F_{k-1}^{(p,1,1,s)} - saF_{n-3}^{(p,1,1,s)}F_{k+1}^{(p,1,1,s)}}{F_{k+1}^{(p,1,1,s)}}$ and

$x_{n-2} = \frac{F_{n-2}^{(p,1,1,s)}b + saF_{n-2}^{(p,1,1,s)}F_{k-1}^{(p,1,1,s)} - saF_{n-4}^{(p,1,1,s)}F_{k+1}^{(p,1,1,s)}}{F_{k+1}^{(p,1,1,s)}}$.

Now, $p^{\chi(n)}x_{n-1} + s^{1-\chi(n)}x_{n-2}$

$$= \frac{\left\{\begin{array}{c}\left(p^{\chi(n)}F_{n-1}^{(p,1,1,s)} + s^{1-\chi(n)}F_{n-2}^{(p,1,1,s)}\right)b + sa\left(p^{\chi(n)}F_{n-1}^{(p,1,1,s)} + s^{1-\chi(n)}F_{n-2}^{(p,1,1,s)}\right)F_{k-1}^{(p,1,1,s)} \\ -sa\left(p^{\chi(n)}F_{n-3}^{(p,1,1,s)} + s^{1-\chi(n)}F_{n-4}^{(p,1,1,s)}\right)F_{k+1}^{(p,1,1,s)}\end{array}\right\}}{F_{k+1}^{(p,1,1,s)}}$$

$$= \frac{\left\{F_n^{(p,1,1,s)}b + saF_n^{(p,1,1,s)}F_{k-1}^{(p,1,1,s)} - sa\left(\begin{array}{c}p^{\chi(n-2)}F_{n-3}^{(p,1,1,s)} \\ +s^{1-\chi(n-2)}F_{n-4}^{(p,1,1,s)}\end{array}\right)F_{k+1}^{(p,1,1,s)}\right\}}{F_{k+1}^{(p,1,1,s)}}$$

$= \frac{F_n^{(p,1,1,s)}b + saF_n^{(p,1,1,s)}F_{k-1}^{(p,1,1,s)} - saF_{n-2}^{(p,1,1,s)}F_{k+1}^{(p,1,1,s)}}{F_{k+1}^{(p,1,1,s)}} = x_n$, which proves the result for every positive integer n.

5 Insertion of terms satisfying the recurrence relation of bifurcating sequence $F_n^{(1,q,r,1)}$

If we consider $p = s = 1$ in (1.1), then we get the sequence $\left\{F_n^{(1,q,r,1)}\right\}$ whose terms are defined by the recurrence relation

$$F_n^{(1,q,r,1)} = q^{\chi(n)}F_{n-1}^{(1,q,r,1)} + r^{1-\chi(n)}F_{n-2}^{(1,q,r,1)},$$

where $F_0^{(1,q,r,1)} = 0, F_1^{(1,q,r,1)} = 1$.This can be written in the form

$$F_n^{(1,q,r,1)} = \begin{cases} F_{n-1}^{(1,q,r,1)} + rF_{n-2}^{(1,q,r,1)} \text{ ; when } n \text{ is odd} \\ qF_{n-1}^{(1,q,r,1)} + F_{n-2}^{(1,q,r,1)} \text{ ; when } n \text{ is even} \end{cases} (n \geq 2) \qquad (5.1)$$

with the initial conditions $F_0^{(1,q,r,1)} = 0, F_1^{(1,q,r,1)} = 1$. First few terms of this sequence are shown in Table 7.

Diwan, Shah [4] studied this sequence and obtained the Binet-type explicit formula for the terms of this sequence as

$$F_n^{(1,q,r,1)} = q^{1-\chi(n)}\left\{\frac{(\alpha-1)^{\chi(n)}\alpha^{\left[\frac{n}{2}\right]} - (\beta-1)^{\chi(n)}\beta^{\left[\frac{n}{2}\right]}}{\alpha-\beta}\right\},$$

where $\alpha = \frac{(r+q+1)+\sqrt{(r+q+1)^2-4q}}{2}, \beta = \frac{(r+q+1)-\sqrt{(r+q+1)^2-4q}}{2}$ with $\alpha - \beta = \sqrt{(r+q+1)^2-4q}, \alpha + \beta = r + q + 1, \alpha\beta = r$.

n	$F_n^{(1,q,r,1)}$
0	0
1	1
2	q
3	$r+q$
4	$q+rq+q^2$
5	$q+2qr+r^2+q^2$
6	$q+rq+2rq^2+2q^2+r^2q+q^3$

Table 7

In this section, we find the formula for the inserted terms between the numbers a and b so that terms of the sequence $a, x_1, x_2, x_3, \dots, x_k, b$ satisfies the recurrence relation (5.1).

When we consider the sequence a, x_1, b, using $F_2^{(1,q,r,1)} = qF_1^{(1,q,r,1)} + F_0^{(1,q,r,1)}$, we get $b = qx_1 + a$. Thus $x_1 = \frac{b-a}{q}$.

When we consider the sequence as a, x_1, x_2, b, so that it satisfies (5.1), we observe that $x_1 = \frac{x_2-a}{q}$ and $x_2 = b - rx_1$. This gives $x_2 = \frac{qb+ra}{r+q}$ and $x_1 = \frac{b-a}{r+q}$.

Further, considering the sequence a, x_1, x_2, x_3, b, it is now easy to observe that $x_1 = \frac{x_2-a}{q}$, $x_2 = x_3 - rx_1$, $x_2 = x_3 - rx_1$ and $x_3 = \frac{b-x_2}{q}$. Solving these three equations in three variables x_1, x_2, x_3 we get $x_1 = \frac{b-a(q+1)}{q^2+q+rq}$, $x_2 = \frac{bq+rqa}{q^2+q+rq}$, $x_3 = \frac{bq+br-ra}{q^2+q+rq}$.

If we continue extending the above finite sequence one more time, then we get the sequence a, x_1, x_2, x_3, x_4, b. Using the similar approach as above, we obtain

$$x_1 = \frac{b-a(1+r+q)}{q^2+r^2+2rq+q}, x_2 = \frac{bq+ar^2+arq}{q^2+r^2+2rq+q}, x_3 = \frac{bq+rb-ar}{q^2+r^2+2rq+q}, x_4 = \frac{bq^2+rbq+qb+ar^2}{q^2+r^2+2rq+q}.$$

We mention these results in the Table 8. From this table, we observe that there is a similar pattern for the first inserted term in case of any number of inserted terms. We thus generalize it for the case of any number of inserted terms and when the terms $x_1, x_2, x_3, \dots, x_k$ are inserted between given two positive integers a and b, we can now write the general formula for x_1 as

$$x_1 = \frac{b - a\left\{\left[\left(\frac{1-r}{q}\right)F_{k-1}^{(1,q,r,1)} + F_k^{(1,q,r,1)}\right]^{\chi(k)}\left[\frac{1}{q}F_k^{(1,q,r,1)}\right]^{1-\chi(k)}\right\}}{F_{k+1}^{(1,q,r,1)}}. \qquad (5.2)$$

Number of inserted term	Formula			
	x_1	x_2	x_3	x_4
1	$\frac{b-a}{q}$	---	---	---
2	$\frac{b-a}{r+q}$	$\frac{qb+ra}{r+q}$	---	---
3	$\frac{b-a(q+1)}{q^2+q+rq}$	$\frac{bq+rqa}{q^2+q+rq}$	$\frac{bq+br-ra}{q^2+q+rq}$	---
4	$\frac{b-a(1+r+q)}{q^2+r^2+2rq+q}$	$\frac{bq+ar^2+arq}{q^2+r^2+2rq+q}$	$\frac{bq+rb-ar}{q^2+r^2+2rq+q}$	$\frac{bq^2+rbq+qb+ar^2}{q^2+r^2+2rq+q}$

Table 8

Next, when the terms $x_1, x_2, x_3, \dots, x_k$ are inserted between given two positive integers a and b, we obtain the general formula for n^{th} inserted term $(1 \le n \le k)$ using the recurrence relation (5.1).

Theorem 5.1 When the terms $x_1, x_2, x_3, \dots, x_k$ are inserted between given integers a and b, so that all x_i's satisfy the recurrence relation of $F_n^{(1,q,r,1)}$, the general formula for any arbitrary term x_n $(1 \le n \le k)$ is given by

$$x_n = \frac{\left\{\begin{array}{c} F_n^{(1,q,r,1)}b - F_n^{(1,q,r,1)}a\left\{\left[\left(\frac{1-r}{q}\right)F_{k-1}^{(1,q,r,1)} + F_k^{(1,q,r,1)}\right]^{\chi(k)}\left[\frac{1}{q}F_k^{(1,q,r,1)}\right]^{1-\chi(k)}\right\} \\ +a\left\{\left[\left(\frac{1-r}{q}\right)F_{n-2}^{(1,q,r,1)} + F_{n-1}^{(1,q,r,1)}\right]^{1-\chi(n)}\left[\frac{1}{q}F_{n-1}^{(1,q,r,1)}\right]^{\chi(n)}\right\}F_{k+1}^{(1,q,r,1)} \end{array}\right\}}{F_{k+1}^{(1,q,r,1)}}.$$

Proof. We use the principle of mathematical induction to prove the result. By (5.2), we have $x_1 = \frac{b - a\left\{\left[\left(\frac{1-r}{q}\right)F_{k-1}^{(1,q,r,1)} + F_k^{(1,q,r,1)}\right]^{\chi(k)}\left[\frac{1}{q}F_k^{(1,q,r,1)}\right]^{1-\chi(k)}\right\}}{F_{k+1}^{(1,q,r,1)}}$, which proves the result for $n = 1$. We next assume that it is true for all positive integers not exceeding n. Then the following holds:

$$x_{n-1} = \frac{\left\{\begin{array}{c} F_{n-1}^{(1,q,r,1)}b - F_{n-1}^{(1,q,r,1)}a\left\{\left[\left(\frac{1-r}{q}\right)F_{k-1}^{(1,q,r,1)} + F_k^{(1,q,r,1)}\right]^{\chi(k)}\left[\frac{1}{q}F_k^{(1,q,r,1)}\right]^{1-\chi(k)}\right\} \\ +a\left\{\left[\left(\frac{1-r}{q}\right)F_{n-3}^{(1,q,r,1)} + F_{n-2}^{(1,q,r,1)}\right]^{1-\chi(n-1)}\left[\frac{1}{q}F_{n-2}^{(1,q,r,1)}\right]^{\chi(n-1)}\right\}F_{k+1}^{(1,q,r,1)} \end{array}\right\}}{F_{k+1}^{(1,q,r,1)}}$$

$$x_{n-2} = \frac{\left\{\begin{array}{c} F_{n-2}^{(1,q,r,1)}b - F_{n-2}^{(1,q,r,1)}a\left\{\left[\left(\frac{1-r}{q}\right)F_{k-1}^{(1,q,r,1)} + F_k^{(1,q,r,1)}\right]^{\chi(k)}\left[\frac{1}{q}F_k^{(1,q,r,1)}\right]^{1-\chi(k)}\right\} \\ +a\left\{\left[\left(\frac{1-r}{q}\right)F_{n-4}^{(1,q,r,1)} + F_{n-3}^{(1,q,r,1)}\right]^{1-\chi(n-2)}\left[\frac{1}{q}F_{n-3}^{(1,q,r,1)}\right]^{\chi(n-2)}\right\}F_{k+1}^{(1,q,r,1)} \end{array}\right\}}{F_{k+1}^{(1,q,r,1)}}$$

Now, $q^{1-\chi(n)}x_{n-1} + r^{\chi(n)}x_{n-2}$

$$= \frac{\left\{\begin{array}{c} \left(q^{1-\chi(n)}F_{n-1}^{(1,q,r,1)} + r^{\chi(n)}F_{n-2}^{(1,q,r,1)}\right)b - a\left(q^{1-\chi(n)}F_{n-1}^{(1,q,r,1)} + r^{\chi(n)}F_{n-2}^{(1,q,r,1)}\right) \\ \left\{\left[\left(\frac{1-r}{q}\right)F_{k-1}^{(1,q,r,1)} + F_{k}^{(1,q,r,1)}\right]^{\chi(k)}\left[\frac{1}{q}F_{k}^{(1,q,r,1)}\right]^{1-\chi(k)}\right\} \\ +a\left\{\begin{array}{l} q^{1-\chi(n)}\left[\left(\frac{1-r}{q}\right)F_{n-3}^{(1,q,r,1)} + F_{n-2}^{(1,q,r,1)}\right]^{1-\chi(n-1)}\left[\frac{1}{q}F_{n-2}^{(1,q,r,1)}\right]^{\chi(n-1)} \\ +r^{\chi(n)}\left[\left(\frac{1-r}{q}\right)F_{n-4}^{(1,q,r,1)} + F_{n-3}^{(1,q,r,1)}\right]^{1-\chi(n-2)}\left[\frac{1}{q}F_{n-3}^{(1,q,r,1)}\right]^{\chi(n-2)} \end{array}\right\}F_{k+1}^{(1,q,r,1)} \end{array}\right\}}{F_{k+1}^{(1,q,r,1)}}$$

$$= \frac{\left\{\begin{array}{c} F_{n}^{(1,q,r,1)}b - aF_{n}^{(1,q,r,1)}\left\{\left[\left(\frac{1-r}{q}\right)F_{k-1}^{(1,q,r,1)} + F_{k}^{(1,q,r,1)}\right]^{\chi(k)}\left[\frac{1}{q}F_{k}^{(1,q,r,1)}\right]^{1-\chi(k)}\right\} \\ +a\left\{\begin{array}{l} q^{1-\chi(n)}\left[\left(\frac{1-r}{q}\right)F_{n-3}^{(1,q,r,1)} + F_{n-2}^{(1,q,r,1)}\right]^{\chi(n)}\left[\frac{1}{q}F_{n-2}^{(1,q,r,1)}\right]^{1-\chi(n)} \\ +r^{\chi(n)}\left[\left(\frac{1-r}{q}\right)F_{n-4}^{(1,q,r,1)} + F_{n-3}^{(1,q,r,1)}\right]^{1-\chi(n)}\left[\frac{1}{q}F_{n-3}^{(1,q,r,1)}\right]^{\chi(n)} \end{array}\right\}F_{k+1}^{(1,q,r,1)} \end{array}\right\}}{F_{k+1}^{(1,q,r,1)}}$$

We calculate the value of third term of numerator separately. Now, when n is odd, we get

$$q^{1-\chi(n)}\left[\left(\frac{1-r}{q}\right)F_{n-3}^{(1,q,r,1)} + F_{n-2}^{(1,q,r,1)}\right]^{\chi(n)}\left[\frac{1}{q}F_{n-2}^{(1,q,r,1)}\right]^{1-\chi(n)}$$

$$+r^{\chi(n)}\left[\left(\frac{1-r}{q}\right)F_{n-4}^{(1,q,r,1)} + F_{n-3}^{(1,q,r,1)}\right]^{1-\chi(n)}\left[\frac{1}{q}F_{n-3}^{(1,q,r,1)}\right]^{\chi(n)}$$

$$= q\left[\frac{1}{q}F_{n-2}^{(1,q,r,1)}\right] + \left(\frac{1-r}{q}\right)F_{n-4}^{(1,q,r,1)} + F_{n-3}^{(1,q,r,1)}$$

$$= \frac{1}{q}\left[qF_{n-2}^{(1,q,r,1)} - rF_{n-4}^{(1,q,r,1)} + \left(qF_{n-3}^{(1,q,r,1)} + F_{n-4}^{(1,q,r,1)}\right)\right]$$

$$= \frac{1}{q}\left[qF_{n-2}^{(1,q,r,1)} - rF_{n-4}^{(1,q,r,1)} + F_{n-2}^{(1,q,r,1)}\right]$$

$$= \frac{1}{q}\left[(1+q)F_{n-2}^{(1,q,r,1)} - rF_{n-4}^{(1,q,r,1)}\right]$$

$$= \frac{1}{q}\left[(1-r)F_{n-2}^{(1,q,r,1)} + qF_{n-2}^{(1,q,r,1)} + rqF_{n-3}^{(1,q,r,1)}\right]$$

$$= \left(\frac{1-r}{q}\right)F_{n-2}^{(1,q,r,1)} + F_{n-1}^{(1,q,r,1)}$$

$$= \left[\left(\frac{1-r}{q}\right)F_{n-2}^{(1,q,r,1)} + F_{n-1}^{(1,q,r,1)}\right]^{1-\chi(n)}.$$

Also, when n is even, we get

$$q^{1-\chi(n)}\left[\left(\frac{1-r}{q}\right)F_{n-3}^{(1,q,r,1)} + F_{n-2}^{(1,q,r,1)}\right]^{\chi(n)}\left[\frac{1}{q}F_{n-2}^{(1,q,r,1)}\right]^{1-\chi(n)}$$

$$+r^{\chi(n)}\left[\left(\frac{1-r}{q}\right)F_{n-4}^{(1,q,r,1)} + F_{n-3}^{(1,q,r,1)}\right]^{1-\chi(n)}\left[\frac{1}{q}F_{n-3}^{(1,q,r,1)}\right]^{\chi(n)}$$

$$= \left[\left(\frac{1-r}{q}\right)F_{n-3}^{(1,q,r,1)} + F_{n-2}^{(1,q,r,1)}\right] + r\left[\frac{1}{q}F_{n-3}^{(1,q,r,1)}\right]$$

$$= \frac{1}{q}\left[qF_{n-2}^{(1,q,r,1)} + F_{n-3}^{(1,q,r,1)}\right] = \frac{1}{q}\left[F_{n-1}^{(1,q,r,1)}\right] = \left[\frac{1}{q}F_{n-1}^{(1,q,r,1)}\right]^{\chi(n)}$$

Therefore, we have

$$x_n = q^{1-\chi(n)}x_{n-1} + r^{\chi(n)}x_{n-2}$$

$$= \frac{\left\{\begin{array}{c} F_n^{(1,q,r,1)}b - F_n^{(1,q,r,1)}a\left\{\left[\left(\frac{1-r}{q}\right)F_{k-1}^{(1,q,r,1)} + F_k^{(1,q,r,1)}\right]^{\chi(k)}\left[\frac{1}{q}F_k^{(1,q,r,1)}\right]^{1-\chi(k)}\right\} \\ +a\left\{\left[\left(\frac{1-r}{q}\right)F_{n-2}^{(1,q,r,1)} + F_{n-1}^{(1,q,r,1)}\right]^{1-\chi(n)}\left[\frac{1}{q}F_{n-1}^{(1,q,r,1)}\right]^{\chi(n)}\right\}F_{k+1}^{(1,q,r,1)} \end{array}\right\}}{F_{k+1}^{(1,q,r,1)}},$$

which proves the result for every positive integer n.

6 Insertion of terms satisfying the recurrence relation of bifurcating sequence $\boldsymbol{F_n^{(1,1,r,s)}}$

If we consider $p = q = 1$ in (1.1), then we get the sequence $\left\{F_n^{(1,1,r,s)}\right\}$ whose terms are defined by the recurrence relation

$$F_n^{(1,1,r,s)} = F_{n-1}^{(1,1,r,s)} + r^{\chi(n)}s^{1-\chi(n)}F_{n-2}^{(1,1,r,s)},$$

where $F_0^{(1,1,r,s)} = 0, F_1^{(1,1,r,s)} = 1$.This can be written in the form

$$F_n^{(1,1,r,s)} = \begin{cases} F_{n-1}^{(1,1,r,s)} + rF_{n-2}^{(1,1,r,s)} \text{ ; when n is odd} \\ F_{n-1}^{(1,1,r,s)} + sF_{n-2}^{(1,1,r,s)} \text{ ; when n is even} \end{cases} (n \geq 2) \qquad (6.1)$$

with the initial conditions $F_0^{(1,1,r,s)} = 0, F_1^{(1,1,r,s)} = 1$. First few terms of this sequence are shown in Table 9.

n	$F_n^{(1,1,r,s)}$
0	0
1	1
2	1
3	$1 + r$
4	$1 + r + s$
5	$1 + 2r + s + r^2$
6	$1 + rs + 2r + 2s + r^2 + s^2$

Table 9

Diwan, Shah [3] derived the Binet-type explicit formula for the terms of this sequence as

$$F_n^{(1,1,r,s)} = \frac{(\alpha-s)^{\chi(n)}\alpha^{\left[\frac{n}{2}\right]}-(\beta-s)^{\chi(n)}\beta^{\left[\frac{n}{2}\right]}}{\alpha-\beta},$$

where $\alpha = \frac{(r+s+1)+\sqrt{(r+s+1)^2-4rs}}{2}, \beta = \frac{(r+s+1)-\sqrt{(r+s+1)^2-4rs}}{2}$ with $\alpha - \beta = \sqrt{(r+s+1)^2 - 4rs}$, $\alpha + \beta = r + s + 1$ and $\alpha\beta = rs$.

In this section, we find the formula for the inserted terms between the numbers a and b so that terms of the sequence $a, x_1, x_2, x_3, \ldots, x_k, b$ satisfies the recurrence relation (6.1).

When we consider the sequence a, x_1, b, where x_1 is the only inserted term between a and b. Then using $F_2^{(1,1,r,s)} = F_1^{(1,1,r,s)} + sF_0^{(1,1,r,s)}$, we get $b = x_1 + sa$. Thus $x_1 = b - sa$.

When we consider the finite sequence as a, x_1, x_2, b, so that it satisfies (6.1), we observe that $x_1 = x_2 - sa$ and $x_2 = b - rx_1$. This gives $x_2 = \frac{b+rsa}{1+r}$ .This also gives $x_1 = \frac{b-sa}{1+r}$.

Further, considering the sequencea, x_1, x_2, x_3, b, it is now easy to observe that $x_1 = x_2 - sa, x_2 = x_3 - rx_1, x_3 = b - sx_2$. Solving these three equations in three variables x_1, x_2, x_3 we get $x_1 = \frac{b-as(1+s)}{1+r+s}, x_2 = \frac{b-rsa}{1+r+s}, x_3 = \frac{b+br+ars^2}{1+r+s}$.

If we continue extending the above finite sequence one more time, then we get the sequence a, x_1, x_2, x_3, x_4, b. Using the similar approach as above, we obtain $x_1 = \frac{b-sa(1+r+s)}{1+2r+s+r^2}, x_2 = \frac{b+sra+sar^2}{1+2r+s+r^2}, x_3 = \frac{b+br-s^2ra}{1+2r+s+r^2}$ and $x_4 = \frac{b+br+bs+r^2s^2a}{1+2r+s+r^2}$.

We mention these results in the Table 10.

Number of inserted term	Formula			
	x_1	x_2	x_3	x_4
1	$b - sa$	---	---	---
2	$\frac{b - sa}{1 + r}$	$\frac{b + rsa}{1 + r}$	---	---
3	$\frac{b - sa(1 + s)}{1 + r + s}$	$\frac{b - rsa}{1 + r + s}$	$\frac{b + br + ars^2}{1 + r + s}$	---
4	$\frac{b - sa(1 + r + s)}{1 + 2r + s + r^2}$	$\frac{b + sra + sar^2}{1 + 2r + s + r^2}$	$\frac{b + br - s^2ra}{1 + 2r + s + r^2}$	$\frac{b + br + bs + r^2s^2a}{1 + 2r + s + r^2}$

Table 10

From this table, we observe that there is a similar pattern for the first inserted term in case of any number of inserted terms. We thus generalize it for the case of any number of inserted terms and when the terms $x_1, x_2, x_3, \dots, x_k$ are inserted between given two positive integers a and b, we can now write the general formula for x_1 as

$$x_1 = \frac{b - sa\left\{\left(F_k^{(1,1,r,s)}\right)^{1-\chi(k)}\left(\sum_{m=0}^{\left[\frac{k-1}{2}\right]} s^{\left[\frac{k-1}{2}\right]-m} F_{2m}^{(1,1,r,s)} + s^{\left[\frac{k-1}{2}\right]}\right)^{\chi(k)}\right\}}{F_{k+1}^{(1,1,r,s)}}, \tag{6.2}$$

Next, when the terms $x_1, x_2, x_3, \dots, x_k$ are inserted between given two positive integers a and b, we obtain the general formula for n^{th} inserted term $(1 \le n \le k)$ using the recurrence relation (6.1).

Theorem 6.1 When the terms $x_1, x_2, x_3, \dots, x_k$ are inserted between given integers a and b, so that all x_i's satisfy the recurrence relation of $F_n^{(1,1,r,s)}$, the general formula for any arbitrary term x_n $(1 \le n \le k)$ is given by

$$x_n = \frac{\left\{\begin{array}{c} F_n^{(1,1,r,s)} b - F_n^{(1,1,r,s)} sa\left\{\left(F_k^{(1,1,r,s)}\right)^{1-\chi(k)}\left(\sum_{m=0}^{\left[\frac{k-1}{2}\right]} s^{\left[\frac{k-1}{2}\right]-m} F_{2m}^{(1,1,r,s)} + s^{\left[\frac{k-1}{2}\right]}\right)^{\chi(k)}\right\} \\ + saF_{k+1}^{(1,1,r,s)}\left(F_{n-1}^{(1,1,r,s)}\right)^{\chi(n)}\left(\sum_{m=0}^{\left[\frac{n-2}{2}\right]} s^{\left[\frac{n-2}{2}\right]-m} F_{2m}^{(1,1,r,s)} + s^{\left[\frac{n-2}{2}\right]}\right)^{1-\chi(n)} \end{array}\right\}}{F_{k+1}^{(1,1,r,s)}}.$$

Proof. We use the principle of mathematical induction to prove the result. Since by (6.2), we have $x_1 = \dfrac{b - sa\left\{\left(F_k^{(1,1,r,s)}\right)^{1-\chi(k)}\left(\sum_{m=0}^{\left[\frac{k-1}{2}\right]} s^{\left[\frac{k-1}{2}\right]-m} F_{2m}^{(1,1,r,s)} + s^{\left[\frac{k-1}{2}\right]}\right)^{\chi(k)}\right\}}{F_{k+1}^{(1,1,r,s)}}$, which proves the result for $n = 1$. We next assume that it is true for all positive integers not exceeding n. Then the following holds:

$$x_{n-1} = \frac{\left\{\begin{array}{c} F_{n-1}^{(1,1,r,s)} b - F_{n-1}^{(1,1,r,s)} sa\left\{\left(F_k^{(1,1,r,s)}\right)^{1-\chi(k)}\left(\sum_{m=0}^{\left[\frac{k-1}{2}\right]} s^{\left[\frac{k-1}{2}\right]-m} F_{2m}^{(1,1,r,s)} + s^{\left[\frac{k-1}{2}\right]}\right)^{\chi(k)}\right\} \\ + saF_{k+1}^{(1,1,r,s)}\left(F_{n-2}^{(1,1,r,s)}\right)^{\chi(n-1)}\left(\sum_{m=0}^{\left[\frac{n-3}{2}\right]} s^{\left[\frac{n-3}{2}\right]-m} F_{2m}^{(1,1,r,s)} + s^{\left[\frac{n-3}{2}\right]}\right)^{1-\chi(n-1)} \end{array}\right\}}{F_{k+1}^{(1,1,r,s)}},$$

$$x_{n-2}=\frac{\left\{\begin{array}{c}F_{n-2}^{(1,1,r,s)}b-F_{n-2}^{(1,1,r,s)}sa\left\{\left(F_k^{(1,1,r,s)}\right)^{1-\chi(k)}\left(\sum_{m=0}^{\left[\frac{k-1}{2}\right]}s^{\left[\frac{k-1}{2}\right]-m}F_{2m}^{(1,1,r,s)}+s^{\left[\frac{k-1}{2}\right]}\right)^{\chi(k)}\right\}\\ +saF_{k+1}^{(1,1,r,s)}\left(F_{n-3}^{(1,1,r,s)}\right)^{\chi(n-2)}\left(\sum_{m=0}^{\left[\frac{n-3}{2}\right]}s^{\left[\frac{n-3}{2}\right]-m}F_{2m}^{(1,1,r,s)}+s^{\left[\frac{n-3}{2}\right]}\right)^{1-\chi(n-2)}\end{array}\right\}}{F_{k+1}^{R(1,1,r,s)}}.$$

Now, $x_{n-1}+r^{\chi(n)}s^{1-\chi(n)}x_{n-2}$

$$=\frac{\left\{\begin{array}{c}\left(F_{n-1}^{(1,1,r,s)}+r^{\chi(n)}s^{1-\chi(n)}F_{n-2}^{(1,1,r,s)}\right)b-\left(F_{n-1}^{(1,1,r,s)}+r^{\chi(n)}s^{1-\chi(n)}F_{n-2}^{(1,1,r,s)}\right)sa\\ \times\left\{\left(F_k^{(1,1,r,s)}\right)^{1-\chi(k)}\left(\sum_{m=0}^{\left[\frac{k-1}{2}\right]}s^{\left[\frac{k-1}{2}\right]-m}F_{2m}^{(1,1,r,s)}+s^{\left[\frac{k-1}{2}\right]}\right)^{\chi(k)}\right\}\\ +saF_{k+1}^{R(1,1,r,s)}\left\{\begin{array}{c}\left(F_{n-2}^{(1,1,r,s)}\right)^{\chi(n-1)}\left(\sum_{m=0}^{\left[\frac{n-3}{2}\right]}s^{\left[\frac{n-3}{2}\right]-m}F_{2m}^{(1,1,r,s)}+s^{\left[\frac{n-3}{2}\right]}\right)^{1-\chi(n-1)}\\ +r^{\chi(n)}s^{1-\chi(n)}\left(F_{n-3}^{(1,1,r,s)}\right)^{\chi(n-2)}\\ \times\left(\sum_{m=0}^{\left[\frac{n-4}{2}\right]}s^{\left[\frac{n-4}{2}\right]-m}F_{2m}^{(1,1,r,s)}+s^{\left[\frac{n-4}{2}\right]}\right)^{1-\chi(n-2)}\end{array}\right\}\end{array}\right\}}{F_{k+1}^{(1,1,r,s)}}$$

$$=\frac{\left\{\begin{array}{c}F_{n}^{(1,1,r,s)}b-F_{n}^{(1,1,r,s)}sa\times\left\{\left(F_k^{(1,1,r,s)}\right)^{1-\chi(k)}\left(\sum_{m=0}^{\left[\frac{k-1}{2}\right]}s^{\left[\frac{k-1}{2}\right]-m}F_{2m}^{(1,1,r,s)}+s^{\left[\frac{k-1}{2}\right]}\right)^{\chi(k)}\right\}\\ +saF_{k+1}^{(1,1,r,s)}\left\{\begin{array}{c}\left(F_{n-2}^{(1,1,r,s)}\right)^{1-\chi(n)}\left(\sum_{m=0}^{\left[\frac{n-3}{2}\right]}s^{\left[\frac{n-3}{2}\right]-m}F_{2m}^{(1,1,r,s)}+s^{\left[\frac{n-3}{2}\right]}\right)^{\chi(n)}\\ +r^{\chi(n)}s^{1-\chi(n)}\left(F_{n-3}^{(1,1,r,s)}\right)^{\chi(n)}\left(\sum_{m=0}^{\left[\frac{n-4}{2}\right]}s^{\left[\frac{n-4}{2}\right]-m}F_{2m}^{(1,1,r,s)}+s^{\left[\frac{n-4}{2}\right]}\right)^{1-\chi(n)}\end{array}\right\}\end{array}\right\}}{F_{k+1}^{(1,1,r,s)}}$$

We calculate the value of third term of numerator separately. Now, when n is odd, we get

$$\left(F_{n-2}^{(1,1,r,s)}\right)^{1-\chi(n)}\left(\sum_{m=0}^{\left[\frac{n-3}{2}\right]}s^{\left[\frac{n-3}{2}\right]-m}F_{2m}^{(1,1,r,s)}+s^{\left[\frac{n-3}{2}\right]}\right)^{\chi(n)}$$

$$+r^{\chi(n)}s^{1-\chi(n)}\left(F_{n-3}^{(1,1,r,s)}\right)^{\chi(n)}\left(\sum_{m=0}^{\left[\frac{n-4}{2}\right]}s^{\left[\frac{n-4}{2}\right]-m}F_{2m}^{(1,1,r,s)}+s^{\left[\frac{n-4}{2}\right]}\right)^{1-\chi(n)}$$

$$=F_{n-2}^{(1,1,r,s)}+s\left(\sum_{m=0}^{\left[\frac{n-4}{2}\right]}s^{\left[\frac{n-4}{2}\right]-m}F_{2m}^{(1,1,r,s)}+s^{\left[\frac{n-4}{2}\right]}\right)$$

$$=F_{n-2}^{(1,1,r,s)}+s\left(\begin{array}{c}s^{\left[\frac{n-4}{2}\right]}F_0^{(1,1,r,s)}+s^{\left[\frac{n-6}{2}\right]}F_1^{(1,1,r,s)}+s^{\left[\frac{n-8}{2}\right]}F_2^{(1,1,r,s)}\\ +\cdots+F_{n-4}^{(1,1,r,s)}+s^{\left[\frac{n-4}{2}\right]}\end{array}\right)$$

$$= F_{n-2}^{(1,1,r,s)} + s^{\left[\frac{n-2}{2}\right]} F_0^{(1,1,r,s)} + s^{\left[\frac{n-4}{2}\right]} F_1^{(1,1,r,s)} + s^{\left[\frac{n-6}{2}\right]} F_2^{(1,1,r,s)}$$
$$+ \cdots + sF_{n-4}^{(1,1,r,s)} + s^{\left[\frac{n-2}{2}\right]}$$
$$= \sum_{m=0}^{\left[\frac{n-2}{2}\right]} s^{\left[\frac{n-2}{2}\right]-m} F_{2m}^{(1,1,r,s)} + s^{\left[\frac{n-2}{2}\right]}$$
$$= \left(\sum_{m=0}^{\left[\frac{n-2}{2}\right]} s^{\left[\frac{n-2}{2}\right]-m} F_{2m}^{(1,1,r,s)} + s^{\left[\frac{n-2}{2}\right]}\right)^{1-\chi(n)} .$$

Also, when n is even, we get

$$\left(F_{n-2}^{(1,1,r,s)}\right)^{1-\chi(n)} \left(\sum_{m=0}^{\left[\frac{n-3}{2}\right]} s^{\left[\frac{n-3}{2}\right]-m} F_{2m}^{(1,1,r,s)} + s^{\left[\frac{n-3}{2}\right]}\right)^{\chi(n)}$$
$$+ r^{\chi(n)} s^{1-\chi(n)} \left(F_{n-3}^{(1,1,r,s)}\right)^{\chi(n)} \left(\sum_{m=0}^{\left[\frac{n-4}{2}\right]} s^{\left[\frac{n-4}{2}\right]-m} F_{2m}^{(1,1,r,s)} + s^{\left[\frac{n-4}{2}\right]}\right)^{1-\chi(n)}$$
$$= \sum_{m=0}^{\left[\frac{n-3}{2}\right]} s^{\left[\frac{n-3}{2}\right]-m} F_{2m}^{(1,1,r,s)} + s^{\left[\frac{n-3}{2}\right]} + rF_{n-3}^{(1,1,r,s)} .$$

To prove the main result, we need to prove that this value should be $F_{n-1}^{(1,1,r,s)}$. Clearly this result holds for $n = 3$, since $F_2^{(1,1,r,s)} = 1$. Assume that this result holds for all positive integers not exceeding n. Now,

$$F_{n-2}^{(1,1,r,s)} = \sum_{m=0}^{\left[\frac{n-4}{2}\right]} s^{\left[\frac{n-4}{2}\right]-m} F_{2m}^{(1,1,r,s)} + s^{\left[\frac{n-4}{2}\right]} + rF_{n-4}^{(1,1,r,s)} \text{and}$$
$$F_{n-3}^{(1,1,r,s)} = \sum_{m=0}^{\left[\frac{n-5}{2}\right]} s^{\left[\frac{n-5}{2}\right]-m} F_{2m}^{(1,1,r,s)} + s^{\left[\frac{n-5}{2}\right]} + rF_{n-5}^{(1,1,r,s)} .$$

Now, $F_{n-2}^{(1,1,r,s)} + sF_{n-3}^{(1,1,r,s)}$

$$= \sum_{m=0}^{\left[\frac{n-4}{2}\right]} s^{\left[\frac{n-4}{2}\right]-m} F_{2m}^{(1,1,r,s)} + s^{\left[\frac{n-4}{2}\right]} + rF_{n-4}^{(1,1,r,s)}$$
$$+ s\left(\sum_{m=0}^{\left[\frac{n-5}{2}\right]} s^{\left[\frac{n-5}{2}\right]-m} F_{2m}^{(1,1,r,s)} + s^{\left[\frac{n-5}{2}\right]} + rF_{n-5}^{(1,1,r,s)}\right)$$
$$= \sum_{m=0}^{\left[\frac{n-4}{2}\right]} s^{\left[\frac{n-4}{2}\right]-m} F_{2m}^{(1,1,r,s)} + s^{\left[\frac{n-4}{2}\right]} + rF_{n-4}^{(1,1,r,s)}$$
$$+ \sum_{m=0}^{\left[\frac{n-5}{2}\right]} s^{\left[\frac{n-3}{2}\right]-m} F_{2m}^{(1,1,r,s)} + s^{\left[\frac{n-3}{2}\right]} + rsF_{n-5}^{(1,1,r,s)}$$
$$= \left(\sum_{m=0}^{\left[\frac{n-4}{2}\right]} s^{\left[\frac{n-4}{2}\right]-m} F_{2m}^{(1,1,r,s)} + s^{\left[\frac{n-4}{2}\right]} + \sum_{m=0}^{\left[\frac{n-5}{2}\right]} s^{\left[\frac{n-3}{2}\right]-m} F_{2m}^{(1,1,r,s)} + s^{\left[\frac{n-3}{2}\right]}\right)$$
$$+ r\left(F_{n-4}^{(1,1,r,s)} + sF_{n-5}^{(1,1,r,s)}\right)$$
$$= \sum_{m=0}^{\left[\frac{n-3}{2}\right]} s^{\left[\frac{n-3}{2}\right]-m} F_{2m}^{(1,1,r,s)} + s^{\left[\frac{n-3}{2}\right]} + rF_{n-3}^{(1,1,r,s)} = F_{n-1}^{(1,1,r,s)} = \left(F_{n-1}^{(1,1,r,s)}\right)^{\chi(n)} .$$

.

Therefore, we have

$$x_n = x_{n-1} + r^{\chi(n)} s^{1-\chi(n)} x_{n-2}$$
$$= \frac{\left\{ \begin{array}{c} F_n^{(1,1,r,s)} b - F_n^{(1,1,r,s)} sa \left\{ \left(F_k^{(1,1,r,s)}\right)^{1-\chi(k)} \left(\sum_{m=0}^{\left[\frac{k-1}{2}\right]} s^{\left[\frac{k-1}{2}\right]-m} F_{2m}^{(1,1,r,s)} + s^{\left[\frac{k-1}{2}\right]} \right)^{\chi(k)} \right\} \\ + saF_{k+1}^{(1,1,r,s)} \left(F_{n-1}^{(1,1,r,s)}\right)^{\chi(n)} \left(\sum_{m=0}^{\left[\frac{n-2}{2}\right]} s^{\left[\frac{n-2}{2}\right]-m} F_{2m}^{(1,1,r,s)} + s^{\left[\frac{n-2}{2}\right]} \right)^{1-\chi(n)} \end{array} \right\}}{F_{k+1}^{(1,1,r,s)}}$$

, which proves the result for every positive integer n.

7 Conclusion

In this paper we derived the general formula which gives the value of any inserted term x_n $(1 \le n \le k)$ between the given fixed positive integers a, b so that terms of the sequence $a, x_1, x_2, x_3, \ldots, x_k, b$ satisfies the recurrence relation of $F_n(p,q)$ and various bifurcating subsequence of $\left\{F_n^{(p,q,r,s)}\right\}$ by considering some fixed values of p, q, r, s.

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