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Study of feedback queueing system with unreliable waiting server under multiple differentiated vacation policy

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Abstract

This manuscript analyses a queueing system with Bernoulli schedule feedback of customers, unreliable waiting server under differentiated vacations. The unsatisfied customer may again join the queue with probability α, following the Bernoulli schedule. The stationary solution is obtained for the model with aid of the Probability Generating function technique. Some important system performance measures are derived and the graphical behavior of these measures with some parameters is analysed. Finally, to obtain the optimal value of service rate for the model, cost optimization is performed through the quadratic fit approach.

Keywords: Feedback; differentiated vacations; optimization; Bernoulli schedule; unreliable waiting server.

2010 AMS subject classification: 60K25, 60K30.

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1. Introduction

Queueing system with server vacations and feedback is a powerful tool and is successfully applied in many real-life congestion problems such as traffic systems, operating systems, communication systems, manufacturing and production lines. Queueing systems with feedback assume that unsatisfied customers may join the queue again to repeat their request before leaving the system. This concept was first introduced by Takacs [16]. Disney and Konig [4] analysed Bernoulli feedback queueing system. Sharma and Kumar [14] studied the markovian feedback queueing model with the impatient behaviour of customers. Later on, Kalidass and Kasthuri [6] did the pioneering work on M/G/1 queueing model with immediate feedback. Sunder et al. [15] analysed the feedback queueing model with different service stations and vacation policies under reneging.

The vacation queueing model is another remarkable and unavoidable feature due to its widespread applications in real-life situations. The server can go on vacations for many reasons like insufficient workload, server breakdown and some secondary tasks assigned to him, etc. The assumption of the service station being a hundred percent reliable is not a feasible one. The server may suffer a breakdown at any instant of time. Ke, J.C. [7, 8, 9] analysed different queueing models with an unreliable server. Later on, Ke et al. [10] performed pioneer work on finite buffer M/M/C queueing system with server breakdown. Kim et al. [11] analysed M/G/1 queueing model with a working breakdown. Levy and Yechiali [12] first analysed vacation queuing systems. Later on Doshi [3], Tian and Zhang [17] performed a comprehensive survey on vacation queueing models. Researchers analysed queues with different vacation strategies like single vacations, multiple vacations and working vacations. Servi and Finn [13] first introduced the concept of working vacation in which the server remains engaged in some auxiliary works and thereby provides service at a relatively slower rate. Banik et al. [2] analysed multiple working vacation queueing systems. A new class of differentiated vacations is introduced by Ibe and Isijola [5]. Zhang and Zhou [20] studied queueing model with m kinds of differentiated working vacations. The transient solution of a differentiated vacation queueing system was carried out by Vijyashree and Janani [19]. Vyshna Unni and Rose Mary [18] considered multi-server queues with differentiated vacations. Amar Aissani et al. [1] studied differentiated vacation queues under general service times.

In this paper, a queue with waiting and unreliable servers under differentiated vacation policy and Bernoulli feedback of customers with different arrival rates in various states is considered.

2. Mathematical description of model

The queueing model is analysed under the following assumptions:

1. The customers arrive in accordance with the Poisson process. The customers are served on an FCFS basis by the server, the service time is assumed to be exponentially distributed with a mean of $1/\mu$.
2. On getting the service, the customers may decide to leave the queue with probability β or unsatisfied customers may re-join the queue for another service with complementary probability $1-\beta$ $(=\alpha)$.
3. When the system gets empty on serving all the customers, the server keeps on waiting for customers for a random period exponentially distributed with a mean $1/w$ and leaves for vacation only if none arrives in that duration. On returning from type I vacation, if it finds customers waiting for the service, it resumes to active state otherwise it goes for type II vacation. In the same way, on returning from vacation II, it switches to Vacation I or resumes to active state depending on whether the system is empty or customers are waiting in the system respectively. This process continues in the same manner. The period of both the vacations are exponentially distributed but with different means $1/\theta_1$ and $1/\theta_2$ respectively.
4. The server breaks down at any point of time in the active service state. The breakdowns occur according to Poisson distribution with parameter γ. The server is immediately sent for repair in such a situation. The repair time is also assumed to follow an exponential distribution with a mean of $1/\delta$. The customers have to wait for their service until the server gets repaired.
5. The arrival rates of customers in two types of vacations, downstate of server and active state are taken to be $\lambda_1, \lambda_2, \lambda_0, \lambda_3$ respectively.

3. Steady state equations

Denoting the number of customers in the system at any time t by C(t) and the server state at time t by S(t), we observe that {C(t), S(t)} is a continuous Markov chain.

The different possible server states are

$$S(t) = \begin{cases} 0, server\ in\ vacation\ I \\ 1, server\ in\ vacation\ II \\ 2, server\ in\ active\ state \\ 3, server\ under\ repair \end{cases}$$

Denoting the probability of n customers in the i[th] state of server by $p_{n\,i}$, the steady-state equations governing the proposed quasi birth-death model using the Markov process are

$$(\lambda_2 + \theta_2)p_{0\,1} = \theta_1 p_{0\,0} \tag{1}$$

$$(\lambda_2 + \theta_2)p_{n\,1} = \lambda_2 p_{n-1\,1}, \qquad n \geq 1 \tag{2}$$

$$(\lambda_3 + w)p_{0\,2} = \mu\beta p_{1\,2} \tag{3}$$

$$(\lambda_3 + \mu\beta + \gamma)p_{n\,2} = \lambda_3 p_{n-1\,2} + \mu\beta p_{n+1\,2} + \delta p_{n\,3} + \theta_1 p_{n\,0} + \theta_2 p_{n\,1}, \qquad n \geq 1 \tag{4}$$

$$(\lambda_0 + \delta)p_{1\,3} = \gamma p_{1\,2} \tag{5}$$

$$(\lambda_0 + \delta)p_{n\,3} = \gamma p_{n\,2} + \lambda_0 p_{n-1\,3}, \qquad n \geq 2 \tag{6}$$

$$(\lambda_1 + \theta_1)p_{0\,0} = \theta_2 p_{0\,1} + w p_{0\,2} \tag{7}$$

$$(\lambda_1 + \theta_1)p_{n\,0} = \lambda_1 p_{n-1\,0}, \qquad n \geq 1 \tag{8}$$

Defining probability generating functions as

$$H_i(z) = \sum_{n=0}^{\infty} p_{n\,i}\, z^n, \qquad for\ i = 0, 1, 2. \tag{9}$$

$$H_3(z) = \sum_{n=1}^{\infty} p_{n\,3}\, z^n \tag{10}$$

Multiplying system of equations (1) and (2) by appropriate powers of z and summing over n from 1to ∞.

$$H_1(z) = \frac{\theta_1}{(\lambda_2 + \theta_2 - \lambda_2 z)} p_{0\,0} \tag{11}$$

Similarly, equations (7) and (8) yield

$$H_0(z) = \frac{(\lambda_1 + \theta_1)}{(\lambda_1 + \theta_1 - \lambda_1 z)} p_{0\,0} \tag{12}$$

On similar steps from equations (5) and (6) we have

$$(\lambda_0 + \delta - \lambda_0 z) H_3(z) = \gamma H_2(z) - \gamma p_{0\,2}$$

After some rearrangement of terms, we obtain

$$H_3(z) = \frac{\gamma(H_2(z) - p_{0\,2})}{\lambda_0(1-z) + \delta} \tag{13}$$

Where,

$$p_{0\,2} = \frac{1}{w}\left(\lambda_1 + \theta_1 - \frac{\theta_1\theta_2}{\lambda_2 + \theta_2}\right) p_{0\,0}, \qquad Using\ (7) \tag{14}$$

Multiplying equations (3) and (4) by an appropriate power of z and taking summation over n, together with the use of equations (9) and (10) we obtain

$$\begin{aligned}\left(\lambda_3 + \mu\beta + \gamma - \lambda_3 z - \frac{\mu\beta}{z}\right) H_2(z) &= \theta_1 H_0(z) + \theta_2 H_1(z) + \delta H_3(z) - (\lambda_1 + 2\theta_1) p_{0\,0} \\ &\quad + \left(\mu\beta + \gamma - \frac{\mu\beta}{z}\right) p_{0\,2}\end{aligned} \tag{15}$$

Considering equations (13) and (15) simultaneously, we obtain

$$H_2(z) = \frac{\theta_1 H_0(z) + \theta_2 H_1(z) - (\lambda_1 + 2\theta_1) p_{0\,0} + g_2(z) p_{0\,2}}{g_1(z)} \tag{16}$$

Where,

$$g_1(z) = \lambda_3(1-z) + \mu\beta\left(1 - \frac{1}{z}\right) - \frac{\gamma\delta}{\lambda_0(1-z) + \delta} + \gamma \tag{17}$$

$$g_2(z) = \mu\beta\left(1 - \frac{1}{z}\right) - \frac{\gamma\delta}{\lambda_0(1-z) + \delta} + \gamma \tag{18}$$

Taking limits $z \to 1$ in equation (11) and (12) the closed-form expressions for P.G.F’s are

$$H_0(1) = \frac{(\lambda_1 + \theta_1)}{\theta_1} p_{0\,0} \tag{19}$$

$$H_1(1) = \frac{\theta_1}{\theta_2} p_{0\,0} \tag{20}$$

Differentiating equation (11) and (12) and taking limits $z \to 1$

$$H_0'(1) = \frac{\lambda_1(\lambda_1 + \theta_1)}{{\theta_1}^2} p_{0\,0} \tag{21}$$

$$H_1'(1) = \frac{\lambda_2 \theta_1}{{\theta_2}^2} p_{0\,0} \tag{22}$$

$$H_0''(1) = \frac{2{\lambda_1}^2(\lambda_1 + \theta_1)}{{\theta_1}^3} p_{0\,0} \tag{23}$$

$$H_1''(1) = \frac{2{\lambda_2}^2 \theta_1}{{\theta_2}^3} p_{0\,0} \tag{24}$$

Similarly, taking limits in equation (16) and using the L-Hospital rule

$$H_2(1) = \frac{\left(\frac{\lambda_1(\lambda_1 + \theta_1)}{\theta_1} + \frac{\theta_1 \lambda_2}{\theta_2}\right) p_{0\,0} + \left(\mu\beta - \frac{\lambda_0 \gamma}{\delta}\right) p_{0\,2}}{\beta\mu - \lambda_3 - \frac{\lambda_0 \gamma}{\delta}} \tag{25}$$

On taking limit in equation (13),

$$H_3(1) = \frac{\gamma(H_2(1) - p_{0\,2})}{\delta} \tag{26}$$

Now, differentiating $H_2(z)$, taking limits $z \to 1$ and applying L' Hospital rule twice,

$$H_2'(1) = \frac{(\theta_1 H_0''(1) + \theta_2 H_1''(1) + g_2''(1) p_{0\,2}) g_1'(1)}{2(g_1'(1))^2} - \frac{(\theta_1 H_0'(1) + \theta_2 H_1'(1) + g_2'(1) p_{0\,2}) g_1''(1)}{2(g_1'(1))^2} \tag{27}$$

Where,

$$g_1'(1) = \mu\beta - \frac{\lambda_0 \gamma}{\delta} - \lambda_3$$

$$g_2'(1) = \mu\beta - \frac{\lambda_0 \gamma}{\delta}$$

$$g_1''(1) = g_2''(1) = -2\left(\mu\beta + \gamma\left(\frac{\lambda_0}{\delta}\right)^2\right)$$

Similarly, after differentiating equation (11) and taking limits $z \rightarrow 1$, we get

$$H_3'(1) = \frac{\gamma(\delta H_2'(1) + \lambda_0(H_2(1) - p_{0\,2}))}{\delta^2} \tag{28}$$

The closed-form expressions for all the P.G.F's are implicitly expressed in terms of only one probability $p_{0\,0}$.

To obtain $p_{0\,0}$ we use the following normalization condition.

$$\sum_{i=0}^{3} H_i(1) = 1 \tag{29}$$

After some rearrangements of terms, we get:

$$p_{0\,0}\left(\frac{\lambda_1+\theta_1}{\theta_1} + \frac{\theta_1}{\theta_2} + \left(\frac{\gamma+\delta}{\delta}\right)(A+BC) - \frac{\gamma C}{\delta}\right) = 1 \tag{30}$$

Where,

$$A = \frac{\lambda_1\theta_2(\lambda_1+\theta_1) + \lambda_2{\theta_1}^2}{\theta_1\theta_2\left(\mu\beta - \lambda_3 - \frac{\lambda_0\gamma}{\delta}\right)}$$

$$B = \frac{\mu\beta\delta - \lambda_0\gamma}{\delta\left(\mu\beta - \lambda_3 - \frac{\lambda_0\gamma}{\delta}\right)}$$

$$C = \frac{\lambda_1\lambda_2 + \lambda_1\theta_2 + \lambda_2\theta_1}{w(\lambda_2+\theta_2)}$$

$$p_{0\,0} = \left(\frac{\lambda_1+\theta_1}{\theta_1} + \frac{\theta_1}{\theta_2} + \left(\frac{\gamma+\delta}{\delta}\right)(A+BC) - \frac{\gamma C}{\delta}\right)^{-1} \tag{31}$$

4. System performance measures

Expected system length = Mean number of customers in the system

$$= EL_s$$

$$= \sum_{i=0}^{3}\sum_{n=1}^{\infty} n p_{n\,i}$$

Expected system length in breakdown state = Mean number of customers in the system in down-state of the server (repair state)

= EL[B]

$$= \sum_{n=1}^{\infty} np_{n\,3}$$

Probability of server being on vacations $P_v = \sum_{n=1}^{\infty} \sum_{i=0}^{1} p_{n\,i}$

Probability of server in the active (normal) state $P_w = \sum_{n=1}^{\infty} p_{n\,2}$

Probability of server being under repair $P_r = \sum_{n=1}^{\infty} p_{n\,3}$

5. Numerical results

In this section, the sensitivity of different performance measures of the queueing model towards the system parameters is analysed and the observed numerical results are graphically represented with aid of MATLAB software. For the purpose, the parameters are fixed as $\lambda_3 = 2.4$, , λ_1= 2, λ_0= 1.2, μ = 7, w = 0.3, γ = 0.7, $\lambda_2 = 2.2$, δ = 0.6, β = 0.8, θ_1= 0.6, θ_2= 0.8 , unless they are changed in graphs as shown.

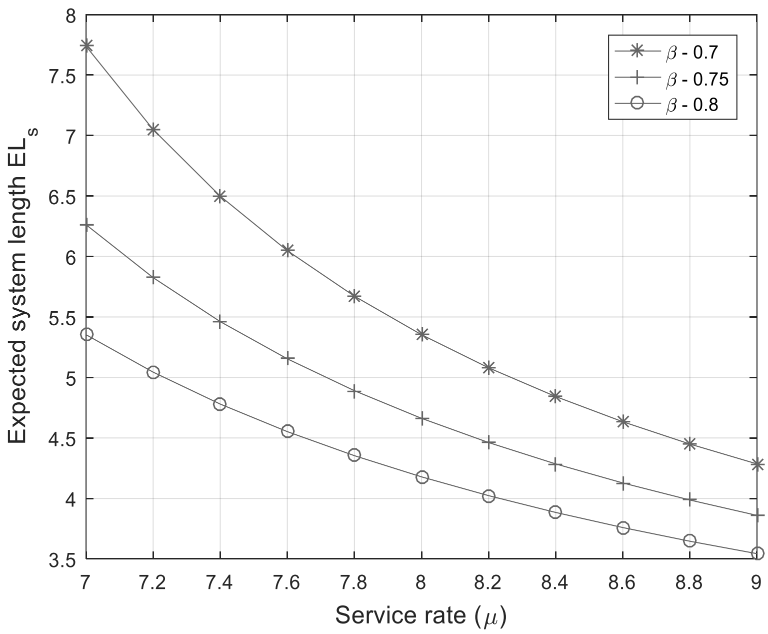

Figure 1: Effect of service rate μ on mean system length

Figure 1 shows variation in mean (expected) system length with μ. The mean system length decreases as μ increases. This is due to a decrease in mean service time with increasing μ. On increasing the probability of leaving the system β by unsatisfied customers on service completion, this system length goes on decreasing further, the reason being that with increasing β, the feedback probability decreases and hence the length of the system.

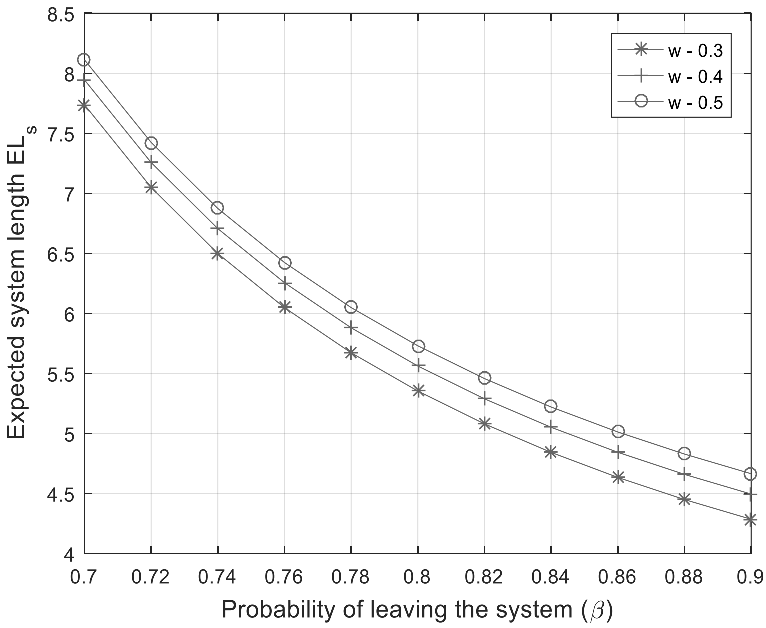

Figure 2: Variation in mean system length versus leaving probability β

Figure 2 illustrates how the expected system length varies with change in leaving probability of customers for different values of w. As w increases, the mean waiting time of the server in normal state decreases and this results in a corresponding increase in mean system length.

Figure 3 reveals the effect of variation in leaving probability β on the probability of the server being on vacation. As we observe from the figure, with an increase in leaving probability, the probability of the server being on vacations increases. This increase is more obvious with increasing values of repair rate δ; this is due to the reason that as δ increases, the mean repair time decreases, thereby increasing the probability of servers being on vacation.

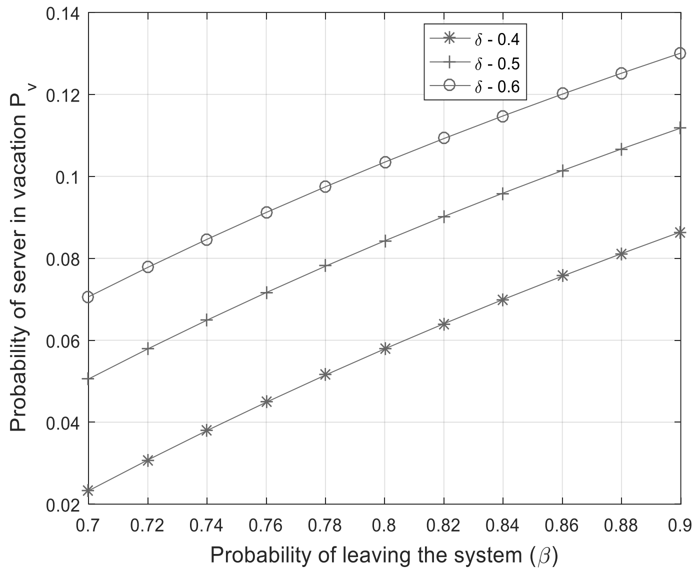

Figure 3: Variation in the probability of server in vacations versus β for different repair rates δ

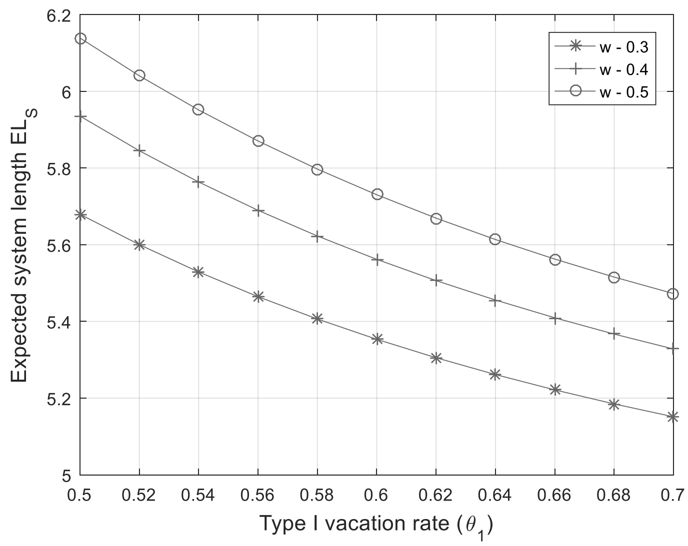

Figure 4: Effect of rate of type I vacation on expected system length

The effect of change in the rate of type I vacation on expected system length is depicted in figure 4. The expected system length decreases as θ_1 increases. This is due to an increase in the duration of type I vacation with decreasing θ_1. The Expected system length further increases as the parameter of waiting for the server w increases. The reason is that with an increase in w, the waiting time of the server in normal state decreases.

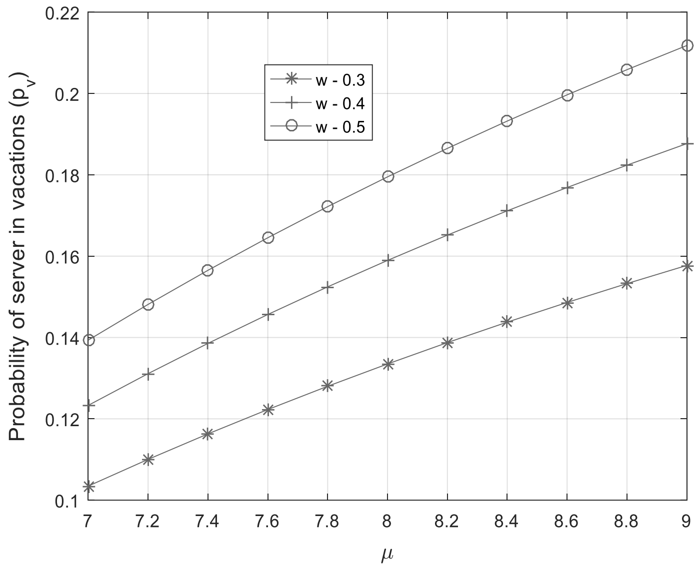

Figure 5: Variation in the probability of server in vacations versus service rate μ

The variation in the probability of the server being on vacation with service rate μ is shown in figure 5. As μ increases the mean service time decreases. This results in faster service thereby increasing the probability of the server being on vacation. This increase is more obvious with increasing the waiting time parameter w. This is because of the corresponding decrease in the mean waiting time of the server in the active (normal) state.

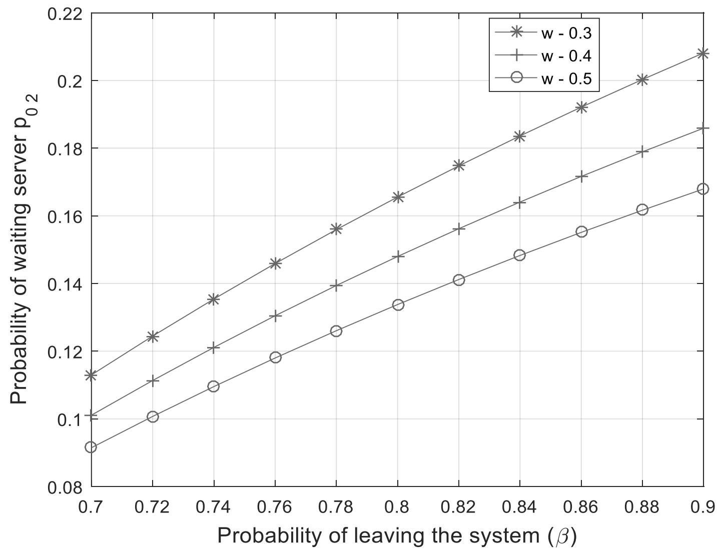

Figure 6: Probability of waiting server versus leaving probability β

Figure 6 represents the effect of leaving probability β of the customer after being served on the probability of waiting server $p_{0\,2}$. The probability of waiting server increases with an increase in the probability of leaving the system b. As w increases, the probability of waiting for server $p_{0\,2}$ decreases. This is due to a decrease in mean waiting time with an increase in w.

6. Cost optimization

In this section, the operating cost function is optimized relative to μ. Here we define some cost elements as

C_{L_S} = Cost per unit time for the customer present in the system.

C_μ = Cost per unit time for service in the active state.

C_{θ_1}= Cost per unit time in the period of type I vacation.

C_{θ_2}= Cost per unit time in the period of type II vacation.

C_γ= Cost per unit time in the breakdown state

C_δ= Cost per unit time for repair

The cost function per unit time is defined as

F(μ) = $EL_S C_{L_S}$ + μC_μ + $\theta_1 C_{\theta_1}$ + $\theta_2 C_{\theta_2}$+ γC_γ+ δC_δ

Fix C_{L_S}= 14, C_μ= 20, C_{θ_1}= 10,C_{θ_2}= 8, C_γ= 7, C_δ = 10 in parabolic method to find the optimal cost F(x) and corresponding value of x. This method starts by generating the quadratic function through calculated points in every iteration. The point at which F(x) is optimum in three-point pattern $\{x_1, x_2, x_3\}$ is given by

$$x_L = \frac{0.5(F(x_1)({x_2}^2 - {x_3}^2) + F(x_2)({x_3}^2 - {x_1}^2) + F(x_3)({x_1}^2 - {x_2}^2))}{F(x_1)(x_2 - x_3) + F(x_2)(x_3 - x_1) + F(x_3)(x_1 - x_2)}$$

This value obtained improves the current three-point pattern by replacing one of the three points. Optimum value up to the desired degree of accuracy is obtained by recursively using the process.

Table 1 shows that optimum value $F(\mu)$ =310.19155 with the permissible error of 10^{-4} for μ= 7.36987. This value verifies the results of Figure 7.

S.No	μ_1	μ_2	μ_3	$F(\mu_1)$	$F(\mu_2)$	$F(\mu_3)$	μ_L
1	7.0	7.50000	8.00000	311.66873	310.34146	313.16952	7.40971
2	7.0	7.40971	7.50000	311.66873	310.20608	310.34146	7.38091
3	7.0	7.38091	7.40971	311.66873	310.19268	310.20608	7.37334
4	7.0	7.37334	7.38091	311.66873	310.19166	310.19268	7.37086
5	7.0	7.37086	7.37334	311.66873	310.19155	310.19166	7.37016
6	7.0	7.37016	7.37086	311.66873	310.19155	310.19155	7.36994
7	7.0	7.36994	7.37016	311.66873	310.19154	310.19155	7.36987

Table 1: Optimal service rate for operating cost by quadratic fit approach

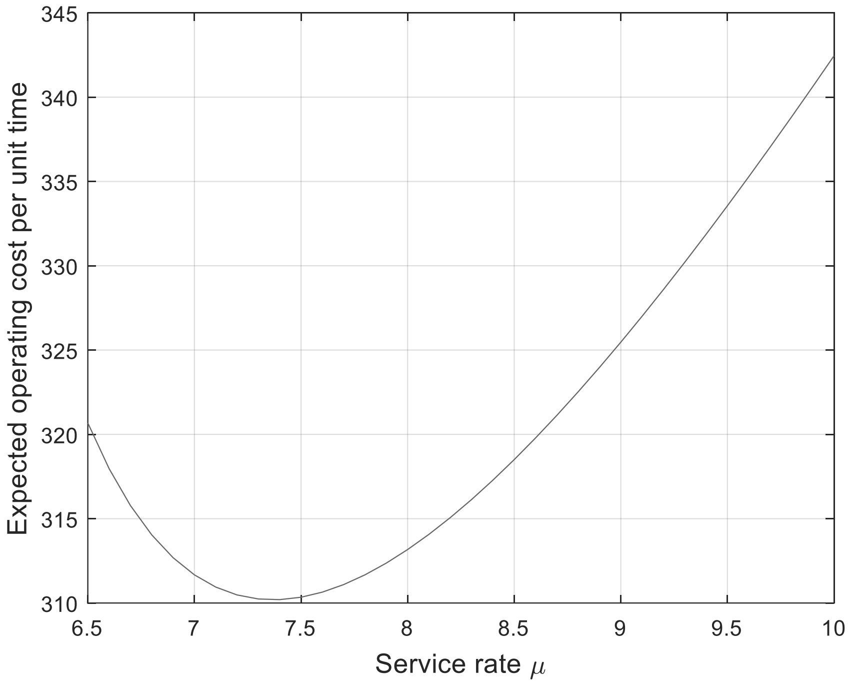

Figure 7: Expected operating cost per unit time versus service rate μ

7. Conclusion

The queueing model under differentiated vacations with an unreliable waiting server and Bernoulli schedule feedback of customers is considered. The impact of state-dependent arrival of customers is studied on the queueing model in steady-state. The sensitivity of some important system measures towards feedback probability, waiting parameter of server, service rate, and duration of vacations is illustrated graphically. Cost is also optimized for the model using the parabolic method. The model can be extended to bulk arrival, general service times for future research.

Conflicts of interests

The authors declare that there is no conflict of interest.

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