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# About two countable families in the finite sets of the Collatz Conjecture

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**Abstract**

With $\boldsymbol{t} \in \mathbb{N}$ we define the sets $\boldsymbol{K_t}$ and $\boldsymbol{K_t^*}$ containing all positive integers that converge to 1 in *t* iterations in the form of Collatz algorithm. The following are the properties of the $\{\boldsymbol{K_t}\}_{\boldsymbol{t}\in\mathbb{N}}$ and $\{\boldsymbol{K_t^*}\}_{\boldsymbol{t}\in\mathbb{N}}$: countability, empty intersection between the elements of the same family, and - at the end of the work - we conjecture that both of the two families are a partition of $\mathbb{N}_{\boldsymbol{0}}$. We demonstrate also that each set $\boldsymbol{K_t}$ and $\boldsymbol{K_t^*}$ is the union of two sets, a set includes even positive integers, the other, if it is non-empty, includes odd positive integers different from 1 and we go on proving that the maximum of each set $\boldsymbol{K_t}$ and $\boldsymbol{K_t^*}$ is $2^t$ and that $\boldsymbol{K_t} \cap \boldsymbol{K_t^*} = \{\boldsymbol{2^t}\}$.

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# 1 Introduction

Let us consider the Collatz conjecture (Leggerini, 2004), also known as the $3n + 1$ problem. We start from a positive integer *n*, if it is even we divide it by two, if it is odd we multiply it by three and add one to it, then we start over by applying the same rules on the number obtained. For example, starting from 3 the sequence is generated: 3, 10, 5, 16, 8, 4, 2, 1. In the second form of the algorithm of $3n + 1$ we calculate $\frac{3n+1}{2}$ if *n* is odd. With 3 we obtain the sequence 3, 5, 8, 4, 2, 1. It is conjectured that, from any positive integer we start, the sequences always arrive at 1 in a finite number of steps. It seems that all trajectories fall into the banal cycle 4, 2, 1 if *n* > *2*. The conjecture has not yet been proven and many mathematicians believe the question be undecidable (Conway, J. H, 1972). By applying the algorithm to a positive integer *n*, a sequence of integers is generated which we will call a **sequence** or **trajectory** of $n$ which we will denote with $T(n)$ (or $T^*(n)$ with the second form of the algorithm). For example $T(5) = \{5,16,8,4,2,1\}$ and $T^*(3) = \{3,5,8,4,2,1\}$. Let $\mathbb{N} = \{0, 1, 2 \dots\}$ and $\mathbb{N}_0 = \{1, 2, 3 \dots\}$. If $i \in \mathbb{N}$ and $n \in \mathbb{N}_0$, we denote by $T_i(n)$ the element of place *i* in the trajectory $T(n)$. If $i = 0$ we set $T_0(n) = n$. The same meaning will have $T_i^*(n)$. For example $T_0(5) = 5, T_3(5) = 4,\ T_2^*(3) = 8$. We define convergent a trajectory that contains the number 1. In any trajectory containing 1 we will ignore the terms subsequent. If the trajectory generated by the integer *n* converges we will say that the number *n* converges. Any number of a trajectory will be treated as a positive integer. The term ”*t-convergent*” will be equivalent to ”convergent in *t* iterations”. We will call the number *t* the convergence time. The notation $k_t$ will indicate that the positive integer *k* is *t-convergent*. In the following TC will be the set of convergence times of the converging positive integers.

# 2 The two forms of Collatz Conjecture

**First form**. With $n \in \mathbb{N}_0$ e $i \in \mathbb{N}_0$ the algorithm is the iteration of the function:

$$T_i(n) = \begin{cases} \frac{T_{i-1}(n)}{2} & \text{if } T_{i-1}(n) \equiv 0(mod2) \\ 3 \cdot T_{i-1}(n) + 1 & \text{if } T_{i-1}(n) \equiv 1(mod2) \end{cases} \quad (2.1)$$

with $T_0(n) = n$ if $i = 0$.

**Second form**. With $n \in \mathbb{N}_0$ and $i \in \mathbb{N}_0$, the algorithm is the iteration of the function:

$$T_i^*(n) = \begin{cases} \frac{T_{i-1}^*(n)}{2} & \text{if } T_{i-1}^*(n) \equiv 0 (mod\ 2) \\ \frac{3T_{i-1}^*(n)+1}{2} & \text{if } T_{i-1}^*(n) \equiv 1 (mod\ 2) \end{cases} \tag{2.2}$$

with $T_0^*(n) = n\ \ if\ i = 0$.

# 3 Construction of the sets K

Let us put in the same set $K_t$ the totality of positive integers *t-convergent* with the algorithm in the first form:

$$\forall t \in \mathbb{N}, K_t = \{k \in N_0 : k = k_t\}. \tag{3.1}$$

For example, applying the algorithm in the first form:

$K_0 = \{1\}$ because 1 converges to 1 in zero iterations;
$K_1 = \{2\}$ because 2 converges to 1 in an iteration;
$K_2 = \{4\}$ because 4 converges to 1 in two iterations;
… … … …

If the Collatz algorithm is used in the second form, in (3.1) we will add to $K_t$ and its elements the symbol $*$, that is:

$$\forall t \in \mathbb{N}, K_t^* = \{k^* \in \mathbb{N}_0 : k^* = k_t^*\}. \tag{3.2}$$

**Proposition 3.1. (Basic)**
*$\forall t \in \mathbb{N}, K_t$ e $K_t^*$ are non-empty.*

*Proof*. Trivially: whatever $t \in \mathbb{N}$, the number $2^t$ converges to 1 in *t* iterations, hence in $K_t$ there is at least $2^t$. For the same reason $k_t^*$ is also non-empty. □

We consider the set *TC* of all times of convergence. Since each $t \in \mathbb{N}$ can be associated with a $K_t$ and a $K_t^*$ by means of $2^t$ and vice versa, we can state that $TC = \mathbb{N}$ and that the families $\{K_t\}_{t\in\mathbb{N}}$ and $\{K_t^*\}_{t\in\mathbb{N}}$ are countable.

The following corollaries then hold.

**Corollary 3.2.**
*Any positive or null integer is a time of convergence.*

**Corollary 3.3.**
*Each of the families* $\{K_t\}_{t\in\mathbb{N}}$ *and* $\{K_t^*\}_{t\in\mathbb{N}}$ *is countable.*

**Proposition 3.4.**
*If* $t_1$ *and* $t_2$*, with* $t_1 \neq t_2$*, are in the set TC, then it results:*

*i*) $K_{t_1} \cap K_{t_2} = \emptyset$
*i**) $K_{t_1}^* \cap K_{t_2}^* = \emptyset$ .

Proof. *i*) **Algorithm in the first form.** By Proposition 3.1, $K_{t_1}$and $K_{t_2}$ are non-empty. Assume that $K_{t_1} \cap K_{t_2} \neq \emptyset$ , with $t_1 \neq t_2$. If $k \in K_{t_1} \cap K_{t_2}$ then $k$ must converge in the same number of iterations, so $t_1 = t_2$, against the hypothesis. Therefore $K_{t_1} \cap K_{t_2} = \emptyset$ . •

*i**) **Algorithm in the second form**. The proof is similar to the previous one: just insert the asterisk to the sets $K_t$ . □

Each family $\{K_t\}_{t\in\mathbb{N}}$ and $\{K_t^*\}_{t\in\mathbb{N}}$ divides $\mathbb{N}_0$ into classes that we cannot consider at the moment of equivalence.

# 4 Decomposition of sets K

Let $t \in \mathbb{N}_0$. Applying the first form of the Collatz algorithm we will prove that each set $K_t$ is formed by a set $A_t$ and a set $B_t$, that is $K_t = A_t \cup B_t$ with $A_t$ containing only even numbers and $B_t$ empty or containing only odd numbers different from 1. Applying the second form of the Collatz algorithm we will prove that $K_t^* = A_t^* \cup B_t^*$ with $A_t^*$ containing only even numbers and $B_t^*$ empty or containing only odd numbers different from 1. We will also prove that the elements of $K_t$ can be obtained from all the elements of $K_{t-1}$ and the elements of $K_t^*$ can be obtained from all elements of $K_{t-1}^*$. If $t = 0$ it is $K_0 = K_0^* = \{1\}$ and therefore $2K_0 = 2K_0^* = \{2\} = K_1 = K_1^*$. Some $B$ sets are empty such as sets $B_1, B_1^*, B_2, B_2^*, B_3, B_3^*, B_4, B_4^*, B_6, B_8, B_{10}$. I don't know if there are other empty B sets. In this study $A_0 = \{1\}$ , $A_0^* = \{1\}$ , $B_0 = \emptyset$ and $B_0^* = \emptyset$ .

Let $t \in \mathbb{N}$. Here we will assume that $K_{t+1}$ is made up of two sets of numbers:

1) by the doubles of the numbers of $K_t$;
2) from the integers $b \neq 1$ which are odd solutions in $\mathbb{N}_0$ of the equation $3b + 1 = k_t$ , with $k_t \in K_t$ , $k_t$ even and $k_t \neq 4$;

and that $K_{t+1}^*$ is formed by two sets of numbers:

1*) by the doubles of the numbers of $K_t^*$;
2*) from the integers $b^* \neq 1$ which are odd solutions in $\mathbb{N}_0$ of the equation $\frac{3b^*+1}{2} = k_t^*$ , with $k_t^* \in K_t^*$ and $k_t^* \neq 2$.

Called $P$ the set of even positive integers, we denote by $2K_t$ (*set of even derivatives of the first type or set of even derivatives of* $K_t$ *or set of doubles of the first type*) the set obtained by doubling all the numbers of $K_t$:

$$\forall t \in \mathbb{N}, 2K_t = \{a \in P: a = 2k_t, k_t \in K_t\}. \quad (4.1)$$

We denote by $2K_t^*$ (*set of even derivatives of the second type or set of even derivatives of* $K_t^*$ *or set of doubles of the second type*) the set obtained by doubling all the numbers of $K_t^*$:

$$\forall t \in \mathbb{N}, 2K_t^* = \{a^* \in P: a^* = 2k_t^*, k_t^* \in K_t^*\}\ . \quad (4.2)$$

We denote by $B_{t+1}$ (*set of odd derivatives of* $K_t$ *or set of odd derivatives of the first type*) the numbers with the property 2) and by $B_{t+1}^*$ (*set of odd derivatives of* $K_t^*$ *to set of odd derivatives of second type*) numbers with the property 2$^*$). Called $D$ the set of integers odd positive, the set of odd derivatives of $K_t$ we have:

$$\forall t \in \mathbb{N}, B_{t+1} = \{b \in D - \{1\}: 3b + 1 = k_t, k_t \in K_t \cap P, k_t \neq 4\} \quad (4.3)$$

while the set of odd derivatives of $K_t^*$ is

$$\forall t \in \mathbb{N}, B_{t+1}^* = \left\{b^* \in D - \{1\}: \frac{3b^*+1}{2} = k_t^*, k_t^* \in K_t^*, k_t^* \neq 2\right\}. \quad (4.4)$$

**Theorem 4.1.** (Theorem of the inclusion of doubles)
*The even derivative of* $K_t$ *(*$K_t^*$*) is contained in* $K_{t+1}(K_{t+1}^*)$, *that is*:

$$\text{a) } \forall t \in \mathbb{N}, 2K_t \subseteq K_{t+1} \qquad \text{b) } \forall t \in \mathbb{N}, 2K_t^* \subseteq K_{t+1}^*. \quad (4.5)$$

*Proof*. By Corollary 3.2 every $t$ is a time of convergence. Given $t \in \mathbb{N}$, we consider $K_t$ (*which is non-empty by Proposition 3.1*). Trivially: $\forall k_t \in K_t$, the trajectory $T(k_t) = \{k_t, \dots, 4,2,1\}$ is contained in the trajectory $T(2k_t) = \{2k_t, k_t, \dots, 4,2,1\}$. This means that $2k_t$ is *(t+1)-convergent*, so $2k_t \in K_{t+1}$. •

If $K_{t+1}$ is devoid of odd numbers, only the sign of equality holds. To prove it, let's suppose that $K_{t+1}$ is devoid of odd numbers and that, absurdly, it contains an even number $a_{t+1}$ which does not is double of any number of $K_t$. Since the even $a_{t+1}$ is also is *(t+1)-convergent*, the trajectory $T(a_{t+1}) = \left\{a_{t+1}, \frac{a_{t+1}}{2}, \dots, 4,2,1\right\}$ will contain the trajectory $T\left(\frac{a_{t+1}}{2},\right) = \left\{\frac{a_{t+1}}{2}, \dots, 4,2,1\right\}$ so $\frac{a_{t+1}}{2}$ is *t-convergent*, that is $\frac{a_{t+1}}{2} \in K_t$, against our hypothesis. It follows that $2K_t$ coincides with $K_{t+1}$ if this is devoid of odd. Then the relation a) of (4.5) holds for the arbitrariness of *t*. •

In the case of $2K_t^*$ proceed in the same way, *mutatis mutandis*. □

**Theorem 4.2.** (Odd derivative theorem of the first type)
*Let $k_t$ be even and $k_t \neq 4$ . If there is a positive integer b satisfying the equation*

$$3b + 1 = k_t \tag{4.6}$$

*then*

$$b = \frac{k_t - 1}{3} \tag{4.7}$$

*belongs to $B_{t+1}$.*

*Proof*. Let *b* and $K_t$ satisfy the hypotheses. Since *b* is odd and different from 1, its successor is $k_t$, because to *b* is applied (2.1), so the trajectory $T(b) = T\left(\frac{k_t-1}{3}\right) = \left\{\frac{k_t-1}{3}, k_t, \dots, 4,2,1\right\}$ contains the trajectory $T(k_t) = \{k_t, \dots, 4,2,1\}$. This means that *b* converges in *t + 1* iterations, that is $b \in B_{t+1}$. □

Recall that an odd derivative $B_t$ either is empty or is formed only by odd positive different from 1.

**Theorem 4.3.** (Theorem of strict inclusion of odd derivatives of the first type)
*The odd derivative of $K_t(K_t^*)$ is strictly contained in $K_{t+1}(K_{t+1}^*)$, that is:*

$$\text{a) } \forall t \in \mathbb{N}, B_{t+1} \subset K_{t+1} \qquad \text{b) } \forall t \in \mathbb{N}, B_{t+1}^* \subset K_{t+1}^*. \tag{4.8}$$

*Proof*. By Proposition 3.1 every $K_{t+1}(K_{t+1}^*)$ is non-empty because it contains at least the even number $2^{t+1}$, therefore $B_{t+1}$ even if it were empty could not coincide with $K_{t+1}(K_{t+1}^*)$. □

**Theorem 4.4.** (Theorem of the union of even and odd derivatives of the first type) *The set $K_{t+1}$ is the union of the set of doubles of $K_t$ and of the odd derivative of $K_t$, that is:*

$$\forall t \in \mathbb{N}, K_{t+1} = 2K_t \cup B_{t+1}. \tag{4.9}$$

*(remarkable equality, algorithm in first form)*

*Proof*. Let us consider $K_t$, with $t \in \mathbb{N}$. It is necessary to demonstrate that

**1)** there are no other even integers *(t+1)-convergent* beyond those of $2K_t$;
**2)** the odd numbers *(t+1)-convergent* are only those of $B_{t+1}$.

**We prove 1).** We denote by $A_{t+1}$ the totality of even positive integers converging in *t + 1* iterations that we know to be non-empty (each $A_t$ contains at least $2^t$). It immediately turns out that $\forall t \in \mathbb{N}, 2K_t \subseteq A_{t+1}$.
We show that

$$\forall t \in \mathbb{N}, 2K_t = A_{t+1}. \tag{4.10}$$

If for a fixed $t \in \mathbb{N}$ there were an even $a_{t+1} \in A_{t+1}$ that was not double of any positive integer of $K_t$, it would be absurd because the trajectory $T(a_{t+1}) = \left\{a_{t+1}, \frac{a_{t+1}}{2}, \dots 4,2,1\right\}$ would contain the trajectory $T\left(\frac{a_{t+1}}{2}\right) = \left\{\frac{a_{t+1}}{2}, \dots ,4,2,1\right\}$ whose seed at $\frac{a_{t+1}}{2} \in K_t$ and whose double $a_{t+1}$ is in $A_{t+1}$, against the hypothesis. Hence the strict inclusion cannot hold and, by the arbitrariness of *t*, (4.10) is true. •

**We prove 2).** With the same fixed $t \in \mathbb{N}$, we denote by $\beta_{t+1}$ the totality of the odd positive integers converging in *t + 1* iterations . Obviously we have $B_{t+1} \subseteq \beta_{t+1}$.
We show that

$$\forall t \in \mathbb{N}, B_{t+1} = \beta_{t+1} . \tag{4.11}$$

If for the fixed *t*, $\beta_{t+1} = \emptyset$, then also $B_{t+1} = \emptyset$ and therefore $K_{t+1} = 2K_t$, that is (4.9) for the arbitrariness of *t*.
Otherwise, for fixed *t*, let $\beta_{t+1} \neq \emptyset$ . If there was an $b_{t+1} \in \beta_{t+1}$ not coming by any even of $K_t$, that is such that $b_{t+1} \notin B_{t+1}$ , then an absurdity would follow because the trajectory $T(b_{t+1}) = \{b_{t+1}, 3b_{t+1} + 1, \dots ,4,2,1\}$ would contain the trajectory $T(3b_{t+1} + 1) = \{3b_{t+1} + 1, \dots ,4,2,1\}$ whose even seed $3b_{t+1} + 1 = k_t \in K_t$ , therefore, by Theorem 4.2, $b_{t+1} \in B_{t+1}$ against the hypothesis. For this reason strict inclusion cannot be valid and, due to the arbitrariness of *t* (4.11) is true. •

From **1)** and **2)** follows the remarkable equality (4.9). □

By (4.10), (4.9) becomes

$$\forall t \in \mathbb{N}, K_{t+1} = A_{t+1} \cup B_{t+1} \,. \tag{4.12}$$

(*remarkable equality, algorithm in the first form*)

If for a given *t* the derivative $B_{t+1}$ of $K_t$ is empty, we have

$$K_{t+1} = A_{t+1}. \tag{4.13}$$

We now find the numbers of $B^*_{t+1}$.

**Theorem 4.5.** (Theorem of the odd derivative of the second type)
*Let* $k^*_t \in \mathbb{N}_0, k^*_t \neq 2$ *. If there exists the positive integer b satisfying the equation*

$$3b^* + 1 = 2k^*_t \tag{4.14}$$

*then*

$$b^* = \frac{2k^*_t - 1}{3} \tag{4.15}$$

*belongs to* $B^*_{t+1}$.

*Proof*. Let $k^*_t$ and $b^*$ satisfy the hypotheses. Since $b^*$ is odd and different from 1, its successor is $k^*_t$, because (2.2) is applied to $b^*$, so the trajectory $T(b^*) = T\left(\frac{2k^*_t - 1}{3}\right) = \left\{\frac{2k^*_t - 1}{3}, k^*_t, \ldots, 4,2,1\right\}$ contains the trajectory $T(k^*_t) = \{k^*_t, \ldots, 4,2,1\}$ . This means that $b^*$ converges in *t + 1* iterations, that is $b^* \in B^*_{t+1}$. □

Recall that an odd derivative $B^*_t$ o is either empty or is formed only by odd positive integers different from 1.

As shown for (4.10) it results

$$\forall t \in \mathbb{N}, 2K^*_t = A^*_{t+1} \tag{4.16}$$

where $A^*_{t+1}$ is the totality of the even positive integers *(t+1)-convergent*, that is of the doubles of the numbers of $K^*_t$. Equation (4.16) is demonstrated how it is done for the first part of the proof of the Theorem 4.4 by adding the asterisk $*$ to the $2K_t$ and $A_t$ sets. Equation (4.16) will occur in the proof of first part of Theorem 4.6.

**Theorem 4.6.** (Theorem of the union of even and odd derivatives of the second type) *The set $K^*_{t+1}$ is the union of the set of doubles of $K^*_t$ and the odd derivative of $K^*_t$, that is:*

$$\forall t \in \mathbb{N}, K^*_{t+1} = 2K^*_t \cup B^*_{t+1} \,. \tag{4.17}$$

(*remarkable equality, algorithm in the second form*)

Proof. We proceed as in the proof of Theorem 4.4 adding the asterisk $*$ to all the sets and considering, in the second part, $\left(\frac{3b^*_{t+1}+1}{2}\right)$ as successor of $b^*_{t+1} \in \beta^*_{t+1}$ . □

By (4.16), (4.17) can be written

$$\forall t \in \mathbb{N}, K^*_{t+1} = A^*_{t+1} \cup B^*_{t+1} \tag{4.18}$$

(*remarkable equality, algorithm in the second form*)

and if, for a certain *t*, the derivative $B^*_{t+1}$ of $K^*_{t+1}$ it is empty, then

$$K^*_{t+1} = A^*_{t+1} \,. \tag{4.19}$$

# 5 Examples

To obtain the set $K_{t+1}$ it will be necessary to double all the numbers $k_t$ of $K_t$ in order to have $A_{t+1}$ and it will be necessary to determine all the numbers $b \in B_{t+1}$ starting from the even numbers $k_t$ of $K_t$, that is, it will be necessary to verify if $k_t - 1$ is divisible by three when $k_t$ is even with $k_t \neq 4$ (Theorem 4.2 and definition of $B_{t+1}$ in (4.3)).

► We determine the sets $K_8$ and $K_9$.

$K_8$

We use the set $K_7 = \{3,20,21,128\}$ . We have $A_8 = 2K_7 = \{6,40,42,256\}$ . It turns out $B_8 = \emptyset$ since none of the equations

(1) 3b + 1 = 20

(2) 3b + 1 = 128

has solutions in $\mathbb{N}_0$. Hence $K_8 = A_8 \cup \emptyset = \{6,40,42,256\}$.

$K_9$
We use the set $K_8 = \{6,40,42,256\}$. We have $A_9$= $2K_8$ = {12, 80, 84, 512}. We solve in $\mathbb{N}_0$ the following equations:

(1) 3b + 1 = 6

(2) 3b + 1 = 40

(3) 3b + 1 = 42

(4) 3b + 1 = 256.

The first and third equations have no solutions in $\mathbb{N}_0$. The second and fourth equations have as solutions in $\mathbb{N}_0$ 13 and 85 respectively, therefore $B_9$ = {13, 85}.
Thus $K_9 = A_9 \cup B_9 = \{12,80,84,512\} \cup \{13,85\} = \{12,13,80,84,85,512\}$ .

In the same way they are obtained

$K_{10} = \{4 - 26 - 160 - 168 - 170 - 1024\}$

$K_{11} = \left\{48 - 52 - \underline{53} - 320 - 336 - 340 - \underline{341} - 2048\right\}$

$K_{12} = \left\{\underline{17} - 96 - 104 - 106 - \underline{113} - 640 - 672 - 680 - 682 - 4096\right\}$

…

The underlined numbers are the odd derivatives of the previous set.

To obtain the set $K^*_{t+1}$ it will be necessary to double all the numbers $k^*_t$ of $K^*_t$ in order to have $A^*_{t+1}$ and it will be necessary to determine all the numbers $b^* \in B^*_{t+1}$ starting from each $k^*_t$ of $K^*_t$, that is, it will be necessary to verify whether $2k^*_t - 1$ is divisible by three when $k^*_t \neq 2$ (Theorem 4.5 and definition of $B^*_{t+1}$ in (4.4)).

► We determine the sets $K^*_5$ and $K^*_6$.

$K^*_5$
We consider $K^*_4 = \{5,16\}$. Its even derivative is $A^*_5 = \{10,32\}$. Of the two equations

(1) $\frac{3b^*+1}{2} = 5$

(2) $\frac{3b^*+1}{2} = 16$

only the first admits in $\mathbb{N}_0$ the solution $b^*$= 3 therefore $B_5^* = \{3\}$ e $K_5^* = A_5^* \cup B_5^* = \{3,10,32\}$.

$K_6^*$

We consider $K_5^* = \{3,10,32\}$. Its even derivative is $A_6^* = \{6,20,64\}$. Of the three equations

(1) $\frac{3b^*+1}{2} = 3$

(2) $\frac{3b^*+1}{2} = 10$

(3) $\frac{3b^*+1}{2} = 32$

only the third has solution $b^* = 21$ in $\mathbb{N}_0$. Hence $B_6^* = \{21\}$ e $K_6^* = A_6^* \cup B_6^* = \{6,20,21,64\}$.

In the same way they are obtained

$K_7^* = \{12 - 13^* - 40 - 42 - 128\}$

$K_8^* = \{24 - 26 - 80 - 84 - 85^* - 256\}$

$K_9^* = \{17^* - 48 - 52 - 53^* - 160 - 168 - 170 - 512\}$

$K_{10}^* = \{11^* - 34 - 35^* - 96 - 104 - 106 - 113^* - 320 - 336 - 340 - 341^* - 1024\}$

...

The numbers with an asterisk are the odd derivatives of the previous set.

# 6 The maxima of $K_t$ and $K_t^*$

By examining the sets K, we can suppose that the number $2^t$ is the maximum of every set $K_t$ and of every $K_t^*$. This is confirmed by the subsequent Theorem 6.2. The following Lemma 6.1 contains some obvious conclusions.

**Lemma 6.1.**
*i)* If $k_t \in K_t$ then $2k_t \in A_{t+1}, \forall t \in \mathbb{N}$
*ii)* If $a_t \in A_t$ then $2a_t \in A_{t+1}, \forall t \in \mathbb{N}_0$
*i∗)* If $k_t^* \in K_t^*$ then $2k_t^* \in A_{t+1}^*, \forall t \in \mathbb{N}$
*ii∗)* If $a_t^* \in A_t^*$ then $2a_t^* \in A_{t+1}^*, \forall t \in \mathbb{N}_0$ .

*Proof*. Recall that (4.10) and (4.16) hold.

*i)* Let $k_t \in K_t$, with $t \in \mathbb{N}$. The trajectory $T(k_t)$ is contained in the trajectory $T(2k_t) = \{2k_t, k_t, \dots, 4,2,1\}$ because $2k_t$ is an even that converges in $t + 1$ iterations, that is $2k_t \in A_{t+1}$. •

*ii)* Let $a_t \in A_t$, with $t \in \mathbb{N}_0$. Since $A_t \subseteq K_t$ is also $a_t \in K_t$. Applying *i)* it follows that $2a_t \in A_{t+1}$ $\forall t \in \mathbb{N}_0$. •

The *i∗)* and *ii∗)* prove to be the *i)* and *ii)* respectively, just asterisking the sets $K_t$, $A_t$ and their elements. □

**Theorem 6.2.** (Maxima theorem of $K_t$ and $K_t^*$)

$i)$ $\forall t \in \mathbb{N}_0, max(K_t) = 2^t$

*i*)* $\forall t \in \mathbb{N}_0, max(K_t^*) = 2^t$.

Proof. *i)* We will proceed by induction using the remarkable equality (4.12). If $t = 1$ then max $max(K_1) = 2^1 = 2$. Let us fix a $t > 1$ and let, by inductive hypothesis

$$max(K_t) = 2^t. \tag{6.1}$$

We will prove that it is also $max(K_{t+1}) = 2^{t+1}$. To do this, it will be necessary to prove that

**1**) max $max(A_{t+1}) = 2^{t+1}$

and

**2**) every number of $B_{t+1}$ is less than $2^{t+1}$.

***First part***

**1**) We show that every number of $A_{t+1}$ is less than or equal to $2^{t+1}$ and that $2^{t+1}$is in $A_{t+1}$. Let $k_t \in K_t$. Then, by hypothesis (6.1)

$$\forall k_t \in K_t, k_t \leq 2^t. \tag{6.2}$$

By the *i)* of Lemma 6.1

$$2k_t \in A_{t+1}. \tag{6.3}$$

From (6.2) it follows that

$$\forall k_t \in K_t, 2k_t \leq 2^{t+1}. \tag{6.4}$$

Since for the inductive hypothesis (6.1) it is $2^t \in K_t$, then, for the remarkable equality (4.12), we have $2^t \in A_t$, from which, for the *ii)* of Lemma 6.1, it follows that

$$2^{t+1} \in A_{t+1}. \tag{6.5}$$

From (6.3), (6.4) and (6.5) we obtain that $max(A_{t+1}) = 2^{t+1}$. •

***Second part***

**2**) If $B_{t+1} = \emptyset$ from (4.12) it follows that $K_{t+1} = A_{t+1}$ and from $max(A_{t+1}) = 2^{t+1}$ (***First part***) it follows that $max(K_{t+1}) = 2^{t+1}$. Let $B_{t+1} \neq \emptyset$. We show that every element $b_{t+1}$ of $B_{t+1}$ is less than $2^{t+1}$. The numbers of $B_{t+1}$ are the odd numbers of the form (4.7):

$$b_{t+1} = \frac{k_t - 1}{3} con\, k_t \in K_t\ and\ k_t\ even \tag{6.6}$$

but, from $k_t - 1 < k_t$ we get that

$$\frac{k_t - 1}{3} < k_t \tag{6.7}$$

then from (6.6), (6.7) and (6.1) it follows that

$$b_{t+1} < k_t \leq 2^t \tag{6.8}$$

and therefore: $\forall b_{t+1} \in B_{t+1}$, $b_{t+1} < 2^{t+1}$, that is **2**). From the first and the second part it follows that all the numbers of $K_{t+1}$ are less than or equal to $2^{t+1}$and this proves the *i).* •

*i**) We will proceed by induction using the remarkable equality (4.18). If $t = 1$ then $max(K_1^*) = 2^1$=2. Let, by inductive hypothesis, be

$$max(K_t^*) = 2^t \text{ con } t > 1 \text{ .} \quad (6.9)$$

We will prove that it is also $max(K_{t+1}^*) = 2^{t+1}$. To do this, it will be necessary to prove that

**1***) $max(A_{t+1}^*) = 2^{t+1}$

and

**2***) *every number of* $B_{t+1}^*$ *is less than* $2^{t+1}$ .

**First part** $*$

**1***) The proof is similar to that of the first part of *i)*, just adding the asterisk $*$ to the sets $A_{t+1}$, $K_{t+1}$ and their elements. Therefore $2^{t+1}$ is the maximum of $A_{t+1}^*$ and **1***) is proved. •

**Second part** $*$

**2***) If $B_{t+1}^* \neq \emptyset$ from 4.18) it follows that $K_{t+1}^* = A_{t+1}^*$ and from $max(A_{t+1}^*) = 2^{t+1}$ (***First part*****) it follows that $max(K_{t+1}^*) = 2^{t+1}$.

Let $B_{t+1}^* \neq \emptyset$. We show that every $b_{t+1}^*$ of $B_{t+1}^*$ is less than $2^{t+1}$. The numbers of $B_{t+1}^*$ are the odd numbers of the form (4.15):

$$b_{t+1}^* = \frac{2k_t^* - 1}{3}, \text{ with } k_t^* \in K_t^* \quad (6.10)$$

but, from $2k_t^* - 1 < 2k_t^*$ we get that

$$\frac{2k_t^* - 1}{3} < 2k_t^* \quad (6.11)$$

and for the inductive hypothesis (6.9) we also have that

$$\forall k_t^* \in K_t^*, 2k_t^* \leq 2^{t+1} \text{ .} \quad (6.12)$$

Finally for (6.10), (6.11), (6.12) we can write that $b^*_{t+1} < 2k^*_t \leq 2^{t+1}$, then $\forall t > 1$ all numbers $b^*_{t+1}$ di $B^*_{t+1}$ are less than $2^{t+1}$. The **2***) is thus proved. •

From **1***) and from **2***) it follows that all integers of $K^*_{t+1}$ are less than or equal to $2^{t+1}$ and so *i*)* is also proved. □

The following corollaries immediately follow from Theorem 6.2.

**Corollary 6.3.**

*i)* $\forall t \in \mathbb{N}, max(2K_t) = 2^{t+1}$

*i∗)* $\forall t \in \mathbb{N}, max(2K^*_t) = 2^{t+1}$ .

**Corollary 6.4.**

*i)* $\forall t \in \mathbb{N}_0, max(A_t) = 2^t$

*i∗)* $\forall t \in \mathbb{N}_0, max(A^*_t) = 2^t$ .

Theorem 6.2 provides indications on the type of numbers contained in the sets K: either there is only $2^t$ or there are positive integers less than or equal to $2^t$ and this means that each set K is finite. Therefore, the following corollary can also be stated.

**Corollary 6.5.**

$\forall t \in \mathbb{N}, K_t$ *and* $K^*_t$ *are finite.*

Each set K is formed by the finite numerical sets A and B. It follows that if B is non-empty then it has a maximum. Therefore the following corollary holds.

**Corollary 6.6.**

*i) If for* $t \in \mathbb{N}_0$ *is* $B_t \neq \emptyset$ *then* $\exists\ max(B_t)$

*i*) If for* $t \in \mathbb{N}_0$ *is* $B^*_t \neq \emptyset$ *then* $\exists\ max(B^*_t)$ .

In the following paragraph 7 we will investigate the maxima of the sets B.

# 7 On the maxima of the sets B

We give a strict increase of the maxima of the sets B.

**Proposition 7.1.**
*i) If* $B_{t+1} \neq \emptyset$ *then* $\exists k_t \in K_t, k_t \neq 4, k_t$ *even* $: max(B_{t+1}) < \frac{k_t - 1}{2}$
*i*) If* $B^*_{t+1} \neq \emptyset$ *then* $\exists k^*_t \in K^*_t, k^*_t \neq 2: max(B^*_{t+1}) < \frac{2k^*_t - 1}{2}$ .

*Proof*. *i*) If $B_{t+1} \neq \emptyset$, then by definition of $B_{t+1}$ in correspondence of every odd $b_{t+1} \in B_{t+1}$ will exist an even number $k_t \in K_t$ with $k_t \neq 4$ such that $b_{t+1} = \frac{k_t - 1}{3}$ but $\frac{k_t - 1}{3} < \frac{k_t - 1}{2}$, then $b_{t+1} < \frac{k_t - 1}{2}$. Then, in particular, *i*) holds also for the maximum of $B_{t+1}$. •

*i*)* If $B^*_{t+1} \neq \emptyset$, then by definition of $B^*_{t+1}$ in correspondence of every odd $b^*_{t+1} \in B^*_{t+1}$ will exist an even number $k^*_t \in K^*_t$ with $k^*_t \neq 2$ such that $b^*_{t+1} = \frac{2k^*_t - 1}{3}$ but $\frac{2k^*_t - 1}{3} < \frac{2\,k^*_t - 1}{2}$, then $b^*_{t+1} < \frac{2\,k^*_t - 1}{2}$. Then, in particular, also for the maximum of $B^*_{t+1}$ holds *i*)*. □

From Proposition 7.1 follows the following corollary which gives a plus a bit more large of the maxima of the sets B.

**Corollary 7.2.**
*i) If* $B_{t+1} \neq \emptyset$, *then* $max(B_{t+1}) < 2^t$
*i*) If* $B^*_{t+1} \neq \emptyset$, *then* $max(B^*_{t+1}) < 2^t$ .

Proof. *i)* If $B_{t+1} \neq \emptyset$, then the inequality *i)* of Proposition 7.1 holds and also $\frac{k_t - 1}{2} < k_t$ but, by Theorem 6.2, the maximum of $K_t$ is $2^t$, so $max(B^*_{t+1}) < 2^t$. •

*i*)* If $B^*_{t+1} \neq \emptyset$, then the inequality *i*)* of Proposition 7.1 holds and also $\frac{2k^*_t - 1}{2} < k^*_t$ but, by Theorem 6.2, the maximum of $K^*_t$ is $2^t$, so $max(B^*_{t+1}) < 2^t$. □

In some cases it is possible to determine the maximum of the sets B. Let's see how. The numbers of $B_{t+1}$and of $B^*_{t+1}$ come from the integer solutions, if they exist, of the equations

$$b_{t+1} = \frac{k_t - 1}{3} \text{ with } k_t \in K_t, k_t \text{ even and } k_t \neq 4 \qquad (7.1)$$

$$b^*_{t+1} = \frac{2k^*_t - 1}{3} \text{ with } k^*_t \in K^*_t \text{ and } k^*_t \neq 2 \qquad (7.2)$$

by the Theorems, respectively, 4.2 and 4.5. In fact, the largest odd integer that can be obtained from (7.1), if we substitute the maximum of $K_t$ for $k_t$, is $\frac{2^t-1}{3}$, which is integer if $2^t - 1$ is divisible by three. Likewise, the largest odd integer which can be obtained from (7.2), if we replace $k^*_t$ by the maximum of $K^*_t$, is $\frac{2^{t+1}-1}{3}$, which is integer if $2^{t+1} - 1$ is divisible by 3. We can therefore state the following theorem.

**Theorem 7.3.**
*i) If* $2^t - 1 \equiv 0 (mod\ 3)$*, with* $t \in \mathbb{N}_0$ *and* $t > 2$*, then* $max(B_{t+1}) = \frac{2^t-1}{3}$

*i*) If* $2^{t+1} - 1 \equiv 0 (mod\ 3)$*, with* $t \in \mathbb{N}_0$ *and* $t > 1$*, then* $max(B^*_{t+1}) = \frac{2^{t+1}-1}{3}$.

SECOND DEMONSTRATION OF THE THEOREM 7.3

*Proof. i)* By hypothesis the number $2^t$ is *t-convergent* and the equation $3b + 1 = 2^t$ is satisfied by $b = \frac{2^t-1}{3}$ which is different from 1 because $t > 2$, therefore, by theorem 4.2 it is $b \in B_{t+1}$. Assume that $\exists \beta \in B_{t+1}: b < \beta$ that is, taking into account the form of $b$ and $\beta$, we suppose that it is $\frac{2^t-1}{3} < \frac{k_t-1}{3}$ with $k_t \in K_t$ e $k_t$ even; from this it follows that $2^t < k_t$, absurd thing because the maximum of $K_t$ is $2^t$. Then it must turn out $\forall \beta \in B_{t+1}: \beta \leq b$, that is the thesis. •

*i*)* By hypothesis the number $2^t$ is *(t+1)-convergent* and the equation $3b + 1 = 2^{t+1}$ is satisfied by $b = \frac{2^{t+1}-1}{3}$ which is different from 1 because $t > 1$, therefore, by theorem 4.5 it is $b^* \in B^*_{t+1}$. Assume that $\exists \beta^* \in B^*_{t+1}: b^* < \beta^*$ that is, taking into account the form of $b^*$ and $\beta^*$, supposing it is $\frac{2^{t+1}-1}{3} < \frac{2k^*_t-1}{3}$ with $k^*_t \in K^*_t$, from this it follows that $2^t < k^*_t$, which is absurd because the maximum of $K^*_t$ is $2^t$. Then it must turn out $\forall \beta^* \in B^*_{t+1}: \beta^* \leq b^*$ that is the thesis. □

For example:

- ...

- for $t = 14$ risults $2^{14} - 1 \equiv 0(mod3)$, then $max(B_{15}) = max(B^*_{14}) = 5461$
- for $t = 16$ risults $2^{16} - 1 \equiv 0(mod3)$, then $max(B_{17}) = max(B^*_{16}) = 21845$
- for $t = 18$ risults $2^{18} - 1 \equiv 0(mod3)$, then $max(B_{19}) = max(B^*_{18}) = 87381$

- ....

# 8 On the intersection of $K_t$ and $K^*_t$

In this paragraph we will prove that the intersection of the sets $K_t$ and $K^*_t$ is $\{2^t\}$.

**Lemma 8.1.**
*The intersection of the odd derivatives of the first type t-convergent and of the even derivatives of the second type t-convergent is empty, that is*

$$\forall t \in \mathbb{N}_0,\ B_t \cap A^*_t = \emptyset. \tag{8.1}$$

*Proof.* Obviously, because an odd derivative either is empty or is made up of odd integers different from 1 and an even derivative contains only even numbers. □

**Lemma 8.2.**
*The intersection of the odd derivatives of the second type t-convergent and of the even derivatives of the first type t-convergent is empty, that is*

$$\forall t \in \mathbb{N}_0, B^*_t \cap A_t = \emptyset. \tag{8.2}$$

*Proof*. Obviously, because an odd derivative either is empty or is made up of odd integers different from 1 and an even derivative contains only even numbers. □

**Lemma 8.3.**
*The intersection of the odd derivatives of the first type t-convergent and of the odd derivatives of the second type t-convergent is empty, that is*

$$\forall t \in \mathbb{N}_0, B_t \cap B_t^* = \emptyset. \tag{8.3}$$

*Proof*. Trivially, if $t = 1$ the sets $B_1$ and $B_1^*$ are both empty. Assume absurdly that for $t > 1$ it results $B_t \cap B_t^* \neq \emptyset$ and consider every $n_t \in B_t \cap B_t^*$.

From

$$n_t \in B_t = \{n_t \in D - \{1\}: 3n_t + 1 = k_{t-1}, k_{t-1} \in K_{t-1} \cap P, k_{t-1} \neq 4, k_{t-1} - 1 \equiv 0(mod\ 3)\}$$

follows that $n_t$ is an odd integer of the form (4.7), that is $n_t = \frac{k_{t-1}-1}{3}$.

From

$$n_t \in B_t^* = \left\{n_t \in D - \{1\}: \frac{3n_t + 1}{2} = k_{t-1}^*, k_{t-1}^* \in K_{t-1}^*, k_{t-1}^* \neq 2, 2k_{t-1}^* - 1 \equiv 0(mod\ 3)\right\}$$

it follows that $n_t$ is an odd integer of the form (4.15), that is $n_t = \frac{2k_{t-1}^*-1}{3}$.

By equating the two expressions of $n_t$ we have $\frac{k_{t-1}-1}{3} = \frac{2k_{t-1}^*-1}{3}$ and therefore

$$k_{t-1} = 2k_{t-1}^*. \tag{8.4}$$

Equality (8.4) is manifestly absurd because $k_{t-1}$ is *(t-1)-convergent* and $2k_{t-1}^*$ is *t-convergent*. Therefore it makes no sense to suppose that the intersection $B_t \cap B_t^*$ for $t > 1$ is non-empty and (8.3) is proved. □

**Lemma 8.4.**
*The intersection of $K_t$ and $K_t^*$ is equal to the intersection of the even derivatives of $K_{t-1}$ and of the derivatives even of $K_t^*$, that is*

$$\forall t \in \mathbb{N}_0, K_t \cap K_t^* = A_t \cap A_t^*. \tag{8.5}$$

*Proof*. We will use the notable equalities 4.12) and 4.18). We have $\forall t \in \mathbb{N}_0$:

$$K_t \cap K_t^* = (A_t \cup B_t) \cap (A_t^* \cup B_t^*) =$$

$$= \big((A_t \cup B_t) \cap A_t^*\big) \cup \big((A_t \cup B_t) \cap B_t^*\big) =$$

$$= (A_t \cap A_t^*) \cup (B_t \cap A_t^*) \cup (A_t \cap B_t^*) \cup (B_t \cap B_t^*). \qquad (8.6)$$

The thesis follows by applying, in order, Lemmas 8.1, 8.2 and 8.3 to the second, third and fourth intersection in the last line of (8.6). □

**Lemma 8.5**.
*The intersection of the even derivatives of the first and second type t-convergent is* $\{2^t\}$*, that is:*

$$\forall t \in \mathbb{N}_0, A_t \cap A_t^* = \{2^t\}. \qquad (8.7)$$

*Proof*. Applying the equalities (4.10) and (4.16) to the intersection $A_t \cap A_t^*$ we have:

$$\forall t \in \mathbb{N}_0, A_t \cap A_t^* = 2K_{t-1} \cap 2K_{t-1}^* = 2(K_{t-1} \cap K_{t-1}^*). \qquad (8.8)$$

Applying Lemma 8.4 to the intersection in the last parenthesis of (8.8) we have

$$\forall t \in \mathbb{N}_0, 2(K_{t-1} \cap K_{t-1}^*) = 2(A_{t-1} \cap A_{t-1}^*) =$$
$$= 2(2K_{t-2} \cap 2K_{t-2}^*) = 2^2(K_{t-2} \cap K_{t-2}^*). \qquad (8.9)$$

Applying Lemma 8.4 again to the intersection in the last parenthesis of (8.9) and iterating, we obtain

$$\forall t \in \mathbb{N}_0, 2^2(K_{t-2} \cap K_{t-2}^*) = 2^2(A_{t-2} \cap A_{t-2}^*) = \cdots = 2^{t-1}(K_1 \cap K_1^*). \qquad (8.10)$$

Finally, applying Lemma 8.4 again to the intersection in the last parenthesis of (8.10), we have

$$\forall t \in \mathbb{N}_0, 2^{t-1}(K_1 \cap K_1^*) = 2^{t-1}(A_1 \cap A_1^*) = 2^{t-1}(2K_0 \cap 2K_0^*) =$$

$$= 2^t(K_0 \cap K_0^*) = 2^t(\{1\} \cap \{1\}) = \{2^t\}. \quad □ \qquad (8.11)$$

**Theorem 8.6.**
*The intersection between* $K_t$ *and* $K_t^*$ *is equal to* $\{2^t\}$*, that is*

$$\forall t \in \mathbb{N}_0, K_t \cap K_t^* = \{2^t\}. \qquad (8.12)$$

*Proof*. Applying Lemma 8.4 to the intersection $K_t \cap K_t^*$, we have (8.5). Applying the Lemma 8.5 at the intersection $A_t \cap A_t^*$ we obtain (8.12). □

# 9 Conclusions

Collatz's conjecture can be re-proposed using the sets K and their first properties. We have seen that the sets $K_t$ and $K_t^*$ are non-empty (Basic 3.1) and they are also two by two disjoint (Corollary 3.4). So, if the following coverage equalities of $\mathbb{N}_0$ were also true:

a) $\bigcup_{t=0}^{+\infty} K_t = \mathbb{N}_0\,, \quad t \in \mathbb{N}$ b) $\bigcup_{t=0}^{+\infty} K_t^* = \mathbb{N}_0\,, \quad t \in \mathbb{N}$

we could say that each of the families $\{K_t\}_{t\in\mathbb{N}}$ and $\{K_t^*\}_{t\in\mathbb{N}}$ is a partition of $\mathbb{N}_0$. In this case the Collatz conjecture would be proved.

# References

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