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Uniqueness of an entire function sharing a polynomial with its linear differential polynomial

Imrul Kaish*
Nasir Uddin Gazi†

Abstract

In this paper we consider an entire function when it shares a polynomial with its linear differential polynomial. Our result is an improvement of a result of P.Li.
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*Department of Mathematics and Statistics, Aliah University, Kolkata, West Bengal 700160, India; imrulksh3@gmail.com.
†Department of Mathematics and Statistics, Aliah University, Kolkata, West Bengal 700160, India; nsrgazi@gmail.com
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1 Introduction, Definitions and Results

Let f be a non-constant meromorphic function defined in the open complex plane $\mathbb{C}$ and $a = a(z)$ be a polynomial. We denote by $E(a; f)$ the set of zeros of $f - a$, counted with multiplicities and by $\overline{E}(a; f)$ the set of distinct zeros of $f - a$.

If for two non-constant meromorphic functions f and g, we have $E(a; f) = E(a; g)$, we say that f and g share a CM and if $\overline{E}(a; f) = \overline{E}(a; g)$, we say that f and g share a IM.

We denote by $S(r, f)$ any function satisfying $S(r, f) = o\{T(r, f)\}$, as $r \to \infty$, possibly outside of a set with finite measure.

For an entire function f, we define $deg(f)$ in the following way:

$deg(f) = \infty$, if f is a transcendental entire function and $deg(f)$ is the degree of the polynomial, if f is a polynomial.

The investigation of uniqueness of an entire function sharing two values introduced by L. A. Rubel and C. C. Yang [Rubel and Yang, 1977] in 1977. Following is their result.

Theorem A. *[Rubel and Yang, 1977] Let f be a non-constant entire function. If $E(a; f) = E(a; f^{(1)})$ and $E(b; f) = E(b; f^{(1)})$, for distinct finite complex numbers a and b, then $f \equiv f^{(1)}$.*

In 1979 E. Mues and N. Steinmetz [Mues and Steinmetz, 1979] tried to improve Theorem A by considering IM sharing of values. They proved the following theorem.

Theorem B. *[Mues and Steinmetz, 1979]. Let f be a non-constant entire function and a, b be two distinct finite complex values. If $\overline{E}(a; f) = \overline{E}(a; f^{(1)})$ and $\overline{E}(b; f) = \overline{E}(b; f^{(1)})$, then $f \equiv f^{(1)}$.*

In 1986 G. Jank, E. Mues and L. Volkmann [Jank et al., 1986] considered an entire function sharing a nonzero value with its derivatives and they proved the following result.

Theorem C. *[Jank et al., 1986] Let f be a non-constant entire function and a be a non-zero finite value. If $\overline{E}(a; f) = \overline{E}(a; f^{(1)}) \subset \overline{E}(a; f^{(2)})$, then $f \equiv f^{(1)}$.*

H. Zhong [Zhong, 1995] tried to improve Theorem C by taking higher order derivatives. By the following example he concluded that in Theorem C the second derivative cannot be straight way replaced by any higher order derivatives.

Example 1.1. *[Zhong, 1995] Let $k(\geq 3)$ be a positive integer and $\omega(\neq 1)$ be a $(k-1)$th root of unity. If $f = e^{\omega z} + \omega - 1$, then f, $f^{(1)}$, and $f^{(k)}$ share the value ω CM, but $f \not\equiv f^{(1)}$.*

Considering two consecutive higher order derivatives H. Zhong [Zhong, 1995] improved Theorem C in another direction. The following is the improved result.

Theorem D. *[Zhong, 1995] Let f be a non-constant entire function and a be a non-zero finite value. If $E(a;f) = E(a;f^{(1)})$ and $\overline{E}(a;f) \subset \overline{E}(a;f^{(n)}) \cap \overline{E}(a;f^{(n+1)})$ for $n(\geq 1)$, then $f \equiv f^{(n)}$.*

For further discussion we need the following notation. Let f be a non-constant meromorphic function, $a = a(z)$ be a polynomial and A be a set of complex numbers. We denote by $n_A(t,a;f)$, the number of zeros of $f-a$, counted according to their multiplicities which lie in $A \cap \{z : |z| \leq r\}$. The integrated counting function $N_A(r,a;f)$ of the zeros of $f-a$ which lie in $A \cap \{z : |z| \leq r\}$ is defined as

$$N_A(r,a;f) = \int_0^r \frac{n_A(t,a;f) - n_A(0,a;f)}{t} dt + n_A(0,a;f)\log r,$$

where $n_A(0,a;f)$ denotes the multiplicity of zeros of $f-a$ at origin. $\overline{N}_A(r,a;f)$ be the reduced counting function of zeros of $f-a$ in $A \cap \{z : |z| \leq r\}$. Clearly if $A = \mathbb{C}$ then $N_A(r,a;f) = N(r,a;f)$ and $\overline{N}_A(r,a;f) = \overline{N}(r,a;f)$.

For standard definitions and notations of the value distribution theory we refer the reader to [Hayman, 1964] and [Yang and Yi, 2003].

Recently I. Lahiri and I. Kaish [Lahiri and Kaish, 2017] improved Theorem D by considering a shared polynomial. They proved the following result.

Theorem E. *[Lahiri and Kaish, 2017] Let f be a non-constant entire function and $a = a(z)(\not\equiv 0)$ be a polynomial with $deg(a) \neq deg(f)$. Suppose that $A = \overline{E}(a;f)\Delta\overline{E}(a;f^{(1)})$ and $B = \overline{E}(a,f^{(1)}) \backslash \{\overline{E}(a,f^{(n)}) \cap \overline{E}(a,f^{(n+1)})\}$, where $\triangle$ denotes the symmetric difference of sets and $n(\geq 1)$ is an integer. If*

(i) $N_A(r,a;f) + N_A(r,a;f^{(1)}) = O\{logT(r,f)\}$,

(ii) $N_B(r,a;f^{(1)}) = S(r,f)$ *and*

(iii) each common zero of $f-a$ and $f^{(1)}-a$ has the same multiplicity,

then $f = \lambda e^z$, where $\lambda(\neq 0)$ is a constant.

Throughout the paper we denote by $L = L(f)$ a nonconstant linear differential polynomial generated by f of the form

$$L = L(f) = a_1 f^{(1)} + a_2 f^{(2)} + ... + a_n f^{(n)}, \tag{1}$$

where $a_1, a_2, ..., a_n(\neq 0)$ are constants.

Considering Linear differential polynomial P.Li [Li, 1999] improved Theorem D in the following way.

Theorem F. *[Li, 1999]. Let f be a non-constant entire function and L be defined in (1) and a be a non-zero finite complex number. If $\overline{E}(a; f) = \overline{E}(a; f^{(1)}) \subset \overline{E}(a; L) \cap \overline{E}(a; L^{(1)})$ then $f = f^{(1)} = L$.*

In this paper we extend Theorem D and Theorem F in the following way

Theorem 1.1. *Let f be a non-constant entire function, L be defined in (1) and $a = a(z)(\not\equiv 0)$ be a polynomial with $deg(a) \neq deg(f)$. Suppose that $A = \overline{E}(a; f)\Delta\overline{E}(a; f^{(1)})$ and $B = \overline{E}(a, f^{(1)})\backslash\{\overline{E}(a, L^{(p)}) \cap \overline{E}(a, L^{(q)})\}$ where p, q are integers satisfying $q > p \geq deg(a)$.*

If

(i) $N_A(r, a; f) + N_A(r, a; f^{(1)}) = O\{\log T(r, f)\}$,

(ii) $N_B(r, a; f^{(1)}) = S(r, f)$ *and*

(iii) each common zero of $f - a$ and $f^{(1)} - a$ has the same multiplicity,

then $f = L = \lambda e^z$, where $\lambda(\neq 0)$ is a constant.

Putting $A = B = \emptyset$ we get the following corollary.

Corolary 1.1. *Let f be a non-constant entire function, L be defined in (1) and $a = a(z)(\not\equiv 0)$ be a polynomial with $deg(a) \neq deg(f)$. If $E(a; f) = E(a; f^{(1)})$ and $\overline{E}(a, f^{(1)}) \subset \overline{E}(a, L^{(p)}) \cap \overline{E}(a, L^{(q)})$ where p, q are integers satisfying $q > p \geq deg(a)$, then $f = L = \lambda e^z$, where $\lambda(\neq 0)$ is a constant.*

Remark 1.1. *If in Corollary 1.1, a is a non-zero constant and $p = deg(a) = 0, q = p + 1$ then it is a particular form of Theorem F.*

Remark 1.2. *If in (1), $a_1 = a_2 = ...a_{n-1} = 0$ and $a_n = 1$ then $L = f^{(n)}$ and if in Corollary 1.1, a is a non-zero constant and $p = deg(a), q = p + 1$, then Corollary 1.1 is the Theorem D.*

Remark 1.3. *It is an open problem whether the Theorem 1.1 is valid or not if we omit the condition $p \geq deg(a)$.*

2 Lemmas

In this section we present some necessary lemmas.

Lemma 2.1. *[Lahiri and Kaish, 2017]. Let f be a transcendental entire function of finite order and $a = a(z)(\not\equiv 0)$ be a polynomial and $A = \overline{E}(a; f)\Delta\overline{E}(a; f^{(1)})$.*

If

(i) $N_A(r,a;f) + N_A(r,a;f^{(1)}) = O\{\log T(r,f)\}$,

(ii) *each common zero of* $f - a$ *and* $f^{(1)} - a$ *has the same multiplicity,*

then $m(r,a;f) = S(r,f)$.

Lemma 2.2. *[Lain, 1993]. Suppose* f *be an entire function,* $a_0, a_1, ...a_n$ *are polynomials and* a_0, a_n *are not identically zero. Then each solution of the linear differential equation* $a_n f^{(n)} + a_{n-1} f^{(n-1)} + ... + a_0 f = 0$ *is of finite order.*

Lemma 2.3. *[Hayman, 1964]. Let* f *be a non-constant meromorphic function and* a_1, a_2, a_3 *be three distinct meromorphic functions satisfying* $T(r, a_\nu) = S(r, f)$ *for* $\nu = 1, 2, 3$ *then*

$T(r,f) \leq \overline{N}(r,0;f-a_1) + \overline{N}(r,0;f-a_2) + \overline{N}(r,0;f-a_3) + S(r,f)$.

Lemma 2.4. *Let* f *be a transcendental entire function and* $a = a(z)(\not\equiv 0)$ *be a polynomial. Also let* $L(f)$*,* $L(a)$ *be the linear differential polynomials generated by* f *and* a *respectively. Suppose*

$h = \dfrac{(a - a^{(1)})(L^{(p)}(f) - L^{(p)}(a)) - (a - L^{(p)}(a))(f^{(1)} - a^{(1)})}{f - a}$,

$A = \overline{E}(a;f) \backslash \overline{E}(a;f^{(1)})$ *and* $B = \overline{E}(a, f^{(1)}) \backslash \{\overline{E}(a, L^{(p)}) \cap \overline{E}(a, L^{(q)})\}$*, where* p, q *are integers satisfying* $0 \leq p < q$.

If

(i) $N_A(r,a;f) + N_B(r,a;f^{(1)}) = S(r,f)$,

(ii) *each common zero of* $f - a$ *and* $f^{(1)} - a$ *has the same multiplicity,*

(iii) h *is a transcendental entire or meromorphic,*

then $m(r,a,f^{(1)}) = S(r,f)$.

Proof. Since $a - a^{(1)} = (f^{(1)} - a^{(1)}) - (f^{(1)} - a)$, if z_0 be a common zero of $f - a$ and $f^{(1)} - a$ with multiplicity $r(\geq 2)$, then z_0 is a zero of $a - a^{(1)}$ with multiplicity $r - 1$. So

$$N_{(2}(r,a;f) \leq 2N(r,0;a - a^{(1)}) + N_A(r,a;f) = S(r,f), \tag{2}$$

where $N_{(2}(r,a;f)$ be the counting function of multiple zeros of $f - a$.

Using (2) and from the hypothesis we get

$$\begin{aligned} N(r,h) &\leq N_A(r,a;f) + N_B(r,a;f^{(1)}) + N_{(2}(r,a;f) + S(r,f) \\ &= S(r,f) \end{aligned}$$

Since $m(r,h)=S(r,f)$, we have $T(r,h)=S(r,f)$
From $h=\dfrac{(a-a^{(1)})(L^{(p)}(f)-L^{(p)}(a))-(a-L^{(p)}(a))(f^{(1)}-a^{(1)})}{f-a}$, we get

$$\begin{aligned} f &= a+\frac{1}{h}\{(a-a^{(1)})(L^{(p)}(f)-L^{(p)}(a))-(a-L^{(p)}(a))(f^{(1)}-a^{(1)})\} \\ &= a+\frac{1}{h}\{(a-a^{(1)})(L^{(p)}(f)-a)-(a-L^{(p)}(a))(f^{(1)}-a)\}. \end{aligned} \tag{3}$$

Case 1. Let $p>0$. Differentiating (3) we get

$$\begin{aligned} f^{(1)} &= a^{(1)}+(\frac{1}{h})^{(1)}\{(a-a^{(1)})(L^{(p)}(f)-a)-(a-L^{(p)}(a))(f^{(1)}-a)\}+ \\ & \frac{1}{h}\{(a^{(1)}-a^{(2)})(L^{(p)}(f)-a)+(a-a^{(1)})(L^{(p+1)}-a^{(1)})\}- \\ & \frac{1}{h}\{(a^{(1)}-L^{(p+1)}(a))(f^{(1)}-a)+(a-L^{(p)}(a))(f^{(2)}-a^{(1)})\}. \end{aligned}$$

This implies
$(f^{(1)}-a)\{1+(\frac{1}{h})^{(1)}(a-L^{(p)}(a))+\frac{1}{h}(a^{(1)}-L^{(p+1)}(a))\}$
$=a^{(1)}-a+(\frac{1}{h})^{(1)}(a-a^{(1)})(L^{(p)}(f)-a)+\frac{1}{h}(a^{(1)}-a^{(2)})(L^{(p)}(f)-a)+\frac{1}{h}(a-a^{(1)})(L^{(p+1)}(f)-a^{(1)})-\frac{1}{h}(a-L^{(p)}(a))(f^{(2)}-a^{(1)})$
$=a^{(1)}-a+(\frac{a-a^{(1)}}{h})^{(1)}(L^{(p)}(f)-L^{(p-1)}(a))+(\frac{a-a^{(1)}}{h})^{(1)}(L^{(p-1)}(a)-a)+\frac{a-a^{(1)}}{h}(L^{(p+1)}(f)-L^{(p)}(a))+\frac{a-a^{(1)}}{h}(L^{(p)}(a)-a^{(1)})-\frac{1}{h}(a-L^{(p)}(a))(f^{(2)}-a^{(1)})$,
or,
$(f^{(1)}-a)\{1+(\frac{a-L^{(p)}(a)}{h}))^{(1)}\}=(a^{(1)}-a)+\{(\frac{a-a^{(1)}}{h})(L^{(p-1)}(a)-a)\}^{(1)}+(\frac{a-a^{(1)}}{h})^{(1)}(L^{(p)}(f)-L^{(p-1)}(a))+\frac{a-a^{(1)}}{h}(L^{(p+1)}(f)-L^{(p)}(a))-\frac{1}{h}(a-L^{(p)}(a))(f^{(2)}-a^{(1)})$,
or

$$\begin{aligned} \frac{1}{f^{(1)}-a} &= \frac{h_1}{h_2}-\frac{1}{h_2}(\frac{a-a^{(1)}}{h})^{(1)}(\frac{L^{(p)}(f)-L^{(p-1)}(a)}{f^{(1)}-a}) \\ & +(\frac{a-a^{(1)}}{hh_2})(\frac{L^{(p+1)}(f)-L^{(p)}(a)}{f^{(1)}-a}) \\ & -\frac{1}{hh_2}(a-L^{(p)}(a))(\frac{f^{(2)}-a^{(1)}}{f^{(1)}-a}), \end{aligned} \tag{4}$$

where $h_1=1+(\frac{a-L^{(p)}(a)}{h})^{(1)}$,
$h_2=a^{(1)}-a+\{(\frac{a-a^{(1)}}{h})(L^{(p-1)}(a)-a)\}^{(1)}$.
We now verify that $h_1\not\equiv 0, h_2\not\equiv 0$.

If $h_1 \equiv 0$, then $1 + (\frac{a-L^{(p)}(a)}{h})^{(1)} \equiv 0$. Integrating we get $\frac{1}{h} = \frac{c_1 - z}{a-L^{(p)}(a)}$, where c_1 is a constant. This is a contradiction, because h is transcendental.

If $h_2 \equiv 0$, then $a^{(1)} - a + \{(\frac{a-a^{(1)}}{h})(L^{(p-1)}(a) - a)\}^{(1)} \equiv 0$. Integrating we get $h = \frac{(a-a^{(1)})(L^{(p-1)}(a)-a)}{P(z)}$, where $P(z)$ is a polynomial. This is again a contradiction. Therefore $h_1 \not\equiv 0, h_2 \not\equiv 0$. Again $T(r, h_1) + T(r, h_1) = S(r, f)$, since $T(r, h) = S(r, f)$.

Now from (4) and using Lemma of logarithmic derivative we get $m(r, a; f^{(1)}) = m(r, \frac{1}{f^{(1)}-a}) = S(r, f)$.

Case 2. Let $p = 0$. Then $L^{(p)}(f) = L(f)$.

Suppose $L(f) = a_1 f^{(1)} + a_2 f^{(2)} + ... + a_n f^{(n)}$
and $L(a) = a_1 a^{(1)} + a_2 a^{(2)} + ... + a_n a^{(n)}$, where $a_1, a_2, ..., a_n (\neq 0)$ are constant, $n (\geq 1)$ be an integer.

From the definition of h we get
$f = a + \frac{1}{h}\{(a - a^{(1)})(L(f) - a) - (a - L(a))(f^{(1)} - a)\}$
Differentiating we get

$$\begin{aligned} f^{(1)} &= a^{(1)} + (\frac{1}{h})^{(1)}\{(a - a^{(1)})(L(f) - a) - (a - L(a))(f^{(1)} - a)\} \\ &+\frac{1}{h}\{(a^{(1)} - a^{(2)})(L(f) - a) + (a - a^{(1)})(L^{(1)}(f) - a^{(1)})\} \\ &-\frac{1}{h}\{(a^{(1)} - L^{(1)}(a))(f^{(1)} - a) - (a - L(a))(f^{(2)} - a^{(1)})\}. \end{aligned}$$

This implies

$(f^{(1)} - a)\{1 + (\frac{a-L(a)}{h})^{(1)}\} = (a^{(1)} - a) + (\frac{a-a^{(1)}}{h})^{(1)}(L(f) - a) + \frac{a-a^{(1)}}{h}(L^{(1)}(f) - a^{(1)}) - \frac{a-L(a)}{h}(f^{(2)} - a^{(1)}) = (a^{(1)} - a) + (\frac{a-a^{(1)}}{h})^{(1)}(L(f) - L_1(a)) + (\frac{a-a^{(1)}}{h})^{(1)}(L_1(a) - a) + (\frac{a-a^{(1)}}{h})(L^{(1)}(f) - L(a)) + (\frac{a-a^{(1)}}{h})(L(a) - a^{(1)}) - \frac{a-L(a)}{h}(f^{(2)} - a^{(1)}) = (a^{(1)} - a) + \{(\frac{a-a^{(1)}}{h})(L_1(a) - a)\}^{(1)} + (\frac{a-a^{(1)}}{h})^{(1)}(L(f) - L_1(a)) + (\frac{a-a^{(1)}}{h})(L^{(1)}(f) - L(a)) - \frac{a-L(a)}{h}(f^{(2)} - a^{(1)})$

Or,

$$\begin{aligned} \frac{1}{f^{(1)} - a} &= \frac{h_3}{h_4} - \frac{1}{h_4}(\frac{a - a^{(1)}}{h})^{(1)}(\frac{L(f) - L_1(a)}{f^{(1)} - a}) \\ &+(\frac{a - a^{(1)}}{hh_4})(\frac{L^{(1)}(f) - L(a)}{f^{(1)} - a}) - (\frac{a - L(a)}{hh_4})(\frac{f^{(2)} - a^{(1)}}{f^{(1)} - a}), \quad (5) \end{aligned}$$

where
$L_1(a) = a_1 a + a_2 a^{(1)} + ... + a_n a^{(n-1)}$,
$h_3 = 1 + (\frac{a-L(a)}{h})^{(1)}$ and
$h_4 = a^{(1)} - a + \{(\frac{a-a^{(1)}}{h})(L_1(a) - a)\}^{(1)}$

Similarly as in Case 1, $h_3 \not\equiv 0$, $h_4 \not\equiv 0$. Also $T(r, h_3) + T(r, h_4) = S(r, f)$. Therefore from (5) and using Lemma of logarithmic derivative we get

$m(r, a; f^{(1)}) = m(r, \frac{1}{f^{(1)}-a}) = S(r, f)$.

This completes the proof of the lemma.

□

Lemma 2.5. *Let f be a transcendental entire function, $a = a(z)(\not\equiv 0)$ be a polynomial and $L = L(f)$ be define in (1). Suppose*

(i) $N_A(r, a; f) + N_A(r, a; f^{(1)}) = S(r, f)$, where $A = \overline{E}(a; f)\Delta\overline{E}(a; f^{(1)})$

(ii) $N_B(r, a; f^{(1)})) = S(r, f)$, where $B = \overline{E}(a, f^{(1)})\backslash\{\overline{E}(a, L^{(p)}) \cap \overline{E}(a, L^{(q)})\}$ p, q are integers satisfying $q > p \geq deg(a)$,

(iii) each common zero of $f - a$ and $f^{(1)} - a$ has the same multiplicity,

(iv) m(r, a; f) = S(r, f), then $f = L = \lambda e^z$, where $\lambda(\neq 0)$ is a constant.

Proof. Let

$$\alpha = \frac{f^{(1)} - a}{f - a}, \tag{6}$$

From the hypothesis we get,

$$N(r, \alpha) \leq N_A(r, a; f) + S(r, f) = S(r, f)$$

and

$$\begin{aligned} m(r, \alpha) &= m(r, \frac{f^{(1)} - a}{f - a}) \\ &= m(r, \frac{f^{(1)} - a^{(1)} + a^{(1)} - a}{f - a}) \\ &\leq m(r, a; f) + S(r, f) \\ &= S(r, f). \end{aligned}$$

Therefore $T(r, \alpha) = S(r, f)$.

From (6) we get

$$\begin{aligned} f^{(1)} &= \alpha f + a(1 - \alpha) \\ &= \alpha_1 f + \beta_1, \end{aligned}$$

where $\alpha_1 = \alpha$ and $\beta_1 = a(1 - \alpha)$

Differentiating we get,

$$f^{(2)} = \alpha_2 f + \beta_2,$$

where $\alpha_2 = \alpha_1^{(1)} + \alpha_1\alpha_1$ and $\beta_2 = \beta_1^{(1)} + \alpha_1\beta_1$.

Similarly,

$$f^{(k)} = \alpha_k f + \beta_k,$$

where $\alpha_{k+1} = \alpha_k^{(1)} + \alpha_1\alpha_k$ and $\beta_{k+1} = \beta_k^{(1)} + \alpha_k\beta_1$.

Clearly $T(r,\alpha_k) + T(r,\beta_k) = S(r,f)$, because $T(r,\alpha) = S(r,f)$.

Now

$$\begin{aligned} L^{(p)} &= \sum_{k=1}^{n} a_k f^{(p+k)} \\ &= (\sum_{k=1}^{n} a_k\alpha_{p+k})f + (\sum_{k=1}^{n} a_k\beta_{p+k}) \\ &= \mu_1 f + \nu_1, \end{aligned} \tag{7}$$

where $\mu_1 = \sum\limits_{k=1}^{n} a_k\alpha_{p+k}$, $\nu_1 = \sum\limits_{k=1}^{n} a_k\beta_{p+k}$

$$\begin{aligned} L^{(q)} &= \sum_{k=1}^{n} a_k f^{(q+k)} \\ &= (\sum_{k=1}^{n} a_k\alpha_{q+k})f + (\sum_{k=1}^{n} a_k\beta_{q+k}) \\ &= \mu_2 f + \nu_2, \end{aligned} \tag{8}$$

where $\mu_2 = \sum\limits_{k=1}^{n} a_k\alpha_{q+k}$, $\nu_2 = \sum\limits_{k=1}^{n} a_k\beta_{q+k}$.

Clearly $T(r,\mu_i) + T(r,\nu_i) = S(r,f)$, $i = 1, 2$.

Let $D = \overline{E}(a; f) \cap \overline{E}(a; f^{(1)}) \cap \overline{E}(a; L^{(p)}) \cap \overline{E}(a; L^{(q)})$.

Note that $D \neq \emptyset$, because otherwise, $N(r,a;f) = S(r,f)$. Then from the hypothesis $T(r,f) = S(r,f)$, a contradiction.

Let $z_1 \in D$ then $f(z_1) = f^{(1)}(z_1) = L^{(p)}(z_1) = L^{(q)}(z_1) = a(z_1)$.

Now from (7) and (8) we get $a(z_1) = \mu_1(z_1)a(z_1) + \nu_1(z_1)$ and $a(z_1) = \mu_2(z_1)a(z_1) + \nu_2(z_1)$

If $\mu_1 a + \nu_1 - a \not\equiv 0$, then

$$\begin{aligned} N(r,a;f) &\leq N_A(r,a;f) + N_B(r,a;f^{(1)}) + N_D(r,a;f) + S(r,f) \\ &\leq N_A(r,0;\mu_1 a + \nu_1 - a) + S(r,f) \\ &= S(r,f), \end{aligned}$$

a contradiction. Therefore

$$\mu_1 a + \nu_1 - a \equiv 0. \tag{9}$$

Similarly

$$\mu_2 a + \nu_2 - a \equiv 0. \tag{10}$$

From (9) and (10) we get $\mu_1 \equiv \mu_2 \equiv 1$ and $\nu_1 \equiv 0 \equiv \nu_2$.

Then from (7)

$$L^{(p)} \equiv f. \tag{11}$$

Also $\mu_1 \equiv 1$ implies

$$\sum_{k=1}^{n} a_k \alpha_{p+k} \equiv 1. \tag{12}$$

From (12) we see that α has no pole. Because if α has a pole of order $d(\geq 1)$ then the left hand side of (12) has a pole of order $(p+k)d$ but the right hand side is a constant.

Again by simple calculation from (12) we get

$$a_n \alpha^{n+p} + P[\alpha] \equiv 0. \tag{13}$$

where $P[\alpha]$ is a differential polynomial in α with degree not exceeding $(n+p-1)$.

If α is transcendental entire, then by Clunie's Lemma we have $m(r,\alpha) = S(r,\alpha)$, a contradiction.

If α is a nonconstant polynomial then left hand side of (13) is also a nonconstant polynomial, which is again a contradiction.

Therefore α is a constant.

Now from $\frac{f^{(1)}-a}{f-a} = \alpha$, we get $f^{(1)} - \alpha f = a(1-\alpha)$.

Integrating we get

$$\begin{aligned} e^{-\alpha z} f &= (1-\alpha) \int a e^{-\alpha z} dz \\ &= (1-\alpha) P(z) e^{-\alpha z} + \lambda, \end{aligned}$$

where $\lambda(\neq 0)$ is a constant and $P(z)$ is a polynomial of degree atmost $deg(a)$,

or, $f = (1-\alpha)P(z) + \lambda e^{\alpha z}$.

Now $f^{(r+1)} = \lambda \alpha^{r+1} e^{\alpha z}$, if $r = deg(a)$

Therefore

$$\begin{aligned} L^{(p)} &= \sum_{k=1}^{n} a_k f^{(p+k)} \\ &= (\sum_{k=1}^{n} a_k \alpha^{p+k})\lambda e^{\alpha z} \\ &= \lambda e^{\alpha z} \\ &= \frac{f^{(1)}}{\alpha} - \frac{1-\alpha}{\alpha} p^{(1)}(z), \end{aligned} \tag{14}$$

Suppose $\alpha \neq 1$.

Since $D = \overline{E}(a; f) \cap \overline{E}(a; f^{(1)}) \cap \overline{E}(a; L^{(p)}) \cap \overline{E}(a; L^{(q)}) \neq \emptyset$,
we have $f(z_2) = f^{(1)}(z_2) = L^{(p)}(z_2) = L^{(q)}(z_2) = a(z_2)$, for some $z_2 \in D$.
From (14) we get

$$a(z_2) = \frac{a(z_2)}{\alpha} - \frac{1-\alpha}{\alpha} P^{(1)}(z_2)$$

or,

$$a(z_2)(1 - \frac{1}{\alpha}) + \frac{1-\alpha}{\alpha} P^{(1)}(z_2) = 0$$

or,

$$(\alpha - 1)\{a(z_2) - P^{(1)}(z_2)\} = 0$$

or,

$$a(z_2) - P^{(1)}(z_2) = 0.$$

Clearly $a(z) - P^{(1)}(z) \not\equiv 0$, because $deg(P^{(1)}(z))$ is less than $deg(a)$.

$$\begin{aligned} N(r, a; f) &\leq N_A(r, a; f) + N_B(r, a; f^{(1)}) + N_D(r, a; f) + S(r, f) \\ &\leq N(r, 0; a - P^{(1)}) + S(r, f) \\ &= S(r, f). \end{aligned}$$

Then from the hypothesis $T(r, f) = S(r, f)$, a contradiction.
Therefore $\alpha = 1$, so $f = \lambda e^z$.
Again

$$\begin{aligned} L &= \sum_{k=1}^{n} a_k f^{(k)} \\ &= (\sum_{k=1}^{n} a_k \alpha^{k})\lambda e^{\alpha z} \\ &= \lambda e^{z}. \end{aligned}$$

Therefore $f = L = \lambda e^z$.
This completes the lemma.

$\square$

3 Proof of the Main Theorem

Proof. First we claim that f is a transcendental entire function.
If f is a polynomial, then
$T(r, f) = O(\log r)$ and $N_A(r, a; f) + N_A(r, a; f^{(1)}) = O(\log r)$.
Then from the hypothesis we get $O(\log r) = O(\log T(r, f)) = S(r, f)$, which implies $T(r, f) = S(r, f)$, a contradiction. Therefore $A = \emptyset$.
Similarly $N_B(r, a; f^{(1)}) = S(r, f)$ implies $B = \emptyset$.
Therefore $E(a; f) = E(a; f^{(1)})$ and $\overline{E}(a; f^{(1)}) \subset \overline{E}(a; L^{(p)}) \cap \overline{E}(a; L^{(q)})$.
Let $deg(f) = m$ and $deg(a) = r$. If $m \geq r + 1$ then $deg(f - a) = m$ and $deg(f^{(1)} - a) \leq m - 1$ which contradicts that $E(a, f) = E(a, f^{(1)})$.
If $m \leq r - 1$, then $deg(f - a) = deg(f^{(1)} - a) = r$. Since $E(a, f) = E(a, f^{(1)})$, $(f - a) = t(f^{(1)} - a)$, where $t(\neq 0)$ is a constant.
If $t = 1$, then $f = f^{(1)}$, which is a contradiction because f is a polynomial.
If $t \neq 1$ then $tf^{(1)} - f \equiv (t-1)a$, which is impossible because $deg((t-1)a) = r$ and $deg(tf^{(1)} - f) = m$ and $m < r$. Therefore our claim " f is transcendental entire function " is established. Now we prove the result into two cases.
Case 1. Let $f \equiv L^{(p)}$. Then

$$\begin{aligned} m(r, a; f) &= m(r, \frac{a}{f - a}\frac{1}{a}) \\ &\leq m(r, \frac{a}{f - a}) + S(r, f) \\ &= m(r, \frac{a}{f - a} + 1 - 1) + S(r, f) \\ &\leq m(r, \frac{a}{f - a} + 1) + S(r, f) \\ &\leq m(r, \frac{f}{f - a}) + S(r, f) \\ &= m(r, \frac{L^{(p)}}{f - a}) + S(r, f), \end{aligned} \tag{15}$$

since $p \geq deg(a)$, by Lemma of logarithmic derivative, $m(r, \frac{L^{(p)}}{f-a}) = S(r, f)$. So from (15) $m(r, a; f) = S(r, f)$. Therefore by Lemma 5, $f = L = \lambda e^z, \lambda(\neq 0)$ is a constant.

Case 2. Let $f \not\equiv L^{(p)}$. This case can be divided into two subcases.

Subcase 2.1. Let $f^{(1)} \not\equiv L^{(p)}$.

Since $a-a^{(1)} = (f^{(1)}-a^{(1)})-(f^{(1)}-a)$, a common zero of $f-a$ and $f^{(1)}-a$ of multiplicity $s(\geq 2)$ is a zero of $a-a^{(1)}$ with multiplicity $s-1(\geq 1)$.
Therefore $N_{(2}(r,a;f^{(1)} \mid f=a) \leq 2N(r,0;a-a^{(1)}) = S(r,f)$,

where $N_{(2}(r,a;f^{(1)} \mid f=a)$ denotes the counting function (counted with multiplicities) of those multiple zeros of $f^{(1)}-a$ which are also zeros of $f-a$.

Now

$$\begin{aligned} N_{(2}(r,a;f^{(1)}) &\leq N_A(r,a;f^{(1)}) + N_B(r,a;f^{(1)}) + N_{(2}(r,a;f^{(1)} \mid f=a) + S(r,f) \\ &= S(r,f). \end{aligned} \tag{16}$$

Using (16) and from the hypothesis we get

$$\begin{aligned} N(r,a;f^{(1)}) &\leq N_B(r,a;f^{(1)}) + N(r,\frac{a-L^{(p)}(a)}{a-a^{(1)}};\frac{L^{(p)}(f)-L^{(p)}(a)}{f^{(1)}-a^{(1)}}) + S(r,f) \\ &\leq T(r,\frac{a-L^{(p)}(a)}{a-a^{(1)}};\frac{L^{(p)}(f)-L^{(p)}(a)}{f^{(1)}-a^{(1)}}) + S(r,f) \\ &= N(r,\frac{L^{(p)}(f)-L^{(p)}(a)}{f^{(1)}-a^{(1)}}) + S(r,f) \\ &\leq N(r,a^{(1)};f^{(1)}) + S(r,f). \end{aligned} \tag{17}$$

Again

$$\begin{aligned} m(r,a;f) &= m(r,\frac{f^{(1)}-a^{(1)}}{f-a}\frac{1}{f^{(1)}-a^{(1)}}) \\ &\leq m(r,a^{(1)};f^{(1)}) + S(r,f) \\ &= T(r,f^{(1)}) - N(r,a^{(1)};f^{(1)}) + S(r,f) \\ &= m(r,f^{(1)}) - N(r,a^{(1)};f^{(1)}) + S(r,f) \\ &\leq m(r,f) - N(r,a^{(1)};f^{(1)}) + S(r,f) \\ &= T(r,f) - N(r,a^{(1)};f^{(1)}) + S(r,f), \end{aligned}$$

i.e

$$N(r,a^{(1)};f^{(1)}) \leq N(r,a;f) + S(r,f).$$

So from (17) we get

$$N(r,a;f^{(1)}) \leq N(r,a;f) + S(r,f). \tag{18}$$

Also

$$\begin{aligned} N(r,a;f) &\leq N_A(r,a;f) + N(r,a;f \mid f^{(1)}=a) \\ &\leq N(r,a;f^{(1)}) + S(r,f). \end{aligned} \tag{19}$$

From (18) and (19) we get

$$N(r,a;f^{(1)}) \;=\; N(r,a;f)+S(r,f). \tag{20}$$

Let

$h = \dfrac{(a-a^{(1)})(L^{(p)}(f)-L^{(p)}(a))-(a-L^{(p)}(a))(f^{(1)}-a^{(1)})}{f-a}$, which is defined in Lemma 2.4.

Clearly $T(r,h)=S(r,h)$.

Now

$$\begin{aligned}
T(r,f) &= m(r,f)\\
&= m(r,a+\frac{1}{h}\{(a-a^{(1)})(L^{(p)}(f)-L^{(p)}(a))-(a-L^{(p)})(f^{(1)}-a^{(1)})\}\\
&\leq m(r,(a-a^{(1)})L^{(p)}(f)-(a-L^{(p)})f^{(1)})+S(r,f)\\
&\leq m(r,f^{(1)})+S(r,f)\\
&= T(r,f^{(1)})+S(r,f)\\
&= m(r,f^{(1)})+S(r,f)\\
&\leq m(r,f)+S(r,f)\\
&= T(r,f)+S(r,f).
\end{aligned}$$

Therefore

$$T(r,f^{(1)}) \;=\; T(r,f)+S(r,f). \tag{21}$$

If h is transcendental, then by Lemma 2.4, $m(r,a;f^{(1)})=S(r,f)$ and from (20) and (21) $m(r,a;f)=S(r,f)$. So from Lemma 2.5, $f=L=\lambda e^{z}$, $\lambda(\neq 0)$, is a constant.

If h is rational, then by Lemma 2.2 we see that f is of finite order. So by Lemma 2.1 we get $m(r,a;f)=S(r,f)$.

Therefore from Lemma 2.5, $f=L=\lambda e^{z}$, $\lambda(\neq 0)$ is a constant.

Subcase 2.2. Let $f^{(1)} \equiv L^{(p)}$. Now

$$\begin{aligned} m(r,a;f) &= m(r,\frac{a^{(1)}}{f-a}\frac{1}{a^{(1)}}) \\ &\leq m(r,\frac{a^{(1)}}{f-a}) + S(r,f) \\ &= m(r,\frac{f^{(1)}-(f^{(1)}-a^{(1)})}{f-a} + S(r,f) \\ &\leq m(r,\frac{f^{(1)}}{f-a}) + S(r,f) \\ &= m(r,\frac{L^{(p)}}{f-a}) + S(r,f). \end{aligned} \tag{22}$$

Since $p \geq deg(a)$, by Lemma of logarithmic derivative, $m(r,\frac{L^{(p)}}{f-a}) = S(r,f)$, so from (22) $m(r,a;f) = S(r,f)$.

Therefore from Lemma 2.5, we get $f = L = \lambda e^z$, $\lambda(\neq 0)$, is a constant.

This completes the proof of the Main Theorem.

□

4 Conclusions

Finally we arrive at the conclusion that a non-constant entire function sharing a polynomial with its linear differential polynomial with some conditions defined in Theorem (1.1) belongs to the class of functions $\mathfrak{F} = \{\lambda e^z : \lambda \in \mathbb{C} \setminus \{0\}\}$.

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References

W. K. Hayman. *Meromorphic Functions*. The Clarendon Press, Oxford, 1964.

G. Jank, E. Mues, and L. Volkmann. Meromorphe funktionen, die mit ihrer ersten und zweiten ableitung einen endlichen wert teilen. *Complex Var. Theory Appl.*, 6:51–71, 1986.

I. Lahiri and I. Kaish. An entire function sharing a polynomial with its derivatives. *Boll. Unione Mat. Ital.*, 10:229–240, 2017.

I. Lain. *Nevanlinna theory and complex differential equations*. Watter de Gruyter, New York, 1993.

P. Li. Entire functions that share one value with their linear differential polynomials. *Kodai Math. j.*, 22:446–457, 1999.

E. Mues and N. Steinmetz. Meromorphe functionen, die mit ihrer ableitung werte teilen. *Manuscripta Math.*, 29:195–206, 1979.

L. A. Rubel and C. C. Yang. Values shared by an entire function and its derivative. *Lecture Notes in Math.(Springer)*, 599:101–103, 1977.

C. C. Yang and H. X. Yi. *Uniqueness Theory of Meromorphic Functions*. Science Press and Kluwer Academic Publishers, New York, 2003.

H. Zhong. Entire functions that share one value with their derivatives. *Kodai Math. J.*, 18:250–259, 1995.