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On homomorphism of fuzzy multigroups

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Abstract

In this paper, the homomorphism of fuzzy multigroups is briefly delineated and some related results are shown. In particular, we consider the corresponding isomorphism theorems of fuzzy multigroups.

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1 Introduction

Since the inception of the theory fuzzy multisets introduced by Yager (1986), the subject has become an interesting area for researchers in algebra. The foundation of algebraic structures of fuzzy multisets was laid by Shinoj et al. (2015); Ibrahim and Awolola (2015) discussed further some new results which will bring new openings and development of fuzzy multigroup concept. Some group concepts like subgroups, abelian groups, normal subgroups and direct product of groups have been established (Ejegwa, 2018a,b,d, 2019). The idea of homomorphism of fuzzy multigroups and their alpha-cuts have also been discussed (Ejegwa, 2018c, 2020).

In this paper, more results on homomorphism of fuzzy multigroups are established and the corresponding isomorphism theorems of fuzzy multigroups which analogously exist in group setting are discussed.

2 Preliminaries

We recall here some basic definitions and results used in the sequel. We refer the reader to (Miyamoto, 2001; Shinoj et al., 2015; Ibrahim and Awolola, 2015).

Definition 2.1. (Miyamoto, 2001) Let X be a nonempty set. A *fuzzy multiset* U over X is characterized by count membership function $CM_U : X \to [0, 1]$ (giving a multiset of the unit interval $[0, 1]$). An expedient notation for a fuzzy multiset U over X is $U = \{(CM_U(a)/a) \mid a \in X\}$ with $CM_U(a) = \{\mu_U^1(a), \mu_U^2(a), ..., \mu_U^m(a), ...\}$, where $\mu_U^1(a), \mu_U^2(a), ..., \mu_U^m(x), ... \in [0, 1]$ such that $(\mu_U^1(x) \geq \mu_U^2(a) \geq, ..., \geq \mu_U^m(a), ...)$.

If the fuzzy multiset U is finite, then $CM_U(a) = \{\mu_U^1(a), \mu_U^2(a), ..., \mu_U^m(a)\}$,

where $\mu_U^1(a), \mu_U^2(a), ..., \mu_U^m(a) \in [0, 1]$ such that $\mu_U^1(a) \geq \mu_U^2(a) \geq, ..., \geq \mu_U^m(a)$.

The set of all fuzzy multisets over X is denoted by $FMS(X)$. Throughout this paper fuzzy multisets are considered finite.

The usual set operations can be carried over to fuzzy multisets. For instance, let $U, V \in FMS(X)$, then

$$U \subseteq V \iff CM_U(a) \leq CM_V(a), \forall\, a \in X,$$
$$U \cap V = \{CM_U(a) \wedge CM_V(a)/a \mid a \in X\},$$
$$U \cup V = \{CM_U(a) \vee CM_V(a)/x \mid a \in X\}.$$

Definition 2.2 (Shinoj et al., 2015) Let P and Q be two nonempty sets such that $\varphi : P \to Q$ is a mapping. Consider the fuzzy multisets $U \in FMS(P)$ and $V \in FMS(Q)$. Then,

(i) the image of U under φ is denoted by $\varphi(U)$ has the count membership function

$$CM_{\varphi(U)}(b) = \begin{cases} \bigvee_{\varphi(a)=b} CM_U(a), & \varphi^{-1}(b) \neq \emptyset \\ 0, & \varphi^{-1}(b) = \emptyset \end{cases}$$

(ii) the inverse image of V under φ denoted by $\varphi^{-1}(V)$ has the count membership function $CM_{\varphi^{-1}(V)}(a) = CM_V(\varphi(a))$.

Definition 2.4 (Shinoj et al., 2015) Let X be a group. A fuzzy multiset U over X is called a fuzzy multigroup if

(i) $CM_U(ab) \geq CM_U(a) \bigwedge CM_U(b), \ \forall\, a, b \in X$, and

(ii) $CM_U(a^{-1}) = CM_U(a), \ \forall\, a \in X$.

The immediate consequence is that $CM_U(e) \geq CM_U(a)\, \forall\, a \in X$, where e is the identity element of X. The set all fuzzy multigroups is denoted by $FMG(X)$. The next definition can be found in Shinoj et al. (2015) .

Definition 2.5 Let $U \in FMG(X)$. Then U is called an abelian fuzzy multigroup over X if $CM_U(ab) = CM_U(ba), \ \forall\, a, b \in X$. The set $AFMG(X)$ is the set of all abelian fuzzy multigroups over X.

Definition 2.6 Let $U \in FMS(X)$. Then $U_* = \{x \in X \mid C_U(a) = C_U(e)\}$
Remark 2.1 For a fuzzy multigroup over a group X, U_* is a group, certainly a subgroup of X Shinoj et al. (2015).

Proposition 2.1 (Ibrahim and Awolola, 2015) Let $U \in FMG(X)$, then $xU = yU \Longleftrightarrow x^{-1}y \in U_*$.

The following propositions are shown in (Ibrahim and Awolola, 2015) .

Proposition 2.2 Let $U \in FMG(X)$. Then the following assertions are equivalent:

(i) $CM_U(ab) = CM_U(ba), \ \forall\, a, b \in X$,

(ii) $CM_U(aba^{-1}) = CM_U(b),\ \forall\, a, b \in X,$

(iii) $CM_U(aba^{-1}) \geq CM_A(b),\ \forall\, a, b \in X,$

(iv) $CM_U(aba^{-1}) \leq CM_U(b),\ \forall\, a, b \in X.$

Proposition 2.3 Let $U \in FMG(X)$. Then $CM_U(ab^{-1}) = CM_U(e)$ implies $CM_U(a) = CM_U(b)$.

As to the converse problem whether $CM_U(a) = CM_U(b)$ implies $CM_U(ab^{-1}) = CM_U(e)$, we give a counter example. Let $X = \{1, s, t, r\}$ be a klein's 4-group and $U = \{(1, 0.7, 0.6, 0.5, 0.5)/1, (0.6, 0.4, 0.2)/s\}$. We see that U is an abelian fuzzy multigroup over X. Then, while $CM_U(t) = CM_U(r) = 0$, we have $CM_U(tr^{-1}) = CM_U(tr) = CM_U(s) = (0.6, 0.4, 0.2) \neq (1, 0.7, 0.6, 0.5, 0.5) = CM_U(1)$. Thus the converse problem above does not hold.

3 Main Results

Proposition 3.1 Let X be a group such that $\varphi : X \to X$ is an automorphism. If $U \in FMG(X)$, then $\varphi(U) = U$ if and only if $\varphi^{-1}(U) = U$.

Proof. Let $a \in X$. Then $\varphi(a) = a$.
Now $CM_{\varphi^{-1}(U)}(a) = CM_U(\varphi(a)) = CM_U(a)$
$\Longrightarrow \varphi^{-1}(U) = U$
Conversely, let $\varphi^{-1}(U) = U$. Since φ is an automorphism, then

$$\begin{aligned}
CM_{\varphi(U)}(a) &= \bigvee\{CM_U(a') \mid a' \in X,\ \varphi(a') = \varphi(a)\} \\
&= CM_U(\varphi(a)) \\
&= CU_{(\varphi^{-1}(U))}(a) \\
&= CM_U(a)
\end{aligned}$$

Hence, the proof.

Proposition 3.2 Let $\varphi : X \to Y$ be a homomorphism of groups such that $U, V \in FMG(Y)$. If U is a constant on $Ker\varphi$, then $\varphi^{-1}(\varphi(U)) = U$.

Proof. Let $\varphi(a) = b$. Then we have
$CM_{\varphi^{-1}(\varphi(U))}(a) = CM_{\varphi(U)}\varphi(a) = CM_{\varphi(U)}(b) = \bigvee\{CM_U(a) \mid a \in X,\ \varphi(a) =$

$b\}$. Since $\varphi(a^{-1}c) = \varphi(a^{-1})\varphi(c) = (\varphi(a))^{-1}\varphi(c) = b^{-1}b = e', \ \forall\, c \in X$, such that $\varphi(c) = b$, which implies that $a^{-1}c \in Ker\varphi$. Moreover, since U is constant on $Ker\varphi$, then $CM_U(a^{-1}c) = CM_U(e)$. Therefore, $CM_U(a) = CM_U(c)$. This completes the proof.

Proposition 3.3 Let $U \in AFMG(X)$ such that a map $\varphi : X \to X/U$ is defined by $\varphi(a) = aU$. Then φ is a homomorphism with $Ker\varphi = \{a \in X \mid CM_U(a) = CM_U(e)\}$.

Proof. Clearly, φ is a homomorphism. Also,

$$\begin{aligned}
Ker\varphi &= \{a \in X : \varphi(a) = eU\} \\
&= \{a \in X : aU = eU\} \\
&= \{a \in X : CM_U(a^{-1}b) = CM_U(b) \ \forall\, b \in X\} \\
&= \{a \in X : CM_U(a^{-1}) = CM_U(e)\} \\
&= \{a \in X : CM_U(a) = CM_H(e)\} = U_*
\end{aligned}$$

Proposition 3.4 Let $\varphi : X \to Y$ be an epimorphism of groups and $U \in AFMG(X)$, then $X/U_* \cong Y$.

proof. Define $\Psi : X/U_* \to Y$ by $\Psi(xU_*) = \varphi(a) \ \forall\, a \in X$.
Let $aU = bU$ such that $CM_U(a^{-1}b) = CM_U(e)$. This implies that $a^{-1}b \in U_*$. It is easy to show that Ψ is well-defined, homomorphism and epimorphism.

Moreover, $\varphi(a) = \varphi(b)$

$$\begin{aligned}
&\Longrightarrow \varphi(a)^{-1}\varphi(b) = \varphi(e) \\
&\Longrightarrow \varphi(a^{-1})\varphi(b) = \varphi(a^{-1}b) = \varphi(e) \\
&\Longrightarrow a^{-1}b \in U_* \\
&\Longrightarrow CM_U(a^{-1}b) = CM_U(e) \\
&\Longrightarrow aU = bU
\end{aligned}$$

This shows that Ψ is an isomorphism.

Proposition 3.5 If $U, V \in AFMG(X)$ with $CM_U(e) = CM_V(e)$, then $U_*V_*/V \cong U_*/U \cap V$.

Proof. Clearly, for some $x \in U_*V_*$, $a = uv$ such that $u \in U_*$ and $v \in V_*$.
Define $\varphi : U_*V_*/V \to U_*/U \cap V$ by $\varphi(aV) = u(U \cap V)$.
If $aV = bV$ with $b = u_1v_1$, $u_1 \in U_*$ and $v_1 \in V_*$, then

$$\begin{aligned} CM_V(a^{-1}b) &= CM_V((uv)^{-1}u_1v_1) \\ &= CM_V(v^{-1}u^{-1}u_1v_1) \\ &= CM_V(u^{-1}u_1v^{-1}v_1) \\ &= CM_V(e). \end{aligned}$$

Hence, $CM_V(u^{-1}u_1) = CM_V(v^{-1}v_1) = CM_V(e)$. Thus,

$$\begin{aligned} CM_{U\cap V}(u^{-1}u_1) &= CM_U(u^{-1}u_1) \wedge CM_V(u^{-1}u_1) \\ &= CM_U(e) \wedge CM_V(e) \\ &= CM_{U\cap V}(e) \end{aligned}$$

That is, $u(U \cap V) = u_1(U \cap V)$. Therefore, φ is well-defined.

If $aV, bV \in U_*V_*/V$, then $ab = uvu_1v_1$. Since $U \in AFMG(X)$, then $CM_U(vu_1v_1) = CM_U(u_1) \Longrightarrow vu_1v_1 \in U_*$.
Hence, $\varphi(aVbV) = \varphi(abV) = u(vu_1v_1)(U \cap V) = u(U \cap V)vu_1v_1(U \cap V)$ and

$$\begin{aligned} CM_{U\cap V}(u_1^{-1}(vu_1u_1)) &\geq CM_U(u_1^{-1}vu_1v_1) \wedge CM_V(u_1^{-1}vu_1v_1) \\ &= CM_U(u_1^{-1}(vu_1v_1)) \wedge CM_V(v(u_1^{-1}u_1v_1)) \\ &= CM_U(e) \wedge CM_V(e) \\ &= CM_{U\cap V}(e). \end{aligned}$$

Hence, $vu_1v_1(U \cap V) = u_1(U \cap V)$
That is, $\varphi(aVbV) = u(U \cap V)u_1(U \cap V) = \varphi(aV)\varphi(bV)$, and this shows that φ is a homomorphism. Undeniably, it is also epimorphism.

Furthermore, if $a, b \in U_*V_*$ with $a = uv$ and $b = u_1v_1$, $u, u_1 \in U_*$ and $v, v_1 \in V_*$ and $u(U \cap V) = u_1(U \cap V)$, then $CM_{U\cap V}(u^{-1}u_1) = CM_{U\cap V}(e)$
That is, $CM_U(u^{-1}u_1) \wedge CM_V(u^{-1}u_1) = CM_U(e) \wedge CM_V(e)$.
However, $CM_U(e) = CM_V(e)$ and $CM_U(u^{-1}u_1) = CM_U(e)$
$\Longrightarrow CM_V(u^{-1}u_1) = CM_V(e)$.

Therefore,

$$\begin{aligned} CM_V(a^{-1}b) &= CM_V((uv)^{-1}u_1v_1) \\ &= CM_V(v^{-1}u^{-1}u_1u_1) = CM_V(u^{-1}u_1v^{-1}v_1) \\ &\geq CM_V(u^{-1}u_1) \wedge CM_V(v^{-1}v_1) \\ &= CM_V(e) \wedge CM_V(e) = CM_V(e) \end{aligned}$$

$\Longrightarrow CM_V(a^{-1}b) = CM_V(e)$

Thus, $aV = bV$.
Hence, $U_*V_*/V \cong U_*/U \cap V$.

Proposition 3.6 Let $U, V \in AFMG(X)$ such that $U \subseteq V$ and $CM_U(e) = CM_V(e)$. Then $X/V \cong (X/U)/(V_*/U)$.

proof. Define $\varphi : X/U \to X/V$ by $\varphi(aU) = aV \ \forall\, a \in X$ such that $CM_U(a^{-1}b) = CM_U(e) = CM_V(e) \ \ \forall \ aU = bU$. Since $U \subseteq V$, we have $CM_V(a^{-1}b) \geq CM_U(a^{-1}b) = CM_V(e)$ and thus $CM_V(a^{-1}b) = CM_V(e)$, that is, $aV = bV$, which implies that φ is well-defined. It is homomorphism and epimorphism too.

Moreover,

$$\begin{aligned} Ker\varphi &= \{aU \in X/U : \varphi(aU) = eV\} \\ &= \{aU \in X/U : aV = eV\} \\ &= \{aU \in X/U : CM_V(a) = CM_V(e)\} \\ &= \{aU \in X/U : a \in V_*\} = V_*/U. \end{aligned}$$

Thus, $Ker\varphi = V_*/U$ and so $X/V \cong (X/U)/(V_*/U)$.

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