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Frattini submultigroups of multigroups

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Abstract

In this paper, we introduce and study maximal submultigroups and present some of its algebraic properties. Frattini submultigroups as an extension of Frattini subgroups is considered. A few submultigroups results on the new concepts in connection to normal, characteristic, commutator, abelian and center of a multigroup are established and the ideas of generating sets, fully and non-fully Frattini multigroups are presented with some significant results.

Keywords: Maximal, Cyclic Multigroup, Commutator and Generating Set.

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1 Introduction

The term multigroup was first mentioned in [7] as an algebraic structure that satisfied all the axioms of group except that the binary operation is multivalued. This concept was later redefined in [19] via count function of multisets and some of its properties were vividly discussed. The idea of submultigroup and its classes were established in [14]. Concept of maximal subgroups is established in [3], [5] and [17] and some of its properties were investigated. Also, normal and characteristic submultigroup were introduced in [8] and [13] respectively, and some of its properties were presented. Frattini in [11], introduced a special subgroup named Frattini subgroup and some results were obtained. Other related work on Frattini subgroup can be found in [1], [2], [6], [12], [15], [16], [18], [20] and [21]. Furthermore, in [10] Frattini subgroup was represented but in fuzzy environment called Frattini fuzzy subgroup. In this paper, we focus on multiset setting to obtain Frattini submultigroups and finally establish some related results.

In general, the union of submultigroups of a multigroup may not be a multigroup, we therefore establish some conditions under which the union of all maximal submultigroups is a multigroup. When this occur, the Frattini submultigroup obtained from such maximal submultigroups is called "fully Frattini" otherwise it is called "non-fully Frattini". Furthermore, other relevant concepts such as; cyclic multigroup, minimal generating set of a multigroup, generator and non-generator of a multigroup are introduced with reference to Frattini submultigroups. Finally, we study some properties of center of a multigroup, normal, commutator, minimal and characteristic submultigroups.

2 Preliminaries

Definition 2.1 (|23|). Let X be a set. A multiset A over X is just a pair $\langle X,\ C_A\rangle$, where X is a set and $C_A : X \to \mathbb{N}$ is a function. Any ordinary set B is actually a multiset $\langle B, \chi_B\rangle$, where χ_B is its characteristic function.

The set X is called the ground or generic set of the class of all multisets containing objects from X.

Definition 2.2 (|22|). Let A and B be two multisets over X, A is called a submultiset of B written as $A \subseteq B$ if $C_A(x) \leq C_B(x)$ for all $x \in X$. Also, if $A \subseteq B$ and $A \neq B$, then A is called a proper submultiset of B and denoted as $A \subset B$.

Definition 2.3 (|22|). Let A and B be two multisets over X, then A and B are equal if and only if $C_A(x) = C_B(x)$ for all $x \in X$. Two multisets A and B are comparable to each other if $A \subseteq B$ or $B \subseteq A$.

Definition 2.4 (|23|). Suppose that A, $B \in MS(X)$ such that $A = \langle X, C_A \rangle$ and $B = \langle X, C_B \rangle$.

i. Their intersection denoted by $A \cap B$ is the multiset $C = \langle X, C_C \rangle$, where $x \in X$, $C_C(x) = C_A(x) \wedge C_B(x)$.
ii. Their union denoted by $A \cup B$ is the multiset $C = \langle X, C_C \rangle$, where $x \in X$, $C_C(x) = C_A(x) \vee C_B(x)$.
iii. Their sum denoted by $A \oplus B$ is the multiset $C = A = \langle X, C_C \rangle$, where $x \in X$, $C_C(x) = C_A(x) + C_B(x)$.

Definition 2.5 (|19|). Let X be a group and $A \in MS(X)$. A is said to be a multigroup of X if the count function of A or C_A satisfies the following two conditions:

i. $C_A(xy) \geq [C_A(x) \wedge C_A(y)]$, $\forall x, y \in X$.
ii. $C_A(x^{-1}) \geq C_A(x)$, $\forall\, x \in X$,

Where C_A is a function that takes X to a natural number, and $\wedge$ denotes minimum operation.

The set of all multigroups defined over X is denoted by $MG(X)$.

Definition 2.6 (|19|). Let $A \in MG(X)$. Then A^{-1} is defined by $C_A(x) = C_A(x^{-1})$ $\forall x \in X$.

Thus, $A \in MG(X) \Leftrightarrow A^{-1} \in MG(X)$.

Definition 2.7 (|19|). Let $A \in MG(X)$. Then A is said to be abelian or commutative if $C_A(xy) = C_A(yx)$ $\forall\, x, y \in X$.

Definition 2.8 (|19|). Let $A \in MG(X)$. Then the sets A_* and A^* are defined as

$$A_* = \{x \in X | C_A(x) > 0\} \quad \text{and} \quad A^* = \{x \in X | C_A(x) = C_A(e)\},$$

where e is the identity element of X.

Definition 2.9 (|19|). Let $\{A_i\}_{i \in I}$, $I = 1, 2, \ldots, n$ be an arbitrary family of multigroups of a group X. Then

$$C_{\cap A_i}(x) = \bigwedge C_{A_i}(x) \quad \forall\, x \in X.$$

$$C_{\cup A_i}(x) = \bigvee C_{A_i}(x) \ \ \forall \ x \in X.$$

Definition 2.10 (|14|). Let $A \in MG(X)$. Then the center of A is defined as $C(A) = \{x \in A | C_A([x,y]) = C_A(e) \forall \ y \in X\}$.

Definition 2.11 (|9|) Commutator of two Submultigroup: Let A and B be submultigroups of $C \in MG(X)$. Then the commutator of A and B is the multiset (A, B) of X defined as follows:

$$C_{(A,\ B)}(x) = \begin{cases} \bigvee_{x=[a,b]} [C_A(a) \wedge C_B(b)] \\ 0 \qquad\qquad otherwise \end{cases}$$

That is, $C_{(A,\ B)}(x) = \bigvee_{x=aba^{-1}b^{-1}}[C_A(a) \wedge C_B(b)]$. Since the supremum of an empty set is zero.

$C_{(A,\ B)}(x) = 0$ if x is not a commutator.

Definition 2.11 (|4|). Let $A \in MG(X)$. Then the order of A denoted by $O(A)$ is defined as $O(A) = \sum_{x \in X} C_A(x)$. i.e., the total numbers of all multiplicities of its element.

Definition 2.12 (|14|). Let $A \in MG(X)$. A submultiset B of A is called a submultigroup of A denoted by $B \subseteq A$ if B is a multigroup. A submultigroup B of A is a proper submultigroup denoted by $B \subset A$, if $B \subseteq A$ and $A \neq B$.

Definition 2.13 (|14|). Let $A \in MG(X)$. Then a submultigroup B of A is said to be complete if $B_* = A_*$, incomplete if $B_* \neq A_*$, regular complete if B is complete and $C_B(x) = C_B(y) \forall \ x, y \ \in X$ and regular incomplete if B is incomplete and $C_B(x) = C_B(y) \ \forall \ x, y \ \in X$.

Definition 2.14 (|8|). Let $A, B \in MG(X)$ such that $A \subseteq B$. Then A is called a normal submultigroup of B if $C_A(xyx^{-1}) \geq \ C_A(y) \forall x \ y, \in X$.

Definition 2.15 (|10|). Let X and Y be two groups and let $f: X \to Y$ be a homomorphism. Suppose A and B are multigroups of X and Y respectively, then f induces a homomorphism from A to B which satisfies

i. $C_{f(A)}(y_1 y_2) \geq C_{f(A)}(y_1) \wedge C_{f(A)}(y_2) \ \ \forall y_1, y_2 \in Y.$

ii. $C_{f(B)}(f(x_1 x_2)) \geq C_{(B)}(f(x_1)) \wedge C_{(B)}(f(x_2)) \ \forall x_1, x_2 \in X$ where

i. the image of A under f denoted by $f(A)$, is a multiset of Y defined by

$$C_{f(A)}(y) = \begin{cases} \vee_{x \in f^{-1}(y)} C_A(x), & f^{-1}(y) \neq \emptyset. \\ 0, & otherwise. \end{cases}$$

for each $y \in Y$.

ii. the inverse image of B under f denoted by $f^{-1}(B)$, is a multiset of X defined by

$C_{f^{-1}(B)}(x) = C_B(f(x)) \forall\, x \in X$.

Definition 2.16 (|10|). Let X and Y be groups and let $A \in MG(X)$ and $B \in MG(Y)$ respectively. Then a homomorphism f from X to X is called an automorphism of A onto A if f is both injective and surjective, that is, bijective.

Definition 2.17 (|13|)**.** Let $A, B \in MG(X)$ such that $A \subseteq B$. Then A is called a characteristic (fully invariant) submultigroup of B if

$C_{A^\theta}(x) = C_A(x) \ \forall\, x \in X$ for every automorphism, θ of X.

That is, $\theta(A) \subseteq A$ for every $\theta \in Aut(X)$.

3 Frattini Submultigroups and their Properties

In this section we propose the concept of minimal, maximal, Frattini, commutator submultigroups, cyclic, fully and non-fully Frattini multigroup and generating set of a multigroup with some illustrative examples.

Definition 3.1

a. Minimal Submultigroup: Let G be a group and $A \in MG(X)$. Then a non trivial proper submultigroup denoted by $\underline{B}$ of A is said to be minimal if there exists no other non-trivial submultigroup C of A such that $C \subseteq \underline{B}$.

Remark 3.1. Every minimal complete submultigroup of a multigroup is unique.

b. Maximal Submultigroup: Let X be a group and $A \in MG(X)$. Then a proper normal submultigroup denoted by $\bar{B}$ of A is said to be maximal if there exists no other proper submultigroup C of A such that $\bar{B} \subseteq C$.

c. **Frattini Submultigroup**: Let X be a group. Suppose A is a multigroup of X and, A_1, A_2, ..., A_n (or simply A_i for $i = 1,2,..,n$) are maximal submultigroups of A. Then the Frattini submultigroup of A denoted by $\Phi(A)$ is the intersection of A_i defined by

$$C_{\Phi(A)}(x) = C_A(x) \wedge C_{A_2}(x) \wedge ... \wedge C_{A_n}(x)\ \forall x \in X$$

or simply by

$$C_{\Phi(A)}(x) = \wedge_{i=1}^{n} C_{A_i}(x)\ \forall x\ \in X.$$

Remark 3.2.

i. Let X be a non abelian group and $A \in MG(X)$. If K is a normal submultigroup of A with an incomplete maximal submultigroups and M_i for each i are the maximal subgroups of A, then the maximal submultigroups of K are submultigroups of M_i.

ii. Let $A \in MG(X)$. If K is a submultigroup of A and $\Phi(K)$ is the Frattini submultigroup of K then, $\Phi(K) \sqsubseteq A$.

d. **Commutator Submultigroup of a Multigroup:** Let $A \in MG(X)$ such that the commutator subgroup of X is given as $X' = \{[a,b]: a,b \in X\}$. Then the commutator submultigroup of A denoted by A' is defined as

$$C_{(A,A)}(x) = \begin{cases} \max\ min\ \{\{C_A(a), C_A(b)\} : x = [a,b], \forall\ a,b \in X\} \\ \quad\quad 0 \quad\quad\quad\quad\quad otherwise\ x \in A \end{cases}$$

e. Let $A \in MG(X)$. Then the sets $[\Phi(A)]_*$ and $[\Phi(A)]^*$ are defined as

$[\Phi(A)]_* = \{x \in X | C_{\Phi(A)}(x) > 0\}$ and

$[\Phi(A)]^* = \{x \in X | C_{\Phi(A)}(x) = C_{\Phi(A)}(e)\}$, where e is the identity element of X.

Remark 3.3.

i. The commutator submultigroup of every abelian multigroup is $\{e\}_n$

ii. Let $A \in MG(X)$ and A' be the commutator submultigroup of A. Then $A' \trianglelefteq A$.

Remark 3.4. Let $A, B, C \in MG(X)$ such that $B, C \sqsubseteq A$ and D' be the commutator submultigroup of B and C. Then $D' \trianglelefteq\ \Phi(A)$.

f. **Cyclic Multigroup:** Let $X = \langle a \rangle$ be a group generated by a. Then a multigroup A over X is said to be a cyclic multigroup if $\exists\ n \in \mathbb{N}$ such that $C_A(na) = C_A(x)\ \forall\ x \in X$. The element a is then called the generator of A otherwise, a non generator of A.

g. **Generating Set of a Multigroup:** Let X be a cyclic group and $A \in MG(X)$. A subset S of X is said to be a generating set for A if all elements of A and its inverses can be expressed as a finite product of elements in S with $C_A(nS) = C_A(x)\ \forall\, x \in X$ and for some $n \in \mathbb{N}$.

h. **Minimal Generating Set of a Multigroup:** Let X be a cyclic group and $A \in MG(X)$. A subset B of X is termed minimal generating set of A if $\exists\, n \in \mathbb{N}$ such that $C_A(nB) = C_A(x)$ and there is no proper subset C of B with $C_A(nC) = C_A(x), \forall\, x \in X$.

i. **Fully Frattini Multigroup:** Let $A \in MG(X)$. Then A is called fully Frattini if the union of the maximal submultigroups equals A. Otherwise, it is called non-fully Frattini. In addition, every multigroup without incomplete maximal submultigroup is called trivial fully Frattini.

4. Some Results on Frattini Submultigroups

In this section, we present some results on Frattini submultigroup of multigroups.

Theorem 4.1. Let $A \in MG(X)$ with complete maximal submultigroups. Then every minimal submultigroup of A is a submultigroup of $\Phi(A)$.

Proof. Suppose $\Phi(A)$ is the Frattini submultigroup of A then A has maximal submultigroup say $A_i\ \forall\, i$ such that $\Phi(A) = \bigwedge_{i=1}^{n} A_i$. Since A is multigroup over X, A has a minimal submultigroup say $\underline{A}$. If $\underline{A}$ is not a submultigroup of $\Phi(A)$ then there exist at least an element $x \in \Phi(A)$ such that $C_{\Phi(A)}(x) < C_{\underline{A}}(x)\ \ \forall\, x \in X$ which contradicts the fact that $\underline{\boldsymbol{A}}$ is a minimal submultigroup of A . Hence $\underline{A}$ is a submultigroup of $\Phi(A)$.

Theorem 4.2 If $\theta(\Phi(A)) \subset \Phi(A)$ for all $\theta \in Aut(A)$, then $\Phi(A)$ is characteristic in A.

Proof. Since θ is an automorphism, the inverse θ^{-1} is also an automorphism of A. Hence we have $\theta^{-1}\, \theta\big(\Phi(A)\big) \subset \theta\big(\Phi(A)\big)$.

Applying θ, we have $\theta\theta^{-1}\,\big(\Phi(A)\big) \subset \theta\big(\Phi(A)\big)$. Then we obtain

$\Phi(A) = \theta\theta^{-1}\,\big(\Phi(A)\big) \subset \theta\big(\Phi(A)\big) \subset\ \Phi(A)$. By this fact, equality holds and so $\theta\big(\Phi(A)\big) = \Phi(A)$. Hence the Frattini submultigroup is characteristic in A.

Theorem 4.3 Every Frattini submultigroup of a multigroup is characteristic.

Proof. By Theorem 4.2, it suffices to proof that $\theta(\Phi(A)) \subset \Phi(A)$ for every automorphism $\theta \in Aut(A)$.

Let $x \in \theta(\Phi(A))$. Then there exists $y \in \Phi(A)$ such that $x = \theta(y)$.

To show that $x \in \Phi(A)$, we consider an arbitrary element $g \in A$. Then since θ is an automorphism, we have $A = \theta(A)$. Thus there exists g' in A such that $g = \theta(g')$.

We have $C_A(xg) = C_A(\theta(y)\theta(g'))$

$= C_A(\theta(yg'))$ (Since θ is a homomorphism)

$= C_A(\theta(g'y))$ (Since $y \in \Phi(A)$)

$= C_A(\theta(g')\theta(y))$ (Since θ is a homomorphism)

$= C_A(gx)$

Since this is true for all $g \in A$ it follows that $x \in \Phi(A)$, and thus $\theta(\Phi(A)) \subset \Phi(A)$. Hence the result.

Theorem 4.4 Every Frattini submultigroup of a multigroup is abelian.

Proof. Let $A \in MG(X)$ and $\Phi(A)$ be the Frattini submultigroup of A. It follows that $\Phi(A)$ is a normal submultigroup of A by definition 2.14 Consequently,

$$C_{\Phi(A)}(xyx^{-1}) = C_{\Phi(A)}(y) \forall\, x, y \in X.$$

Thus, $C_{\Phi(A)}(xy) = C_{\Phi(A)}(yx) \forall\, x, y \in X$.

Hence, the result follows by Definition *2.7*

Theorem 4.5 Every $\Phi(A)$ is a normal submultigroup of A.

Proof. Let $A \in MG(X)$ and $\Phi(A)$ be the Frattini submultigroup of A. Then

$C_{\Phi(A)}(e) \geq C_{\Phi(A)}(x)\ \forall\ x \in X$, since $C_{A_i}(e) \geq C_{A_i}(x)\ \forall\ x \in X$.

Now, let $x, y \in X$, then since $\Phi(A)$ is a multigroup over X by *definition 2.14*, we get

$$C_{\Phi(A)}(xy^{-1}) \geq C_{\Phi(A)}(x) \wedge C_{\Phi(A)}(y)$$

Now we proof that $\Phi(A)$ is a normal submultigroup of A. Let $x, y \in X$, then it follows that

$$\begin{aligned} C_{\Phi(A)}(yxy^{-1}) &= C_{\Phi(A)}((yx)y^{-1}) \\ &= C_{\Phi(A)}(x(yy^{-1})) \\ &= C_{\Phi(A)}(xe) \geq C_{\Phi(A)}(x). \end{aligned}$$

Hence, the result by Definition *2.14*

Theorem 4.6 Let A be a multigroup over a non-Abelian group X, then $C(A) \trianglelefteq [\Phi(A)]_*$.

Proof. $C(A) \neq \emptyset$, since at least $e \in C(A)$. Let $x, y \in C(A)$ then for all $z \in [\Phi(A)]_*$, $C_A([x,z]) = C_A(e)$ and $C_A([y,z]) = C_A(e)$. Consequently,

$$\begin{aligned} C_A([xy,\ z]) &= C_A([x,z]^y[y,z]) \text{ where } [x,z]^y = yx^{-1}z^{-1}xzy^{-1} \\ &\geq C_A([x,z]^y) \wedge C_A([y,z]) \\ &\geq C_A([x,z]^y) \text{ since } C_A([y,z]) = C_A(e) \\ &= C_A(y[x,z]y^{-1}) = C_A([x,z]) = C_A(e). \end{aligned}$$

Thus $xy \in C(A)$.

Now, let $x \in C(A)$. Then $C_A([x,z]) = C_A(e) \forall\ z \in [\Phi(A)]_*$.

Hence, $C_A([x^{-1},z]) = C_A(xz^{-1}x^{-1}z) = C_A(xz^{-1}x^{-1}zxx^{-1})$

$$\begin{aligned} &= C_A(z^{-1}x^{-1}zxx^{-1}x) = C_A([z,x]) \\ &= C_A([x,z]^{-1}) = C_A([x,z]) = C_A(e). \end{aligned}$$

Thus, $x^{-1} \in C(A)$ therefore, $C(A)$ is a subgroup of $\Phi(A)_*$. To show that $C(A)$ is a normal subgroup of $[\Phi(A)]_*$. Let $x \in [\Phi(A)]_*$ and $y \in \mathrm{C(A)}$

Then $xyx^{-1} = (xy)x^{-1} = (yx)x^{-1} = y \in \mathrm{C(A)}$.

Thus, $x \in [\Phi(A)]_*$ and $y \in C(A) \Rightarrow xyx^{-1} \in C(A)$. Hence, $C(A) \trianglelefteq [\Phi(A)]_*$

Remark 4.1 If A is a multigroup over an abelian group X, $[\Phi(A)]_*$ is the root set of Frattini submultigroup of A and $C(A)$ is the center of A then $[\Phi(A)]_*$ is a normal subgroup of $C(A)$.

Theorem 4.7. If A is a multigroup over a non-Abelian group X and K is a normal submultigroup of A, then $\Phi(K) \sqsubseteq \Phi(A)$.

Proof. Clearly, $\Phi(K)$ and $\Phi(A)$ are submultigroups of A.

Let M_i be the maximal submultigroups of K and M_j be the maximal submultigroups of A for each i and j. Then by *Remark* 3.2 (i) we have

$$M_i \sqsubseteq M_j \text{ for each } i \text{ and } j.$$

$$|(\cap M_i)| \leq |(\cap M_j)|$$

$\Longrightarrow |\Phi(K)| \leq |\Phi(A)|$

Therefore, $\Phi(K) \subseteq \Phi(A)$.

To show that $\Phi(K) \sqsubseteq \Phi(A)$, suppose $M_i \cap M_j = \{e\}_n$ then the result holds trivially. But if $M_i \cap M_j \neq \{e\}_n$ then for any element $a \in M_i$, $C_{M_i}(a) \leq C_{M_j}(a)$ for each i and j. Therefore, $\Phi(K) \sqsubseteq \Phi(A)$.

Theorem 4.8 If A is a regular multigroup over a group X. Then $A' \trianglelefteq \Phi(A)$.

Proof. Since A' is a multigroup over X, then $C_{A'}(xy^{-1}) \geq [C_{A'}(x) \wedge C_{A'}(y)]$ $\forall\, x, y \in X$.

Let $a \in A'$ and $b \in \Phi(A)$ then $C_{A'}(ab^{-1}) \geq [C_{A'}(a) \wedge C_{A'}(b)]$. Thus $ab^{-1} \in A'$.

Therefore A' is a submultigroup of $\Phi(A)$.

Now by Theorem 4.5, $\Phi(A) \trianglelefteq A$ and clearly $A' \trianglelefteq A$. So let $a \in A'$ and $g \in \Phi(A)$, then $C_{A'}(ag) = C_{A'}(ga)$ implies $C_{A'}(gag^{-1}) = C_{A'}(a)$. Hence $A' \trianglelefteq \Phi(A)$.

Theorem 4.9 If $A \in MG(X)$, A' is the commutator submultigroup of A and $\Phi(A)$ is the Frattini submultigroup of A. Then $\Phi(A) \trianglelefteq A'$.

Proof. Since $\Phi(A)$ is a multigroup over X, then $C_{\Phi(A)}(xy^{-1}) \geq [C_{\Phi(A)}(x) \wedge C_{\Phi(A)}(y)]\ \forall\ x, y \in X$.

Let $a \in \Phi(A)$ and $b \in A'$, then $C_{\Phi(A)}(ab^{-1}) \geq [C_{\Phi(A)}(a) \wedge C_{\Phi(A)}(b)]$. Thus $ab^{-1} \in \Phi(A)$. Therefore $\Phi(A)$ is a submultigroup of A'.

Now, by Theorem 4.5, $\Phi(A) \trianglelefteq A$. Let $a \in \Phi(A)$ and $g \in A'$, then $C_{\Phi(A)}(ag) = C_{\Phi(A)}(ga)$ implies $C_{\Phi(A)}(gag^{-1}) = C_{\Phi(A)}(a)$. Hence, $\Phi(A) \trianglelefteq A'$.

Theorem 4.10. Every Frattini submultigroup of a cyclic multigroup is abelian.
Proof. Let $\Phi(A)$ be the Frattini submultigroup of a cyclic multigroup A over a cyclic group X, then there exists $a \in X$ such that $\forall x, y \in X$ we have $C_{\Phi(A)}(na) = C_{\Phi(A)}(x)$ and $C_{\Phi(A)}(ma) = C_{\Phi(A)}(y)$ for $n, m \in \mathbb{N}$. It now follows that $C_{\Phi(A)}(x + y) = C_{\Phi(A)}(na + ma) = C_{\Phi(A)}\, a(n + m)$

$$= C_{\Phi(A)}\, a(m + n) = C_{\Phi(A)}(ma + na) = C_{\Phi(A)}(y + x).$$

Theorem 4.11. If A is a regular multigroup with an incomplete maximal submultigroups over a cyclic group X. Then $\Phi(A)$ is contained in the set of all non-generators of A. In particular, $[\Phi(A)]^*$ coincide with the set of all non-generators if A has only one maximal submultigroup.

Proof. Let X be a cyclic group and $A \in MG(X)$ and $\Phi(A)$ denotes the Frattini submultigroup of A. Let $Gen(A)$ be the set of all generators of A and $M_{i=1,2,\dots,n}$ be the incomplete maximal submultigroups of A, then for all $x \in Gen(A)$, $x \notin M_{i=1,2,\dots,n}$. In fact, all $y \in M_i$ is a non-generator.

Further, $A^* = Gen(A) \cup M_1{}^* \cup M_2{}^* \cup \dots \cup M_n{}^*$ and $y \in A^* \backslash Gen(A) = \cup M_i{}^*$. Since $C_{\Phi(A)}(x) = \bigwedge_{i=1}^{n} C_{A_i}(x)\ \forall x \in X$, we have that for all $y \in \Phi(A)$, $y \in \cup A_i$. This implies that $\Phi(A) \subseteq \cup M_i$. But $\cup M_i$ is the largest set containing all non generators. Hence $\Phi(A)$ is contained in the set of all non-generators. Suppose A has only one nontrivial maximal submultigroup say B then, $A^* = Gen(A) \cup B^*$ and $y \in B$ $(y \notin Gen(A))$. Since $[\Phi(A)]^* = B^*$, therefore $y \in \Phi(A)$ for all non-generators y. Hence, $[\Phi(A)]^*$ is indeed the set of all non-generators.

Theorem 4.12. If a regular multigroup A over a cyclic group X has two incomplete maximal submultigroups, then the union of its generators coincide with the non-generating set of A^*.

Proof. Let M_1 and M_2 be the maximal submultigroups of A and $Gen(A)$ be the collection of all generators of A. Clearly, $Gen(A) \nsubseteq M_1$ and M_2 (since M_1 and M_2 does not contain any generator). Now, $[n(Gen(A))]$ can be expressed as $[n(Gen(A))] = M^* \, Gen(A)$ if n is odd and $[n(Gen(A))] = M^*$ if n is even with $M^* \cap Gen(A) = \emptyset$ for any maximal submultigroup of A.

Also, $M_1{}^* Gen(A) = A^* \backslash M_2{}^*$. That is, $[n(Gen(A))] = A^* \backslash M_2{}^*$ for odd values of n and for any M but since $A^* = Gen(A) \cup M_1{}^* \cup M_2{}^*$ for even n we have $[n(Gen(A))] = M_1{}^* = A^* \backslash M_2{}^* \neq A^*$ for any M. Hence the result.

Theorem 4.13. If a regular multigroup A over a cyclic group X has two maximal submultigroups, then the union of the non-generators coincide with the generating set of A^*.

Proof. Let M_1 and M_2 be the maximal submultigroups of A and $NGen(A)$ be the collection of all non- generators of A. Clearly, $NGen(A) \subseteq M_1{}^* \cup M_2{}^*$ and so $NGen(A)$ generates A^*.

$$NGen(A) = \{i, j, k, \dots, u\}, \text{ where } i, j, k, \dots, u \in A.$$

$$[2(NGen(A))] = \{i, j, k, \dots, u\}\{i, j, k, \dots, u\}$$

Taking every $i, j \in NGen(A)$, $ij = k \in A$ (for some $k \in A$) if $i, k \neq \{e\}_n$ (i.e., ij generates distinct elements in A).

Since $e \in NGen(A)$, we have that $[n(NGen(A))] = NGen(A) \cup (A \backslash NGen(A))$ for some $n \in \mathbb{N}$. More explicitly,

$[n(NGen(A))] = \{i, j, k, \dots, u\}\{i, j, k, \dots, u\} \dots for\ n\ time$

$= A^* = \{i, j, k, \dots u, p, r, q, \dots, t\}$ where $p, r, q, \dots, t \in Gen(A)$.

This yields $[n(NGen(A))] = NGen(A) \cup Gen(A) = A^*$ for some $n \in \mathbb{N}$.

Theorem 4.14. If a regular multigroup A over a cyclic group X has two incomplete maximal submultigroups and $Gen(A)$ is the set of generators of A, then $[n(NGen(A))]$ form one of the root set of the maximal submultigroup of A for some $n \in \mathbb{N}$.

Proof. Let A be a multigroup over a cyclic group, M_1, M_2 be the maximal submultigroups of A and $i, j, k, \dots, u$ be the generators of A.

Then, $[n(Gen(A))] = \{i, j, k, \dots, u\}\{i, j, k, \dots, u\} \dots for\ n\ time$.

Since $e \notin Gen(A)$, for all $s, t \in Gen(A)$, $s \cdot t \notin Gen(A)$, $s \cdot t \in [2(Gen(A))]$ and $s, t \notin M_{i=1,2}$.

But, $A^* = Gen(A) \cup {M_1}^* \cup {M_2}^*$. Therefore, $s \cdot t \in A^* \backslash Gen(A)$.

In particular, $[2(Gen(A)] \subseteq A^* \backslash Gen(A)$ and $[2(Gen(A)] = {M_i}^*$ for any i.

Remark 4.2

a. A generator of any multigroup over a cyclic group is not contained in any of its maximal submultigroups.

b. The set of non-generators of any multigroup may not be a submultigroup.

Theorem 4.15. If β is a minimal generating set of a multigroup A over a cyclic group X, then $\beta \nsubseteq \Phi(A)$.

Proof. Suppose $\beta \subseteq \Phi(A)$, then $\beta \subseteq M_i$, $\forall\, i$ where M_i is a maximal submultigroup of A. Now, since β contains at least one generator of A, then every M_i contains at least one generator of A which is a contradiction. Hence, $\beta \nsubseteq \Phi(A)$.

Remarks 4.3

i. If A is a multigroup over a cyclic group X. Then the union of all the minimal generating sets of A is equal to A^*.

ii. Every minimal generating set contains a non-generator.

iii. Given a multigroup A over a cyclic group X with order k, If X is a minimal generating set of A then X^m gives $m+1$ elements of A^*.

Theorem 4.16 Every irregular multigroup with complete maximal submultigroups over a group is fully Frattini.

Proof. For multigroup A to be irregular implies $\forall\, x, y \in X$, $C_A(x) \neq C_A(y)$. Now let M_i for each i be the complete maximal submultigroups of A. For M_i to be complete in A implies $M_i{}^* = A^*$. Since M_i is complete in A, then there exists $x \in M_i$ such that $C_{M_i}(x) = C_A(x)$ $\forall\, x \in X$. Therefore $\cup\, M_i = A$ for each i.

Theorem 4.17 Every irregular multigroup with an incomplete maximal submultigroups over a non-cyclic group is fully Frattini.

Proof. X is a non-cyclic group, implies it has no generator and $C_A(x) \neq C_A(y)$ $\forall\, x, y \in X$. Now let M_i for each i be the incomplete maximal submultigroups of A. Then for each $x \in X$, x is contained in at least one of the M_i with $C_{M_i}(x) = C_A(x)$ $\forall\, x \in X$. Therefore $\cup$ for each i.

$M_i = A$

Theorem 4.18. Every cyclic multigroup with incomplete maximal submultigroups is not fully Frattini.

Proof. Suppose X is a cyclic group and A is a multigroup with incomplete maximal submultigroups over X. Where A has set of generators $Gen(A)$. Now let $Gen(A) = \{x_j\}$ for some finite j then $A^* = Gen(A) \cup M_i{}^*$, where $M_i{}^*$are the root sets of all the maximal submultigroups of A for each i and $x_j \notin M_i$.by remark 4.2a, $\cup M_i = A \backslash Gen(A) \neq A$. Hence, A is not fully Frattini.

Theorem 4.19 Every regular multigroup over a group is non-fully Frattini.
Proof. Let X be a group and A be a multigroup over X. For A to be a regular multigroup implies $\forall\, x, y \in X$, $C_A(x) = C_A(y)$. Let M_i for each i be the maximal submultigroup of A. Since A is regular and by Definition $3.1b$, there exists at least an element $x \in X$ such that $C_{M_i}(x) < C_A(x)$ for each i and so $\bigvee_{i=1}^{n} M_i \neq A$.

Remark 4.4. Let A be a nontrivial fully Frattini multigroup over a group X then A has at least three maximalsubmultigroups if $C_A(x) < C_A(e)\ \forall\, x \in X$ where e is the identity element of X.

5 Conclusions

Most results in Frattini subgroup are extended to multigroup. A number of new results were obtained. Notion of cyclic, generators, non-generators, minimal generating sets were introduced and results with reference to Frattini submultigroups were established. Notwithstanding, more properties of maximal and Frattini submultigroups, fully and non-fully Frattini submultigroups and also cyclic multigroups are amenable for further investigation in multigroup framework.

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