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Notes on the Solutions of the First Order Quasilinear Differential Equations

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Abstract

The system of the quasilinear differential first order equations with the antisymetric matrix and the same element $f(t, x(t))$ on the main diagonal have the property that $r'(t) = f(t, x(t))\, r(t)$, where $r(t) \geq 0$ is the polar function of the system. In special cases, when values $f(t, x(t))$ and $g(t, x(t))$ are only dependent on $r^2(t)$, $t \in J_0$ we can find the general solution of the system (1) explicitly.

Keywords: nonlinear; quasilinear; differential equation; differential system;
2010 AMS subject classifications: 34C10.

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1 Introduction

Norkin, S. B. and Tchartorickij, J. A. [1] and Kurzweill, J. [2] investigated the oscillatory properties of the 1,2-nontrivial solutions $x(t)$ of systems of two first order linear differential equations applying polar coordinates. Mamrilla, D. and Norkin, S. B. [3] investigated the oscilatory properties of the 1,2,3-nontrivial solutions $x(t)$ of systems of three first order linear differential equations applying spherical coordinates.
Applying polar (spherical) coordinates, the boundedness and oscillatority of the 1,2 (1,2,3)-nontrivial solutions $x(t)$ of systems of two (three) first order quasi-linear differential equations have been investigated by Mamrilla, D. [4], [5], [6] and Mamrilla, D. and Seman, J. and Vagaská, A. [7], while special attention was paid to the study of the properties of the $x\left(t\right)$ solutions of the systems, the matrix of which has the same element $f\left(t,x\left(t\right)\right)$ on the main diagonal.

This paper gives some asymptotical and oscillatory properties of the solutions to the system of the nonlinear differential equations:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}' = \begin{pmatrix} f\left(t,x\left(t\right)\right) & 0 & g\left(t,x\left(t\right)\right) \\ 0 & f\left(t,x\left(t\right)\right) & 0 \\ -g\left(t,x\left(t\right)\right) & 0 & f\left(t,x\left(t\right)\right) \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \tag{1}$$

where $t>0$, $0\neq f\left(t,x\left(t\right)\right)$, $0\neq g\left(t,x\left(t\right)\right)\in C_0\left(D\equiv J\times R^3,R\right)$.

We assume that each solution

$$\begin{aligned} x\left(t\right) &= \left(x_1\left(t\right),x_2\left(t\right),x_3\left(t\right)\right), \\ x_1\left(t_0\right) &= x_1^0, \\ x_2\left(t_0\right) &= x_2^0, \\ x_3\left(t_0\right) &= x_3^0,\ t_0\in J \end{aligned} \tag{2}$$

exists on the interval J and we denote $h>t_0>0$ the right endpoint of the interval J and $J_0=[t_0,h)$.

We shall denote

$$\begin{aligned} g_1\left(t,x\right) &= f\left(t,x\right)x_1+g\left(t,x\right)x_3, \\ g_2\left(t,x\right) &= f\left(t,x\right)x_2, \\ g_3\left(t,x\right) &= -g\left(t,x\right)x_1+f\left(t,x\right)x_3. \end{aligned} \tag{3}$$

It is known that if $D_0\subset D$ is open nonempty set and derivatives $(\partial g_i(t,x)/\partial x_j)$ are continuous functions on D_0 for every $i,j\in\{1,2,3\}$ then each point $(t_0,x_1^0,x_2^0,x_3^0)\in D_0$ is passed by one and only one integral curve $x\in D$ of the system (1) [3].

Definition 1.1. *The solution $x(t)$ to the system (1) is called $i - trivial$, $i \in \{1,2,3\}$ is fixed, if $x_i(t) = 0$ on the interval J_0. Otherwise $x(t)$ is $i - nontrivial$ solution. If for at least one $i \in \{1,2,3\}$ the solution to the system (1) is $i - nontrivial$, shortly so solution $x(t)$ is said to be $nontrivial$.*

It is obvious that system (1) has $1,2,3 - trivial$ solution; $1,3 - trivial$ and $2 - nontrivial$ solution; $1,3 - nontrivial$ and $2 - trivial$ solution; $1,2,3 - nontrivial$ solution.

Definition 1.2. *The solution $x(t)$ to the system (1) is called $i - positive$ ($i - negative$), $i \in \{1,2,3\}$ is fixed, if $x_i(t)$ is positive (negative) function on the interval J_0.*

Definition 1.3. *The solution $x(t)$ to the system (1) is called $i - nondecreasing$ ($i - nonincreasing$), $i \in \{1,2,3\}$ is fixed, if $x_i(t)$ is nondecreasing (nonincreasing) function on the interval J_0.*

It is obvious that if $f(t,x)\,x_2 \geq 0$ ($f(t,x)\,x_2 \leq 0$) for any point $(t,x) \in D$ then arbitrary solution $x(t)$, $t \in J_0$ to the system (1) is $2 - nondecreasing$ ($2 - nonincreasing$).

Definition 1.4. *The solution $x(t)$ to the system (1) is called $i - bounded$, $i \in \{1,2,3\}$ is fixed, if $x_i(t)$ is the bounded function on interval J_0. At other cases $x(t)$ is $i - unbounded$ one which is called $i - from\ above$ ($i - from\ below$) $unbounded$, $i \in \{1,2,3\}$ is fixed, if $x_i(t)$ is from above (from below) unbounded function on interval J_0.*

It is obvious that if for every continuous function y defined on interval J_0 :

a) $\sup\limits_{y}\left(\int_{t_0}^{h}|f(t,y)\,y_2|dt\right) < \infty$, then any solution $x(t)$, $t \in J_0$ to the system (1) is $2 - bounded$,

b) $\sup\limits_{y}\left(\int_{t_0}^{h} f(t,y)\,y_2 dt\right) = -\infty$ $\left(\inf\limits_{y}\left(\int_{t_0}^{h} f(t,y)\,y_2 dt\right) = \infty\right)$ then there exists a point $t^* \geq t_0$ and $2 - negative$ ($2 - positive$) solution $x(t)$, $t \in [t^*, h)$ to the system (1) such that it is $2 - from\ below$ ($2 - from\ above$) unbounded.

Definition 1.5. *The solution $x(t)$ to the system (1) is called $i - oscillatory$, $i \in \{1,2,3\}$ is fixed, if $x_i(t)$ is the oscillatory function, i. e. if there exists the increasing sequence $\{t_n\}_{n=1}^{\infty}$ such that $t_n \in J_0$, $t_n \to h$ and $x_i(t_n).x_i(t_{n+1}) < 0$ for each $n \in N$. The solution $x(t)$ is called $i - nonoscillatory$ if there exists $h_1 < h$ such that $x_i(t)$ is not changing its sign on the interval $[h_1, h)$, resp. if it has maximally finite number of zero point on the interval $[t_0, h)$.*

2 Main results

Theorem 2.1. *The general solution to the system (1) is generated by the trinity of the functions:*

$$x_1(t) = \left(C_2 \cos\left(\int_{t_0}^{t} g(s, x(s))\, ds\right) - C_3 \sin\left(\int_{t_0}^{t} g(s, x(s))\, ds\right)\right) \times \exp\left(\int_{t_0}^{t} f(s, x(s))\, ds\right),$$

$$x_2(t) = C_1 \exp\left(\int_{t_0}^{t} f(s, x(s))\, ds\right),$$

$$x_3(t) = \left(-C_2 \sin\left(\int_{t_0}^{t} g(s, x(s))\, ds\right) - C_3 \cos\left(\int_{t_0}^{t} g(s, x(s))\, ds\right)\right) \times \exp\left(\int_{t_0}^{t} f(s, x(s))\, ds\right),$$

where $C_i\,(i = 1, 2, 3) \in R$ are arbitrary constants.

Proof. The characteristic quasipolynomial of the system (1) is

$$det\left(A(t, x(t)) - \lambda(t, x(t))\, E\right) = \\ = \left(f(t, x(t)) - \lambda(t, x(t))\right)^3 + g^2(t, x(t))\left(f(t, x(t)) - \lambda(t, x(t))\right) = 0$$

the solutions of which are the functions

$$\lambda_1(t, x(t)) = f(t, x(t)) \quad \text{and}$$

$$\lambda_{2,3}(t, x(t)) = f(t, x(t)) \pm i g(t, x(t)).$$

The fundamental system of the solutions to the system (1) is generated by the vector functions $X_1(t, x(t))$, $ReX_2^c(t, x(t))$, $ImX_2^c(t, x(t))$, where

$$X_1(t, x(t)) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \exp\left(\int_{t_0}^{t} f(s, x(s))\, ds\right)$$

$$X_2^c(t, x(t)) = \left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + i \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}\right) \exp\left(\int_{t_0}^{t} \left(f(s, x(s)) - i g(s, x(s))\right) ds\right),$$

e.g.,

$$X_2(t, x(t)) =$$
$$= \left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \cos\left(\int_{t_0}^{t} g(s, x(s))\, ds \right) + \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \sin\left(\int_{t_0}^{t} g(s, x(s))\, ds \right) \right)$$
$$\times \exp\left(\int_{t_0}^{t} f(s, x(s))\, ds \right),$$

$$X_3(t, x(t)) =$$
$$= \left(\begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \cos\left(\int_{t_0}^{t} g(s, x(s))\, ds \right) - \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \sin\left(\int_{t_0}^{t} g(s, x(s))\, ds \right) \right)$$
$$\times \exp\left(\int_{t_0}^{t} f(s, x(s))\, ds \right).$$

This proves the theorem.□

Corolary 2.1. *If we put $g(t, x(t)) = 1$ in Theorem (2.1), we obtain assertion of Theorem (2.1) in [7].*

Theorem 2.2. *Let for all continuous functions y defined on the interval J_0:*

a) $\sup\limits_{y} \left(\int_{t_0}^{h} |f(s, y)|\, ds \right) < \infty$, *then each solution $x(t)$, $t \in J_0$ to the system (1) is $1, 2, 3 - \mathit{bounded}$,*

b) $\sup\limits_{y} \left(\int_{t_0}^{h} f(s, y)\, ds \right) = -\infty$, *then each solution $x(t)$, $t \in J_0$ to the system (1) is $1, 2, 3 - \mathit{bounded}$ and such that $x_1(t) \to 0$, $x_2(t) \to 0$, $x_3(t) \to 0$ for $t \to h$,*

c) $\inf\limits_{y} \left(\int_{t_0}^{h} f(s, y)\, ds \right) = \infty$, *then each solution $x(t)$, $t \in J_0$ to the system (1) is such that it is $i - \mathit{unbounded}$ at least for one $i \in \{1, 2, 3\}$.*

Proof. Theorem (2.1)implies that the general solution to the system (1) fulfils a condition $x_1^2(t) + x_2^2(t) + x_3^2(t) = (C_1^2 + C_2^2 + C_3^2) \exp\left(2 \int_{t_0}^{t} f(s, x(s))\, ds \right)$, and this implies the assertion of the theorem. □

We assume that for each nontrivial solution $x(t)$, $t \in J_0$ to the system (1) there exists the trinity of the functions $r(t) > 0$, $u(t)$, $v(t) \in C_1(J_0, R)$ such that the coordinates $x_i(t)$, $t \in J_0$, $i = 1, 2, 3$ fulfil [7]:

$$\begin{aligned}
x_1(t) &= r(t)\cos u(t), \\
x_2(t) &= r(t)\sin u(t)\cos v(t), \\
x_3(t) &= r(t)\sin u(t)\sin v(t), \\
r'(t) &= x_1'(t)\cos u(t) + x_2'(t)\sin u(t)\cos v(t) + \\
&+ x_3'(t)\sin u(t)\sin v(t), \\
r(t)\,u'(t) &= -x_1'(t)\sin u(t) + x_2'(t)\cos u(t)\cos v(t) + \\
&+ x_3'(t)\cos u(t)\sin v(t), \\
r(t)\sin u(t)\,v'(t) &= -x_2'(t)\sin v(t) + x_3'(t)\cos v(t).
\end{aligned} \tag{4}$$

The function $r(t)$ is called the polar, $u(t)$ the first angle function and $v(t)$ the second angle function. From this after equivalent arrangement for nontrivial solutions to the system (1) we get:

$$\begin{aligned}
r'(t) &= f(t, x(t))\,r(t), \\
u'(t) &= -g(t, x(t))\sin v(t), \\
\sin u(t)\,v'(t) &= -g(t, x(t))\cos u(t)\cos v(t).
\end{aligned} \tag{5}$$

3 Conclusions

The paper deals with qualitative and quantitative properties of the solutions of special differential equations and systems of differential equations. Non-linear and quasi-linear equations are less researched in mathematical publications, so the goal of this paper was to investigate some asymptotical and oscillatory properties of non-trivial solutions of such differential equations and systems thus contributing to knowledge in this field of research. Special attention was focused on the study of the asymptotic and oscillatory properties of the $x(t)$ solutions of the systems, the matrix of which has the same element on the main diagonal. We have achieved new results due to the investigation of this subject by applying of polar or spherical coordinates.

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