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H_v-Fields, h/v-Fields

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Abstract

In the last decades, the hyperstructures have had a lot of applications in mathematics and in other sciences. These applications range from biomathematics and hadronic physics to linguistic and sociology. For applications the largest class of the hyperstructures, the H_v-*structures*, is used, they satisfy the *weak axioms* where the non-empty intersection replaces the equality. The main tools in the theory of hyperstructures are the fundamental relations which connect, by quotients, the H_v-structures with the corresponding classical ones. These relations are used to define hyperstructures as H_v-fields, H_v-vector spaces and so on, as well. The extension of the reproduction axiom, from elements to fundamental classes, introduces the extension of H_v-structures to the class of h/v-structures. We focus our study mainly in the relation of these classes and we present some constructions on them.
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1 Introduction

The main object in this paper is the largest class of hyperstructures called H_v-*structures* introduced in 1990 [35], which satisfy the weak axioms where the non-empty intersection replaces the equality. Abbreviation: *hyperoperation*=**hope**.

Definition 1.1. *An algebraic hyperstructure is called a set H equipped with at least one* ***hope*** $\cdot : H \times H \to P(H) - \{\varnothing\}$. *We abbreviate by WASS the weak associativity:* $(xy)z \cap x(yz) \neq \varnothing, \forall x, y, z \in H$ *and by COW the weak commutativity:* $xy \cap yx \neq \varnothing, \forall x, y \in H$. *The hyperstructure* $(H, \cdot)$ *is called an* H_v-*semigroup if it is WASS, it is called* $\mathbf{H_v}$**-group** *if it is reproductive* H_v*-semigroup, i.e.,* $xH = Hx = H, \forall x \in H$.

Motivation. The quotient of a group by an invariant subgroup, is a group. F. Marty (1934), *'Sur une generalization de la notion de groupe'.* 8^{eme} *Congres Math. Scandinaves, Stockholm, pp.45-49,* states: the quotient of a group by a subgroup is a hypergroup. The quotient of a group by a partition (or equivalently to any equivalence) is an H_v-group.

In an H_v-semigroup the *powers* are defined by: $h^1 = \{h\}, h^2 = h \cdot h, ..., h^n = h \circ h \circ ... \circ h$, where $(\circ)$ is the *n-ary circle hope*, i.e. take the union of hyperproducts, n times, with all possible patterns of parentheses put on them. An H_v-semigroup $(H, \cdot)$ is *cyclic of period s*, if there is an element h, called *generator*, and a natural number s, the minimum : $H = h^1 \cup h^2 ... \cup h^s$. Analogously the cyclicity for the infinite period is defined [30], [33], [39]. If there is an h and s, the minimum: $H = h^s$, then $(H, \cdot)$, is called *single-power cyclic of period s.*

Definition 1.2. *An* $(R, +, \cdot)$ *is called* $\mathbf{H_v}$*–**ring** if* $(+)$ *and* $(\cdot)$ *are WASS, the reproduction axiom is valid for* $(+)$ *and* $(\cdot)$ *is weak distributive with respect to* $(+)$*:*

$$x(y + z) \cap (xy + xz) \neq \varnothing,\ (x + y)z \cap (xz + yz) \neq \varnothing,\ \forall x, y, z \in R.$$

Let $(R, +, \cdot)$ *be an* H_v*-ring,* $(M, +)$ *be a COW* H_v*-group and there exists an external hope*

$$\cdot : R \times M \to P(M) : (a, x) \to ax$$

such that $\forall a, b \in R$ *and* $\forall x, y \in M$ *we have*

$$a(x + y) \cap (ax + ay) \neq \varnothing,\ (a + b)x \cap (ax + bx) \neq \varnothing,\ (ab)x \cap a(bx) \neq \varnothing,$$

then M is called an H_v*-module over F. In the case of an* H_v*-field F, which is defined later, instead of an* H_v*-ring R, then the* $\mathbf{H_v}$*–**vector space** is defined.*

For more definitions and applications on hyperstructures one can see books [4], [5], [9], [10], [11], [39] and papers as [3], [7], [8], [15], [16], [20], [21], [27], [38], [40], [41], [43], [48], [55], [68].

Definition 1.3. *Let $(H,\cdot),(H,*)$ be H_v-semigroups on the same set H, the hope $(\cdot)$ is called smaller than the $(*)$, and $(*)$ greater than $(\cdot)$, iff there exists an*

$$f \in Aut(H,*) \text{ such that } xy \subset f(x*y),\ \forall x,y \in H.$$

Then we write $\cdot \leq *$ and we say that $(H,*)$ contains $(H,\cdot)$. If $(H,\cdot)$ is a structure then it is called basic structure and $(H,*)$ is called $H_b - structure$.

The Little Theorem. Greater hopes than ones which are WASS or COW, are also WASS or COW, respectively.

This Theorem leads to a partial order on H_v-structures and to posets [39], [42], [43], [21].

Let $(H,\cdot)$ be hypergroupoid. We *remove* $h \in H$, if we take the restriction of $(\cdot)$ in the set $H-\{h\}$. $\underline{h} \in H$ *absorbs* $h \in H$ if we replace h by $\underline{h}$ and h does not appear. $\underline{h} \in H$ merges with $h \in H$, if we take as product of any $x \in H$ by $\underline{h}$, the union of the results of x with both h, $\underline{h}$, and consider h and $\underline{h}$ as one class with representative $\underline{h}$.

The main tool in hyperstructures is the *fundamental relation*. M. Koscas 1970, [20], defined in hypergroups the relation β and its transitive closure β*. This relation is defined in H_v-groups, as well, and connect hyperstructures with the classical structures. T. Vougiouklis [34], [35], [39], [40], [41], [53], [54], [60], introduced the γ* and ϵ* relations, which are defined, in H_v-rings and H_v-vector spaces, respectively. He also named all these relations, *fundamental*. (see also [4], [5], [1], [8], [10], [11]).

Definition 1.4. *The* ***fundamental relations*** *β*, γ* and ϵ*, are defined, in H_v-groups, H_v-rings and H_v-vector spaces, respectively, as the smallest equivalences so that the quotient would be group, ring and vector spaces, respectively.*

Specifying the above motivation we remark that: Let $(G,\cdot)$ be a group and R be an equivalence relation (or a partition) in G, then $(G/R,\cdot)$ is an H_v-group, therefore we have the quotient $(G/R,\cdot)/\beta^*$ which is a group, the *fundamental* one.

The main Theorem to find the fundamental classes is the following:

Theorem 1.1. *Let $(H,\cdot)$ be an H_v-group and denote by U the set of all finite products of elements of H. We define the relation β in H by setting $x\beta y$ iff $\{x,y\} \subset u$ where $u \in U$. Then β* is the transitive closure of β.*

Notation. We denote by [x] the fundamental class of the element $x \in H$. Therefore $\beta^*(x) = [x]$.

Analogous theorems are for H_v-rings, H_v-vector spaces and so on. For proof, see [34], [39]. An element is called **single** [39] if its fundamental class is singleton so, $[x] = \{x\}$.

More general structures can be defined by using the fundamental structures. An application in this direction is the general hyperfield. There was no general definition of a hyperfield, but from 1990 [35] there is the following [38], [39]:

Definition 1.5. *An H_v-ring $(R, +, \cdot)$ is called* $\mathbf{H_v}$***-field*** *if R/γ^* is a field.*

Since the algebras are defined on vector spaces, the analogous to Theorem 1.1, on H_v-vector spaces is the following: Let $(V, +)$ be an H_v-vector space over the H_v-field F. Denote by U the set of all expressions consisting of finite hopes either on F and V or the external hope applied on finite sets of elements of F and V. We define the relation ϵ, in V as follows: $x\epsilon y$ iff $\{x, y\} \in u$ where $u \in U$. Then the relation ϵ^* is the transitive closure of the relation ϵ.

Definition 1.6. *[53], [54], [57]. Let $(L, +)$ be an H_v-vector space over the H_v-field $(F, +, \cdot)$, $\phi : F \to F/\gamma^*$ the canonical map and $\omega_F = \{x \in F : \phi(x) = 0\}$, where 0 is the zero of the fundamental field F/γ^*. Let ω_L be the core of the canonical map $\phi' : L \to L/\epsilon^*$ and denote by the same symbol 0 the zero of L/ϵ^*. Consider the bracket (commutator) hope:*

$$[,] : L \times L \to P(L) : (x, y) \to [x, y]$$

then L is an $\mathbf{H_v}$***-Lie algebra*** *over F if the following axioms are satisfied:*

(L1) *The bracket hope is bilinear, i.e.*
$[\lambda_1 x_1 + \lambda_2 x_2, y] \cap (\lambda_1[x_1, y] + \lambda_2[x_2, y]) \neq \varnothing$
$[x, \lambda_1 y_1 + \lambda_2 y_2] \cap (\lambda_1[x, y_1] + \lambda_2[x, y_2]) \neq \varnothing,$
$\forall x, x_1, x_2, y, y_1, y_2 \in L, \lambda_1, \lambda_2 \in F$

(L2) $[x, x] \cap \omega_L \neq \varnothing, \ \ \forall x \in L$

(L3) $([x, [y, z]] + [y, [z, x]] + [z, [x, y]]) \cap \omega_L \neq \varnothing, \ \ \forall x, y \in L$

In the Definition 1.5, was introduced a new class of which is the following [45] (for a preliminary report see: T. Vougiouklis. A generalized hypergroup, Abstracts AMS, Vol. 19.3, Issue 113, 1998, p.489):

Definition 1.7. *The H_v-semigroup $(H, \cdot)$ is called* ***h/v-group*** *if H/β^* is a group.*

An important and well known class of hyperstructures defined on classical structures are defined as follows [30], [33], [36], [57], [60]:

Definition 1.8. *Let $(G, \cdot)$ be groupoid, then for every $P \subset G, P \neq \varnothing$, we define the following hopes called P-hopes: $\forall x, y \in G$*

$$\underline{P} : x\underline{P}y = (xP)y \cup x(Py),$$

$$\underline{P}_r : x\underline{P}_r y = (xy)P \cup x(yP),$$
$$\underline{P}_l : x\underline{P}_l y = (Px)y \cup P(xy).$$

*The $(G, \underline{P})$,$(G, \underline{P}_r)$, $(G, \underline{P}_l)$ are called **P-hyperstructures**. The most usual case is if $(G, \cdot)$ is semigroup, then $x\underline{P}y = (xP)y \cup x(Py) = xPy$ and $(G, \underline{P})$ is a semihypergroup.*

A **generalization of P-hopes**, used in Santilli's isotheory, is the following [12], [13], [14]: Let $(G, \cdot)$ be abelian group and P a subset of G with $\#P > 1$. We define the hope $(\times_P)$ as follows:

$$x \times_P y = \begin{cases} x \cdot P \cdot y = \{x \cdot h \cdot y | h \in P\} & \text{if } x \neq e \text{ and } c \neq e \\ x \cdot y & \text{if } x = e \text{ or } y = e \end{cases}$$

we call this hope P_e-hope. The hyperstructure $(G, \times_P)$ is abelian H_v-group.

Definition 1.9. *[36]. An H_v-structure is called **very thin** if all hopes are operations except one, which has all hyperproducts singletons except one, which is a subset of cardinality more than one. Therefore, in a very thin H_v-structure in H there exists a hope $(\cdot)$ and a pair $(a, b) \in H^2$ for which $ab = A$, with $cardA > 1$, and all the other products, are singletons.*

From the properties of the very thin hopes the *Attach Construction* is obtained [43], [54]: Let $(H, \cdot)$ be an H_v-semigroup and $v \notin H$. We extend the $(\cdot)$ into $\underline{H} = H \cup \{v\}$ by:

$$x \cdot v = v \cdot x = v, \forall x \in H, \text{ and } v \cdot v = H.$$

The $(\underline{H}, \cdot)$ is an H_v-group, where $(\underline{H}, \cdot)/\beta^* \cong Z_2$ and v is a single.

A class of H_v-structures is the following [47], [49], [57], [60]:

Definition 1.10. *Let $(G, \cdot)$ be groupoid (resp. hypergroupoid) and $f : G \to G$ be a map. We define a hope (∂), called theta-hope, we write ∂-hope, on G as follows*

$x\partial y = \{f(x){\cdot}y, x{\cdot}f(y)\}, \ \forall x, y \in G.$ *(resp.* $x\partial y = (f(x){\cdot}y) \cup (x{\cdot}f(y)), \ \forall x, y \in G)$

If $(\cdot)$ is commutative then ∂ is commutative. If $(\cdot)$ is COW, then ∂ is COW.

If $(G, \cdot)$ is a groupoid (or hypergroupoid) and $f : G \to P(G) - \{\varnothing\}$ be any multivalued map. We define the ∂-hope on G as follows:

$$x\partial y = (f(x) \cdot y) \cup (x \cdot f(y)), \ \forall x, y \in G.$$

The ∂-hopes can be defined in H_v-vector spaces and H_v-Lie algebras:

Let $(\mathbf{A}, +, \cdot)$ be an algebra over the field F. Take any map $f : \mathbf{A} \to \mathbf{A}$, then the *$\partial$-hope on the Lie bracket* $[x, y] = xy - yx$, is defined as follows

$$x\partial y = \{f(x)y - f(y)x, f(x)y - yf(x), xf(y) - f(y)x, xf(y) - yf(x)\}.$$

then $(\mathbf{A}, +, \partial)$ is an H_v-algebra over F, with respect to the ∂-hopes on Lie bracket, where the weak anti-commutativity and the inclusion linearity is valid.

Motivation for the theta-hope is the map *derivative* where only the multiplication of functions can be used. Basic property: if $(G, \cdot)$ is semigroup then $\forall f$, the ∂-hope is WASS.

Example.

(a) In integers $(\mathbf{Z}, +, \cdot)$ fix $n \neq 0$, a natural number. Consider the map f such that $f(0) = n$ and $f(x) = x$, $\forall x \in \mathbf{Z} - \{0\}$. Then $(Z, \partial_+, \partial.)$, where ∂_+ and $\partial.$ are the ∂-hopes refereed to the addition and the multiplication respectively, is an H_v-near-ring, with

$$(\mathbf{Z}, \partial_+, \partial.)/\gamma^* \cong \mathbf{Z}_n.$$

(b) In $(\mathbf{Z}, +, \cdot)$ with $n \neq 0$, take f such that $f(n) = 0$ and $f(x) = x$, $\forall x \in \mathbf{Z} - \{n\}$. Then $(\mathbf{Z}, \partial_+, \partial.)$ is an H_v-ring, moreover, $(\mathbf{Z}, \partial_+, \partial.)/\gamma^* \cong \mathbf{Z}_n$.

Special case of the above is for $n = p$, prime, then $(\mathbf{Z}, \partial_+, \partial.)$ is an H_v-field.

The uniting elements method was introduced by Corsini-Vougiouklis [6] in 1989. With this method one puts in the same class, two or more elements. This leads, through hyperstructures, to structures satisfying additional properties.

The *uniting elements* method is the following: Let **G** be algebraic structure and d, a property which is not valid. Suppose that d is described by a set of equations; then, take the partition in **G** for which it is put together, in the same class, every pair of elements that causes the non-validity of the property d. The quotient by this partition $\mathbf{G}/d$ is an H_v-structure. Then, quotient out the H_v-structure $\mathbf{G}/d$ by the fundamental relation β^*, a stricter structure $(\mathbf{G}/d)/\beta^*$ for which the property d is valid, is obtained.

It is very important if more properties are desired, then we have the following [39]:

Theorem 1.2. *Let $(\mathbf{R}, +, \cdot)$ be a ring, and $F = \{f_1, ..., f_m, f_{m+1}, ..., f_{m+n}\}$ be a system of equations on $\mathbf{R}$ consisting of two subsystems $F_m = \{f_1, ..., f_m\}$ and $F_n = \{f_{m+1}, ..., f_{m+n}\}$. Let σ, σ_m be the equivalence relations defined by the uniting elements procedure using the systems F and F_m respectively, and let σ_n be the equivalence relation defined using the induced equations of F_n on the ring $\mathbf{R}_m = (\mathbf{R}/\sigma_m)/\gamma^*$. Then,*

$$(\mathbf{R}/\sigma)/\gamma^* \cong (\mathbf{R}_m/\sigma_n)\gamma^*.$$

Combining the uniting elements procedure with the enlarging theory or the ∂-theory, we can obtain analogous results [39], [51], [54], [60], [22].

Theorem 1.3. *In the ring $(\mathbf{Z}_n, +, \cdot)$, with n=ms we enlarge the multiplication only in the product of the special elements $0 \cdot m$ by setting $0 \otimes m = \{0, m\}$ and the rest results remain the same. Then*

$$(\mathbf{Z}_n, +, \otimes)/\gamma^* \cong (\mathbf{Z}_m, +, \cdot).$$

Remark that we can enlarge other products as well, for example $2 \cdot m$ by setting $2 \otimes m = \{2, m+2\}$, then the result remains the same. In this case 0 and 1 remain scalars.

Corollary. In the ring $(\mathbf{Z}_n, +, \cdot)$, with n=ps where p is prime, we enlarge only the product $0 \cdot p$ by $0 \otimes p = \{0, p\}$ and the rest remain the same. Then $(\mathbf{Z}_n, +, \otimes)$ is very thin H_v-field.

2 Constructions of H_v-fields and h/v-fields

The class of h/v-groups is more general than the H_v-groups since in h/v-groups the reproductivity is not valid. The r*eproductivity of classes* is valid, i.e. if H is partitioned into equivalence classes, then

$$x[y] = [xy] = [x]y, \forall x, y \in H,$$

because the quotient is reproductive. In a similar way the *h/v-rings, h/v-fields, h/v-modulus, h/v-vector spaces* etc are defined.

Remark 2.1. *From definition of the H_v-field, we remark that the reproduction axiom in the product, is not assumed, the same is also valid for the definition of the h/v-field. Therefore, an H_v-field is an h/v-field where the reproduction axiom for the sum is also valid.*

We know that the reproductivity in the classical group theory is equivalent to the axioms of the existence of the unit element and the existence of an inverse element for any given element. From the definition of the h/v-group, since a generalization of the reproductivity is valid, we have to extend the above two axioms on the equivalent classes.

Definition 2.1. *Let $(H, \cdot)$ be an H_v-semigroup, and denote $[x]$ the fundamental, or equivalent class, of the element $x \in H$. We call* ***unit class*** *the class $[e]$ if we have*

$$([e] \cdot [x]) \cap [x] \neq \varnothing \text{ and } ([x] \cdot [e]) \cap [x] \neq \varnothing, \forall x \in H,$$

and for each element $x \in H$, we call ***inverse class*** *of $[x]$, the class $[x']$, if we have*

$$([x] \cdot [x']) \cap [e] \neq \varnothing \text{ and } ([x'] \cdot [x]) \cap [e] \neq \varnothing.$$

The 'enlarged' hyperstructures were examined in the sense that a new element appears in one result. In enlargement or reduction, most useful are those H_v-structures or h/v-structures with the same fundamental structure [43], [53].

Construction 2.1. *(**a**) Let $(H,\cdot)$ be an H_v-semigroup and $v \notin H$. We extend the $(\cdot)$ into $\underline{H} = H \cup \{v\}$ as follows:*

$$x \cdot v = v \cdot x = v, \forall x \in H, \text{ and } v \cdot v = H.$$

*The $(\underline{H},\cdot)$ is an h/v-group, called **attach**, where $(\underline{H},\cdot)/\beta^* \cong \mathbf{Z}_2$ and v is a single element.*

We have core $(\underline{H},\cdot) = H$. The scalars and units of $(H,\cdot)$ are scalars and units (resp.) in $(\underline{H},\cdot)$. If $(H,\cdot)$ is COW (resp. commutative) then $(\underline{H},\cdot)$ is also COW (resp. commutative).

*(**b**) Let $(H,\cdot)$ be an H_v-semigroup and $\{v_1,\ldots,v_n\} \cap H = \varnothing$, is an ordered set, where if $v_i < v_j$, when $i < j$. Extend $(\cdot)$ in $\underline{H}_n = H \cup \{v_1,\ldots,v_n\}$ as follows:*

$$x \cdot v_i = v_i \cdot x = v_i, v_i \cdot v_j = v_j \cdot v_i = v_j, \forall i < j \text{ and}$$

$$v_i \cdot v_i = H \cup \{v_1,\ldots,v_{i-1}\}, \forall x \in H, i \in \{1,\ldots,n\}.$$

*Then $(\underline{H}_n,\cdot)$ is h/v-group, called **attach elements**, where $(\underline{H}_n,\cdot)/\beta^* \cong Z_2$ and v_n is single.*

*(**c**) Let $(H,\cdot)$ be an H_v-semigroup, $v \notin H$, and $(\underline{H},\cdot)$ be its attached h/v-group. Take an element $0 \notin \underline{H}$ and define in $\underline{H}_o = H \cup \{v, 0\}$ two hopes:*
hypersum (+): $0+0 = x+v = v+x = 0$, $0+v = v+0 = x+y = v$, $0+x = x+0 = v+v = H$, $\forall x, y \in H$
hyperproduct $(\cdot)$: remains the same as in $\underline{H}$ moreover $0\cdot 0 = v\cdot x = x\cdot 0 = 0, \forall x \in \underline{H}$
*Then $(\underline{H}_o,+,\cdot)$ is h/v-field with $(\underline{H}_o,+,\cdot)/\gamma^* \cong Z_3$. (+) is associative, $(\cdot)$ is WASS and weak distributive with respect to (+). 0 is zero absorbing and single but not scalar in (+). $(\underline{H}_o,+,\cdot)$ is called the **attached h/v-field** of the H_v-semigroup $(H,\cdot)$.*

Let us denote by U the set of all finite products of elements of a hypergroupoid $(H,\cdot)$. Consider the relation defined as follows:

$$xLy \text{ iff there exists } u \in U \text{ such that } ux \cap uy \neq \varnothing.$$

Then the transitive closure L^* of L is called *left fundamental reproductivity relation.* Similarly, the *right fundamental reproductivity relation* R^* is defined.

Theorem 2.1. *If $(H, \cdot)$ is a commutative semihypergroup, i.e. the strong commutativity and the strong associativity is valid, then the strong expression of the above L relation: $ux = uy$, has the property: $L^* = L$.*

Proof. Suppose that two elements x and y of H are L* equivalent. Therefore, there are $u_1, \ldots, u_{n+1}$ elements of U, and $z_1, \ldots, z_n$ elements of H, such that

$$u_1x = u_1z_1, u_2z_1 = u_2z_2, \ldots, u_nz_{n-1} = u_nz_n, u_{n+1}z_n = u_{n+1}y.$$

From these relations, using the strong commutativity, we obtain

$$u_{n+1} \ldots u_2u_1x = u_{n+1} \ldots u_2u_1z_1 = u_{n+1} \ldots u_1u_2z_1 =$$

$$= u_{n+1} \ldots u_2u_1z_2 = \cdots = u_{n+1} \ldots u_2u_1y$$

Therefore, setting $u = u_{n+1} \ldots u_2u_1 \in U$, we have $ux = uy$. □

Corollary. Let $(S, \cdot)$ be commutative semigroup which has an element $w \in S$ such that the set wS is finite. Consider the transitive closure L* of the relation L defined by:

xLy iff there exists $z \in S$ such that $zx = zy$.

Then $< S/L^*,^\circ /\beta^*$ is a finite commutative group, where $(^\circ)$ is the induced operation on classes of S/L*.

Open problem: Prove that L*, is the smallest equivalence: H/L*, is reproductive.

We present now the **small non-degenerate $\mathbf{H_v}$-fields** on $(\mathbf{Z}_n, +, \cdot)$ which satisfy the following conditions, appropriate in Santilli's iso-theory:

1. multiplicative very thin minimal,
2. COW (non-commutative),
3. they have the elements 0 and 1, scalars,
4. when an element has inverse element, then this is unique.

Remark that last condition means than we cannot enlarge the result if it is 1 and we cannot put 1 in enlargement. Moreover we study only the upper triangular cases, in the multiplicative table, since the corresponding under, are isomorphic since the commutativity is valid for the underline rings. From the fact that the reproduction axiom in addition is valid, we have always H_v-fields.

Theorem 2.2. *All multiplicative H_v-fields defined on $(\mathbf{Z}_4, +, \cdot)$, which have non-degenerate fundamental field, and satisfy the above 4 conditions, are the following isomorphic cases:*

The only product which is set is $2 \otimes 3 = \{0, 2\}$ or $3 \otimes 2 = \{0, 2\}$.

The fundamental classes are $[0] = \{0, 2\}, [1] = \{1, 3\}$ and we have $(\mathbf{Z}_4, +, \otimes)/\gamma^* \cong (\mathbf{Z}_2, +, \cdot)$.

Example. Let us denote by E_{ij} the matrix with 1 in the ij-entry and zero in the rest entries. Then take the following 2×2 upper triangular H_v-matrices on the above H_v-field $(\mathbf{Z}_4, +, \cdot)$ of the case that only $2 \otimes 3 = \{0, 2\}$ is a hyperproduct:

$$I = E_{11}+E_{22}, a = E_{11}+E_{12}+E_{22}, b = E_{11}+2E_{12}+E_{22}, c = E_{11}+3E_{12}+E_{22},$$

$$d = E_{11}+3E_{22}, e = E_{11}+E_{12}+3E_{22}, f = E_{11}+2E_{12}+3E_{22}, g = E_{11}+3E_{12}+3E_{22},$$

then, we obtain for $\mathbf{X} = \{I, a, b, c, d, e, f, g\}$, that $(\mathbf{X}, \otimes)$ is non-COW H_v-group and the fundamental classes are $\underline{a} = \{a, c\}, \underline{d} = \{d, f\}, \underline{e} = \{e, g\}$ and the fundamental group is isomorphic to $(\mathbf{Z}_2 \times \mathbf{Z}_2, +)$. In this H_v-group there is only one unit and every element has a unique double inverse.

Theorem 2.3. *All multiplicative H_v-fields defined on $(\mathbf{Z}_6, +, \cdot)$, which have non-degenerate fundamental field, and satisfy the above 4 conditions, are the following isomorphic cases:*
We have the only one hyperproduct,

(I) $2 \otimes 3 = \{0, 3\}$ *or* $2 \otimes 4 = \{2, 5\}$ *or*
$3 \otimes 4 = \{0, 3\}$ *or* $3 \otimes 5 = \{0, 3\}$ *or* $4 \otimes 5 = \{2, 5\}$
Fundamental classes: $[0] = \{0, 3\}, [1] = \{1, 4\}, [2] = \{2, 5\},$ *and*
$(\mathbf{Z}_6, +, \cdot)/\gamma^* \cong (\mathbf{Z}_3, +, \cdot)$.

(II) $2 \otimes 3 = \{0, 2\}$ *or* $2 \otimes 3 = \{0, 4\}$ *or* $2 \otimes 4 = \{0, 2\}$ *or* $2 \otimes 4 = \{2, 4\}$ *or*
$2 \otimes 5 = \{0, 4\}$ *or* $2 \otimes 5 = \{2, 4\}$ *or* $3 \otimes 4 = \{0, 2\}$ *or* $3 \otimes 4 = \{0, 4\}$ *or*
$3 \otimes 5 = \{3, 5\}$ *or* $4 \otimes 5 = \{0, 2\}$ *or* $4 \otimes 5 = \{2, 4\}$
Fundamental classes: $[0] = \{0, 2, 4\}, [1] = \{1, 3, 5\},$ *and*
$(\mathbf{Z}_6, +, \otimes)/\gamma^* \cong (\mathbf{Z}_2, +, \cdot)$.

Theorem 2.4. *All multiplicative H_v-fields defined on $(\mathbf{Z}_9, +, \cdot)$, which have non-degenerate fundamental field, and satisfy the above 4 conditions, are the following isomorphic cases:*
We have the only one hyperproduct,
$2 \otimes 3 = \{0, 6\}$ *or* $\{3, 6\}$, $2 \otimes 4 = \{2, 8\}$ *or* $\{5, 8\}$, $2 \otimes 6 = \{0, 3\}$ *or* $\{3, 6\}$,
$2 \otimes 7 = \{2, 5\}$ *or* $\{5, 8\}$, $2 \otimes 8 = \{1, 7\}$ *or* $\{4, 7\}$, $3 \otimes 4 = \{0, 3\}$ *or* $\{3, 6\}$,
$3 \otimes 5 = \{0, 6\}$ *or* $\{3, 6\}$, $3 \otimes 6 = \{0, 3\}$ *or* $\{0, 6\}$, $3 \otimes 7 = \{0, 3\}$ *or* $\{3, 6\}$,
$3 \otimes 8 = \{0, 6\}$ *or* $\{3, 6\}$, $4 \otimes 5 = \{2, 5\}$ *or* $\{2, 8\}$, $4 \otimes 6 = \{0, 6\}$ *or* $\{3, 6\}$,
$4 \otimes 8 = \{2, 5\}$ *or* $\{5, 8\}$, $5 \otimes 6 = \{0, 3\}$ *or* $\{3, 6\}$, $5 \otimes 7 = \{2, 8\}$ *or* $\{5, 8\}$,
$5 \otimes 8 = \{1, 4\}$ *or* $\{4, 7\}$, $6 \otimes 7 = \{0, 6\}$ *or* $\{3, 6\}$, $6 \otimes 8 = \{0, 3\}$ *or* $\{3, 6\}$,
$7 \otimes 8 = \{2, 5\}$ *or* $\{2, 8\}$,
Fundamental classes: $[0] = \{0, 3, 6\}$, $[1] = \{1, 4, 7\}$, $[2] = \{2, 5, 8\}$*, and*
$(\mathbf{Z}_9, +, \otimes)/\gamma^* \cong (\mathbf{Z}_3, +, \cdot)$.

Theorem 2.5. *All H_v-fields defined on $(\mathbf{Z}_{10}, +, \cdot)$, which have non-degenerate fundamental field, and satisfy the above 4 conditions, are the following isomorphic cases:*

(I) We have the only one hyperproduct,
$2 \otimes 4 = \{3, 8\}$, $2 \otimes 5 = \{2, 5\}$, $2 \otimes 6 = \{2, 7\}$, $2 \otimes 7 = \{4, 9\}$, $2 \otimes 9 = \{3, 8\}$, $3 \otimes 4 = \{2, 7\}$, $3 \otimes 5 = \{0, 5\}$, $3 \otimes 6 = \{3, 8\}$, $3 \otimes 8 = \{4, 9\}$, $3 \otimes 9 = \{2, 7\}$, $4 \otimes 5 = \{0, 5\}$, $4 \otimes 6 = \{4, 9\}$, $4 \otimes 7 = \{3, 8\}$, $4 \otimes 8 = \{2, 7\}$, $5 \otimes 6 = \{0, 5\}$, $5 \otimes 7 = \{0, 5\}$, $5 \otimes 8 = \{0, 5\}$, $5 \otimes 9 = \{0, 5\}$, $6 \otimes 7 = \{2, 7\}$, $6 \otimes 8 = \{3, 8\}$, $6 \otimes 9 = \{4, 9\}$, $7 \otimes 9 = \{3, 8\}$, $8 \otimes 9 = \{2, 7\}$.
Fundamental classes: $[0] = \{0, 5\}$, $[1] = \{1, 6\}$, $[2] = \{2, 7\}$, $[3] = \{3, 8\}$, $[4] = \{4, 9\}$ *and* $(\mathbf{Z}_{10}, +, \otimes)/\gamma^* \cong (\mathbf{Z}_5, +, \cdot)$.

(II) The cases where we have two classes $[0] = \{0, 2, 4, 6, 8\}$ *and* $[1] = \{1, 3, 5, 7, 9\}$*, thus we have fundamental field* $(\mathbf{Z}_{10}, +, \otimes)/\gamma^* \cong (\mathbf{Z}_2, +, \cdot)$*, can be described as follows:*
Taking in the multiplicative table only the results above the diagonal, we enlarge each of the products by putting one element of the same class of the results. We do not enlarge setting the element 1, and we cannot enlarge only the product $3 \otimes 7 = 1$. *The number of those H_v-fields is 103.*

Example 2.1. *In order to see how hard is to realize the reproductivity of classes and the unit class and inverse class, we consider the above H_v-field $(\mathbf{Z}_{10}, +, \otimes)$ where we have $2 \otimes 4 = \{3, 8\}$. Then the Multiplicative Table of the hyperproduct is the following:*

$\otimes$	***0***	***1***	***2***	***3***	***4***	***5***	***6***	***7***	***8***	***9***
0	*0*	*0*	*0*	*0*	*0*	*0*	*0*	*0*	*0*	*0*
1	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*
2	*0*	*2*	*4*	*6*	***3,8***	*0*	*2*	*4*	*6*	*8*
3	*0*	*3*	*6*	*9*	*2*	*5*	*8*	*1*	*4*	*7*
4	*0*	*4*	*8*	*2*	*6*	*0*	*4*	*8*	*2*	*6*
5	*0*	*5*	*0*	*5*	*0*	*5*	*0*	*5*	*0*	*5*
6	*0*	*6*	*2*	*8*	*4*	*0*	*6*	*2*	*8*	*4*
7	*0*	*7*	*4*	*1*	*8*	*5*	*2*	*9*	*6*	*3*
8	*0*	*8*	*6*	*4*	*2*	*0*	*8*	*6*	*4*	*2*
9	*0*	*9*	*8*	*7*	*6*	*5*	*4*	*3*	*2*	*1*

On this table it is easy to see that the reproductivity is not valid but it is very hard to see that the reproductivity of classes is valid. We can see the reproductivity of classes easier if we reformulate the Multiplicative Table according to the fundamental classes, $[0] = \{0, 5\}, [1] = \{1, 6\}, [2] = \{2, 7\}, [3] = \{3, 8\}, [4] = \{4, 9\}$. *Then we obtain:*

⊗	***0***	***5***	***1***	***6***	***2***	***7***	***3***	***8***	***4***	***9***
0	*0*	*0*	*0*	*0*	*0*	*0*	*0*	*0*	*0*	*0*
5	*0*	*5*	*5*	*0*	*0*	*5*	*5*	*0*	*0*	*5*
1	*0*	*5*	*1*	*6*	*2*	*7*	*3*	*8*	*4*	*9*
6	*0*	*0*	*6*	*6*	*2*	*2*	*8*	*8*	*4*	*4*
2	*0*	*0*	*2*	*2*	*4*	*4*	*6*	*6*	***3,8***	*8*
7	*0*	*5*	*7*	*2*	*4*	*9*	*1*	*6*	*8*	*3*
3	*0*	*5*	*3*	*8*	*6*	*1*	*9*	*4*	*2*	*7*
8	*0*	*0*	*8*	*8*	*6*	*6*	*4*	*4*	*2*	*2*
4	*0*	*0*	*4*	*4*	*8*	*8*	*2*	*2*	*6*	*6*
9	*0*	*5*	*9*	*4*	*8*	*3*	*7*	*2*	*6*	*1*

From this it is easy to see the unit class and the inverse class of each class.

3 The h/v-representations and applications

H_v-structures are used in Representation Theory of H_v-groups which can be achieved either by generalized permutations or by H_v-matrices [31], [32], [38], [39], [44], [46], [57], [58]. The representations by generalized permutations can be faced by translations [37]. Moreover in hyperstructure theory we can define hyperproduct on non-square ordinary matrices by using the so called helix hopes where we use all entries of them [65], [28], [29] and [13], [14], [66], [67]. Thus, we face the representations of the hyperstructures by non-square matrices as well.

$\mathbf{H_v}$-matrix (or **h/v-matrix**) is a matrix with entries of an H_v-ring or H_v-field (or h/v-ring or h/v-field). The hyperproduct of two H_v-matrices (a_{ij}) and $= (b_{ij})$, of type $m \times n$ and $n \times r$ respectively, is defined in the usual manner and it is a set of $m \times r$ H_v-matrices. The sum of products of elements of the H_v-ring is considered to be the n-ary circle hope on the hyperaddition. The hyperproduct of H_v-matrices is not necessarily WASS.

The problem of the H_v-matrix (or h/v-group) representations is the following:

Definition 3.1. *Let $(H, \cdot)$ be an H_v-group (or h/v-group). Find an H_v-ring (or h/v-ring) $(R, +, \cdot)$, a set M_R=$\{(a_{ij})|a_{ij}{\in}R\}$ and a map $T : H \to M_R : h \mapsto T(h)$ such that*

$$T(h_1h_2) \cap T(h_1)T(h_2) \neq \varnothing, \forall h_1, h_2 \in H.$$

T is an $\mathbf{H_v}$***-matrix (or h/v matrix) representation***.
If $T(h_1h_2) \subset T(h_1)T(h_2), \forall h_1, h_2 \in H$, then T is an inclusion representation.
If $T(h_1h_2) = T(h_1)T(h_2), \forall h_1, h_2 \in H$, then T is a good representation and an induced representation T^* of the hypergroup algebra is obtained. If T is one to one and the good condition is valid then it is called faithful representation.

The main theorem of the theory of representations is the following [31], [32], [38]:

Theorem 3.1. *A necessary condition in order to have an inclusion representation T of an h/v-group $(H,\cdot)$ by $n\times n$, h/v-matrices over the h/v-ring $(R,+,\cdot)$ is the following:*
For all classes $\beta*(x)$, $x\in H$ there must exist elements $a_{ij}\in H, i,j\in\{1,...,n\}$ such that

$$T(\beta*(a))\subset\{A=(a'_{ij})|a'_{ij}\in\gamma*(a_{ij}),i,j\in\{1,...,n\}\}$$

Thus, inclusion representation $T:H\to M_R:a\mapsto T(a)=(a_{ij})$ induces an homomorphic $T*$ of $H/\beta*$ over $R/\gamma*$ by setting
$T*(\beta*(a))=[\gamma*(a_{ij})],\forall\beta*(a)\in H/\beta*$, where $\gamma*(a_{ij})R/\gamma*$ is the ij entry of $T*(\beta*(a))$. $T*$ is called fundamental induced representation of T.

Let T a representation of an h/v-group H by h/v-matrices and $tr_\phi(T(x))=\gamma^*(Tx_{ii})$ be the fundamental trace, then is called *fundamental character*, the mapping

$$X_T:H\to R/\gamma*:x\mapsto X_T(x)=tr_\phi(T(x))=trT^*(x)$$

In representations of H_v-groups there are two difficulties: First to find an H_v-ring or an H_v-field and second, an appropriate set of H_v-matrices. Notice that the more interesting cases are for the small H_v-fields, where the results have one or few elements.

Example 3.1. *In the case of the H_v-field $(\mathbf{Z}_6,+,\otimes)$ where the only one hyperproduct is $2\otimes4=\{2,5\}$ we consider the 2×2 h/v-matrices of type*

$$\underline{i}=E_{11}+iE_{12}+4E_{22},\ \textit{where } i=0,1,2,3,4,5,$$

then an h/v-group is obtained and the multiplicative table of the hyperproduct of those H_v-matrices is given by

$\otimes$	$\underline{0}$	$\underline{1}$	$\underline{2}$	$\underline{3}$	$\underline{4}$	$\underline{5}$
$\underline{0}$	$\underline{0}$	$\underline{1}$	$\underline{2}$	$\underline{3}$	$\underline{4}$	$\underline{5}$
$\underline{1}$	$\underline{4}$	$\underline{5}$	$\underline{0}$	$\underline{1}$	$\underline{2}$	$\underline{3}$
$\underline{2}$	$\underline{2}$	$\underline{0,3}$	$\underline{1,4}$	$\underline{2,5}$	$\underline{0,3}$	$\underline{1,4}$
$\underline{3}$	$\underline{0}$	$\underline{1}$	$\underline{2}$	$\underline{3}$	$\underline{4}$	$\underline{5}$
$\underline{4}$	$\underline{4}$	$\underline{5}$	$\underline{0}$	$\underline{1}$	$\underline{2}$	$\underline{3}$
$\underline{5}$	$\underline{2}$	$\underline{3}$	$\underline{4}$	$\underline{5}$	$\underline{0}$	$\underline{1}$

where the fundamental classes are $(0)=\{\underline{0},\underline{3}\},(1)=\{\underline{1},\underline{4}\},(2)=\{\underline{2},\underline{5}\}$ and the fundamental group is isomorphic to $(\mathbf{Z}_3,+)$. Remark that $(\mathbf{Z}_6,\otimes)$ is an h/v-group which is cyclic where the elements $\underline{2}$ and $\underline{4}$ are generators of period 4. Notice that the hope $(\otimes)$ is a hyperproduct of h/v-matrices although (0) stands for the unit matrix, this is so because the symbolism follows the entry 12.

Example 3.2. *Let us denote by E_{ij} the matrix with 1 in the ij-entry and zero in the rest entries. Then take the following 2×2 upper triangular h/v-matrices on the above h/v-field $(\mathbf{Z}_4, +, \otimes)$ of the case that only $2 \otimes 3 = \{0, 2\}$ is a hyperproduct:*

$$I = E_{11}+E_{22}, a = E_{11}+E_{12}+E_{22}, b = E_{11}+2E_{12}+E_{22}, c = E_{11}+3E_{12}+E_{22},$$

$$d = E_{11}+3E_{22}, e = E_{11}+E_{12}+3E_{22}, f = E_{11}+2E_{12}+3E_{22}, g = E_{11}+3E_{12}+3E_{22},$$

then, we obtain the following multiplicative table for the set X={I,a,b,c,d,e,f,g}

$\otimes$	I	a	b	c	d	e	f	g
I	I	a	b	c	d	e	f	g
a	a	b	c	I	g	d	e	f
b	b	c	I	a	d,f	e,g	d,f	e,g
c	c	I	a	b	e	f	g	d
d	d	e	f	g	I	a	b	c
e	e	f	g	d	c	I	a	b
f	f	g	d	e	I,b	a,c	I,b	a,c
g	g	d	e	f	a	b	c	I

The $(\mathbf{X}, \otimes)$ is non-COW, H_v-group and we can see that the fundamental classes are the $\underline{a} = \{a, c\}$, $\underline{d} = \{d, f\}$, $\underline{e} = \{e, g\}$ and the fundamental group is isomorphic to $(\mathbf{Z}_2 \times \mathbf{Z}_2, +)$. In this H_v-group there is only one unit and every element has a unique double inverse. Only f has one more right inverse element, the d, since $f \otimes d = \{I, b\}$.

Remark that if we need h/v-fields where the elements have at most one inverse element, then we must exclude the case of $2 \otimes 5 = \{1, 4\}$ from (I), and the case $3 \otimes 5 = \{1, 3\}$ from (II).

Last decades H_v-structures have applications in other branches of mathematics and in other sciences. These applications range from biomathematics -conchology, inheritance- and hadronic physics or on leptons to mention but a few. The hyperstructure theory is related to fuzzy theory; consequently, hyperstructures can be widely applicable in industry and production, too [2], [5], [11], [12], [23], [25], [43], [47], [59].

The Lie-Santilli theory on isotopies was born in 1970's to solve Hadronic Mechanics problems. Santilli proposed a 'lifting'of the n-dimensional trivial unit matrix of a normal theory into a nowhere singular, symmetric, real-valued, positive-defined, n-dimensional new matrix. The original theory is reconstructed such as to admit the new matrix as left and right unit. The isofields needed, correspond into the hyperstructures were introduced by Santilli & Vougiouklis in 1996 [25] and they are called *e-hyperfields*, [12], [24], [52], [56], [61].

Definition 3.2. *A hyperstructure $(H, \cdot)$ which contains a unique scalar unit e, is called e-hyperstructure. In an e-hyperstructure, we assume that for every element x, there exists an inverse x^{-1}, i.e. $e \in x \cdot x^{-1} \cap x^{-1} \cdot x$.*

Definition 3.3. *A hyperstructure $(F, +, \cdot)$, where $(+)$ is an operation and $(\cdot)$ is a hope, is called e-hyperfield if the following axioms are valid: $(F, +)$ is an abelian group with the additive unit 0, $(\cdot)$ is WASS, $(\cdot)$ is weak distributive with respect to $(+)$, 0 is absorbing element: $0 \cdot x = x \cdot 0 = 0, \forall x \in F$, there exists a multiplicative scalar unit 1, i.e. $1 \cdot x = x \cdot 1 = x, \forall x \in F$, and $\forall x \in F$ there exists a unique inverse x^{-1}, such that $1 \in x \cdot x^{-1} \cap x^{-1} \cdot x$.*

The elements of an e-hyperfield are called *e-hypernumbers*. In the case that the relation: $1 = x \cdot x^{-1} = x^{-1} \cdot x$, is valid, then we say that we have a *strong e-hyperfield*.

Definition 3.4. *Main e-Construction. Given a group $(G, \cdot)$, where e is the unit, then we define in G, a large number of hopes $(\otimes)$ as follows:*

$$x \otimes y = \{xy, g_1, g_2, ...\}, \forall x, y \in G - \{e\}, \text{ and } g_1, g_2, ... \in G - \{e\}$$

g_1, g_2,... are not necessarily the same for each pair (x,y). $(G, \otimes)$ is an H_v-group, in fact it is an H_b-group which contains the $(G, \cdot)$. $(G, \otimes)$ is an e-hypergroup. Moreover, if for each x,y such that $xy = e$, then $(G, \otimes)$ becomes a strong e-hypergroup

The main e-construction gives an extremely large number of e-hopes.

Example. Consider the quaternions $\mathbf{Q} = \{1, -1, i, -i, j, -j, k, -k\}$, with $i^2 = j^2 = -1, ij = -ji = k$, and denote $\underline{i} = \{i, -i\}, \underline{j} = \{j, -j\}, \underline{k} = \{k, -k\}$. We define a lot of hopes $(*)$ by enlarging few products. For example, $(-1) * k = \underline{k}, k * i = \underline{j}$ and $i * j = \underline{k}$. Then the hyperstructure $(Q, *)$ is a strong e-hypergroup.

The Lie-Santilli admissibility on non-quare matrices [12], [14], [24], [26], [57], [61]:

Construction 3.1. *Let $L = (M_{m \times n}, +)$ be an H_v-vector space of $m \times n$ hypermatrices over the H_v-field $(F, +, \cdot), \phi : F \to F/\gamma^*$, the canonical map and $\omega_F = \{x \in F : \phi(x) = 0\}$, where 0 is the zero of the fundamental field F/γ*. Similarly, let ω_L be the core of the canonical map $\phi' : L \to L/\epsilon^*$ and denote by the same symbol 0 the zero of L/ϵ*. Take any two subsets $R, S \subseteq L$ then a Santilli's Lie-admissible hyperalgebra is obtained by taking the Lie bracket, which is a hope:*

$$[,]_{RS} : L \times L \to P(L) : [x, y]_{RS} = xR^t y - yS^t x.$$

Notice that $[x, y]_{RS} = xR^t y - yS^t x = \{xr^t y - ys^t x | r \in R \text{ and } s \in S\}$

An application, which combines the ∂-structures and fuzzy theory, is to replace in questionnaires the scale of Likert by the bar of Vougiouklis & Vougiouklis [19]:

Definition 3.5. *In every question substitute the Likert scale with 'the bar' whose poles are defined with '0' on the left end, and '1' on the right end:*

0 ________________ *1*

The subjects/participants are asked instead of deciding and checking a specific grade on the scale, to cut the bar at any point s/he feels expresses her/his answer to the specific question

The use of the Vougiouklis & Vougiouklis bar instead of a Likert scale has several advantages during both the filling-in and the research processing. The final suggested length of the bar, according to the Golden Ratio, is 6.2cm, [17], [18], [50], [51], [62], [63], [64].

References

[1] N. Antampoufis, S. Spartalis and T. Vougiouklis, *Fundamental relations in special extensions*, $8^t h$ AHA, Samothraki, Greece 2002, Spanidis, (2003), 81-89.

[2] N. Antampoufis, T. Vougiouklis, and A. Dramalidis, *Geometrical and circle hyperoperations in urban applications*, Ratio Sociologica, V.4, N.2, (2011), 53-66.

[3] J. Chvalina and S. Hoskova , *Modelling of join spaces with proximities by first-order linear partial differential operators*, Ital. J. Pure Appl. Math. 21, (2007), 177-190.

[4] P. Corsini , *Prolegomena of Hypergroup Theory*, Aviani Editore , 1993.

[5] P. Corsini and V. Leoreanu, *Application of Hyperstructure Theory*, Klower Academic Publishers, 2003

[6] P. Corsini and T. Vougiouklis, *From groupoids to groups through hypergroups*, Rendiconti Mat. S. VII, 9, (1989), 173-181.

[7] B. Davvaz, *On H_v-subgroups and anti fuzzy H_v-subgroups*, Korean J. Comp. Appl. Math. V.5, N.1, (1998), 181-190.

[8] B. Davvaz, *A brief survey of the theory of H_v-structures*, 8^{th} AHA, Greece, Spanidis, (2003), 39-70.

[9] B. Davvaz,*Polygroup Theory and Related Systems*, World Scientific, 2013.

[10] B. Davvaz, *Semihypergoup Theory*, Academic Press, 2016.

[11] B. Davvaz and V. Leoreanu, *Hyperring Theory and Applications*, Int. Academic Press, 2007.

[12] B. Davvaz, R.M. Santilli and T. Vougiouklis, *Algebra, Hyperalgebra and Lie?Santilli Theory*, J. Generalized Lie Theory Appl., 9:2, (2015), 1-5.

[13] B. Davvaz, S. Vougiouklis and T. Vougiouklis,*On the multiplicative H_v-rings derived from helix hyperoperations*, Util. Math., 84, (2011), 53-63.

[14] A. Dramalidis, R. Mahjoob and T. Vougiouklis, *P-hopes on non-square matrices for Lie-Santilli Admissibility*, Clifford Algebras Appl. CACAA, V.4, N.4, (2015), 361-372.

[15] A. Dramalidis, and T. Vougiouklis, *Fuzzy H_v-substructures in a two dimensional Euclidean vector space*, 10^{th} AHA, Brno, Czech Republic 2008, (2009), 151-159, and Iranian J. Fuzzy Systems, 6(4), 1-9.

[16] A. Dramalidis, and T. Vougiouklis, *The rational numbers through H_v-structures*, Int. J. Modern Sc. Eng. Techn., ISSN 2347-3755, http://www.ijmset.com., V. 2(7), (2015), 32-41.

[17] P. Kambaki-Vougioukli, A., Karakos, N. Lygeros and T. Vougiouklis, *Fuzzy instead of discrete*, Annals of Fuzzy Math. and Informatics (AFMI), V.2, N.1, (2011), 81-89.

[18] P. Kambaki-Vougiouklis, P. Nikolaidou and T. Vougiouklis, *Questionnaires in Linguistics using the Bar and the H_v-structures*, Studies in Systems, Decision and Control 66, Springer, 2017, (2017), 257-266.

[19] P. Kambaki-Vougioukli and T. Vougiouklis, *Bar instead of scale*, Ratio Sociologica, 3, (2008), 49-56.

[20] M. Koskas, *Groupoides demi-hypergroupes et hypergroupes*, J. Math. Pures Appl., 49 (9), (1970), 155-192.

[21] N. Lygeros and T. Vougiouklis, *The LV-hyperstructures*, Ratio Math., 25, (2013), 59-66.

[22] R. Mahjoob and T. Vougiouklis, *Applications of the uniting elements method*, Italian J. Pure Applied Math., N.36, (2016), 23-34.

[23] A. Maturo, E. Sciarra, I. Tofan , *A formalization of some aspects of the Social Organization by means of fuzzy set theory*, Ratio Sociologica, V. 1, N. 1, (2008), 5-20.

[24] P. Nikolaidou and T. Vougiouklis, *The Lie-Santilli admissible hyperalgebras of type A_n*, Ratio Mathematica 26, (2014), 113-128.

[25] R.M Santilli and T. Vougiouklis, *Isotopies, Genotopies, Hyperstructures and their Appl.*, Proc. New Frontiers: Hyperstructures Related Algebras, Hadronic, (1996), 1-48.

[26] R.M Santilli and T. Vougiouklis, *Lie-admissible hyperalgebras*, Italian J. Pure Applied Math., N.31, (2013), 239-254.

[27] S. Spartalis, A. Dramalides, A. and T. Vougiouklis, *On H_v-Group Rings, Algebras*, Groups and Geometries, 15, (1998), 47-54.

[28] S. Vougiouklis , *H_v-vector spaces from helix hyperoperations*, Int. J. Math. Anal. (New Series), 1(2), (2009), 109-120.

[29] S. Vougioukli and T. Vougiouklis, *Helix-hopes on Finite H_v-fields*, Algebras Groups and Geometries (AGG), V.33, N.4, (2016), 491-506.

[30] T. Vougiouklis , *Cyclicity in a special class of hypergroups*, Acta Un. Car. - Math. Et Ph., V.22, N1, (1981), 3-6.

[31] T. Vougiouklis, *Representations of hypergroups, Hypergroup algebra*, Proc. Convegno: ipergrouppi, altre strutture multivoche appl. Udine, (1985), 59-73.

[32] T. Vougiouklis, *Representations of hypergroups by hypermatrices*, Rivista Mat. Pura Appl., N 2, (1987), 7-19.

[33] T. Vougiouklis, *Generalization of P-hypergroups*, Rend. Circ. Mat. Palermo, S.II, 36, (1987), 114-121.

[34] T. Vougiouklis, *Groups in hypergroups*, Annals Discrete Math. 37, (1988), 459-468.

[35] T. Vougiouklis, *The fundamental relation in hyperrings. The general hyperfield*, Proc. 4th AHA, World Scientific, (1991), 203-211.

[36] T. Vougiouklis, *The very thin hypergroups and the S-construction*, Combinatorics ?88, Incidence Geom. Comb. Str., 2, (1991), 471-477.

[37] T. Vougiouklis, *Representations of hypergroups by generalized permutations*, Algebra Universalis, 29, (1992), 172-183.

[38] T. Vougiouklis, *Representations of H_v-structures*, Proc. Int. Conf. Group Theory 1992, Timisoara, (1993), 159-184.

[39] T. Vougiouklis, *Hyperstructures and their Representations*, Monographs in Math. Hadronic Press, 1994

[40] T. Vougiouklis, *Some remarks on hyperstructures*, Contemporary Math., Amer. Math. Society, 184, (1995), 427-431.

[41] T. Vougiouklis, *A new class of hyperstructures*, JCI& SS,V.20, N.1-4, (1995), 229-239.

[42] T. Vougiouklis, *H_v-groups defined on the same set*, Discrete Math. 155, (1996), 259-265.

[43] T. Vougiouklis, *Enlarging H_v-structures*, Algebras and Comb., ICAC'97, Hong Kong, Springer-Verlag, (1999), 455-463.

[44] T. Vougiouklis, *On H_v-rings and H_v-representations*, Discrete Math., Elsevier, 208/209, (1999), 615-620.

[45] T. Vougiouklis, *The h/v-structures*, J. Discrete Math. Sci. Cryptography, V.6, N.2-3, (2003), 235-243.

[46] T. Vougiouklis, *Finite H_v-structures and their representations*, Rend. Sem. Mat. Messina S.II, V.9, (2003), 245-265.

[47] T. Vougiouklis, *A hyperoperation defined on a groupoid equipped with a map*, Ratio Mathematica, n.15, N.1, (2005), 25-36.

[48] T. Vougiouklis, *On a matrix hyperalgebra*, J. Basic Science V.3/N1, (2006), 43-47.

[49] T. Vougiouklis, *∂-operations and H_v-fields*, Acta Mathematica Sinica. English Series, V.24, N.7, (2008), 1067-1078.

[50] T. Vougiouklis, *Hyperstructures as models in social sciences*, Ratio Sociologica, V.2, N.2, (2009), 21-34.

[51] T. Vougiouklis, *Bar and Theta Hyperoperations*, Ratio Mathematica, 21, (2011), 27-42.

[52] T. Vougiouklis, *The e-hyperstructures*, J. Mahani Math. R. C., V1, N1, (2012), 13-28.

[53] T. Vougiouklis, *The Lie-hyperalebras and their fundamental relations*, Southeast Asian Bull. Math., V.37(4), (2013), 601-614.

[54] T. Vougiouklis, *Enlarged Fundamentally Very Thin H_v-structures*, J. Algebraic Structures and Their Appl. (ASTA), V.1, No1, (2014), 11-20.

[55] T. Vougiouklis, *From H_v-rings to H_v-fields*, Int. J.A.H.A. V.1, N.1, (2014), 1-13.

[56] T. Vougiouklis, *On the isoHv-numbers*, Hadronic J., Dec.5, (2014), 1-18.

[57] T. Vougiouklis, *Hypermathematics, H_v-structures, hypernumbers, hypermatrices and Lie-Santilli admissibility*, American J. Modern Physics, 4(5), (2015), 34-46.

[58] T. Vougiouklis, *Hypermatrix representations of finite H_v-groups*, European J. Combinatorics, V.44 B, (2015), 307-315.

[59] T. Vougiouklis, *Quiver of hyperstructures for Ying's twin universes*, American J. Modern Physics, 4(1-1), (2015), 30-33.

[60] T. Vougiouklis, *On the Hyperstructure Theory*, Southeast Asian Bull. Math., Vol. 40(4), (2016), 603-620.

[61] T. Vougiouklis , *Hypernumbers, Finite Hyper-fields* Algebras Groups and Geometries (AGG), V.33, N.4, (2016), 471-490.

[62] T. Vougiouklis and P. Kambaki-Vougioukli, *On the use of the bar*, China-USA Business Review, V.10, N.6, (2011), 484-489.

[63] T. Vougiouklis and P. Kambaki-Vougioukli, *Bar in Questionnaires*, Chinese Business Review, V.12, N.10, (2013), 691-697.

[64] T. Vougiouklis and P. Vougiouklis, *Questionnaires with the 'bar' in social sciences*. Science & Philosophy, 3(2), (2015), 47-58.

[65] T. Vougiouklis and S. Vougiouklis, *The helix hyperoperations*, Italian J. Pure Appl. Math., 18, (2005), 197-206.

[66] T. Vougiouklis and S. Vougiouklis, *Hyper-representations by non square matrices, Helix-hopes*, American J. Modern Physics, 4(5), (2015), 52-58.

[67] T. Vougiouklis and S. Vougiouklis, *Helix-Hopes on Finite Hyperfields*, Ratio Mathematica, V.31, (2016), 65-78.

[68] J. Zhan, V. Leoreanu-Fotea and T. Vougiouklis,*Fuzzy soft Γ-hypermodules*, U.P.B. Sci. Bull., Series A, Vol. 73, Iss. 3, (2011), 13-28.