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On Rough Sets and Hyperlattices

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Abstract

In this paper, we introduce the concepts of upper and lower rough hyper fuzzy ideals (filters) in a hyperlattice and their basic properties are discussed. Let θ be a hyper congruence relation on L. We show that if μ is a fuzzy subset of L, then $\overline{\theta}(<\mu>) = \overline{\theta}(<\overline{\theta}(\mu)>)$ and $\overline{\theta}(\mu^*) = \overline{\theta}((\overline{\theta}(\mu))^*)$, where $<\mu>$ is the least hyper fuzzy ideal of L containing μ and

$$\mu^*(x) = sup\{\alpha \in [0,1] : x \in I(\mu_\alpha)\}$$

for all $x \in L$. Next, we prove that if μ is a hyper fuzzy ideal of L, then μ is an upper rough fuzzy ideal. Also, if θ is a $\wedge-$complete on L and μ is a hyper fuzzy prime ideal of L such that $\overline{\theta}(\mu)$ is a proper fuzzy subset of L, then μ is an upper rough fuzzy prime ideal. Furthermore, let θ be a $\vee$-complete congruence relation on L. If μ is a hyper fuzzy ideal, then μ is a lower rough fuzzy ideal and if μ is a hyper fuzzy prime ideal such that $\underline{\theta}(\mu)$ is a proper fuzzy subset of L, then μ is a lower rough fuzzy prime ideal.

Keywords: rough set, upper and lower approximations, hyperlattice, hyper fuzzy prime ideal, hyper fuzzy prime filter.

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1 Introduction

In this paper, we are using three basic notions: hyperlattice, rough set and fuzzy subset. Hyperstructure theory was born in 1934 when Marty [14] defined hypergroups as a generalization of groups. Extending lattices (also called hyperlattices) have been recently studied by a number of authors, in particular, Koguep, Nkuimi and Lele [12], Feng and Zou [8], Guo and Xin [9], Rahnamai-Barghi [19], etc.

Rough set theory was introduced by Pawlak in 1982 [17]. Many authors have studied the general properties of generalized rough sets [1, 4, 5].

The concept of fuzzy subsets was first introduced by Zadeh [22] in 1965 and then the fuzzy subsets have been used in the reconsideration of classical mathematics. The relationships between fuzzy subsets and algebraic hyperstructures had been already considered by many researchers (for example [2, 21]). Also, there have been many papers studying the connections and differences of fuzzy subset theory and rough set theory [3, 15, 18]. In recent years, many efforts have been made to compare and combine the three theories [6, 7].

This paper is structured as follows. After the introduction, in Section 2, we recall some basic notions and results on hyperlattices, rough sets and fuzzy subsets. In Section 3, the notions of hyper congruence relation on a hyperlattice are introduced. Next, some important properties of θ-upper approximations of a fuzzy subset will be studied. Also by an example, we show that Theorem 2.15 in [12] is incorrect (see Example 3.20) and a corrected version is considered, Proposition 3.21. Finally, in Section 4, θ-lower approximations of a fuzzy subset on a hyperlattice will be studied.

2 Preliminaries of hyperlattices, rough sets and fuzzy subsets

In the remainder of the paper we use some notation and results from the theory of hyperlattices, rough sets and fuzzy subsets. We present a few basic definitions here.

Let L be a set partially ordered by the binary relation $\leq$. The poset $(L, \leq)$ is a *meet-semilattice* if for all elements x and y of L, the greatest lower bound or the meet of the set $\{x, y\}$, denoted by $x \wedge y$, exists. For x and y in a meet-semilattice L, $x \leq y \Leftrightarrow x = x \wedge y$.

Replacing greatest lower bound with least upper bound results in the dual concept of a *join-semilattice*. The least upper bound of $\{x, y\}$ is called the join of x and y and is denoted by $x \vee y$. A poset L is a *lattice* if and only if it is both a meet- and a join-semilattice.

In this paper, we use the following notion of a hyperlattice.

Let L be a non-empty meet-semilattice and $\vee : L \times L \to \mathcal{P}(L)^*$ be a hyperoperation, where $\mathcal{P}(L)$ is the power set of L and $\mathcal{P}(L)^* = \mathcal{P}(L) \setminus \{\emptyset\}$. Then $(L, \wedge, \vee)$ is a *hyperlattice* [19], if for all $a, b, c \in L$:

1. $a \in a \vee a$ and $a = a \wedge a$.

2. $a \vee b = b \vee a$ and $a \wedge b = b \wedge a$.

3. $(a \vee b) \vee c = a \vee (b \vee c)$ and $(a \wedge b) \wedge c = a \wedge (b \wedge c)$.

4. $a \in [a \wedge (a \vee b)] \cap [a \vee (a \wedge b)]$.

5. $a \in a \vee b \Leftrightarrow a \wedge b = b$.

Where for all non-empty subsets A and B of L, $A \wedge B = \{a \wedge b | a \in A, b \in B\}$, $A \vee B = \bigcup\{a \vee b | a \in A, b \in B\}$.

Throughout this paper, L is a hyperlattice with the least element 0 and the greatest element 1. For $X \subseteq L$ and $x \in L$ we write:

1. $\downarrow X = \{y \in L : y \leq x$ for some $x \in X\}$.

2. $\uparrow X = \{y \in L : y \geq x$ for some $x \in X\}$.

3. $\downarrow x = \downarrow \{x\}$.

4. $\uparrow x = \uparrow \{x\}$.

A pair (L, θ), where θ is an equivalence relation on L, is called an *approximation space* [17] and for $a \in L$, the equivalence class (or coset) of a modulo θ is the set $[a]_\theta = \{x \in L | (a, x) \in \theta\}$ and also for $A \subseteq L$, we put $[A]_\theta = \bigcup_{a \in A} [a]_\theta$.

For an approximation space (L, θ), by an *upper rough approximation* in (L, θ) we mean a mapping $\overline{Apr} : \mathcal{P}(L) \to \mathcal{P}(L)$ which is defined for every $X \in \mathcal{P}(L)$ by

$$\overline{Apr}(X) = \{a \in L : [a]_\theta \cap X \neq \emptyset\}.$$

Also, by a *lower rough approximation* in (L, θ) we mean a mapping $\underline{Apr} : \mathcal{P}(L) \to \mathcal{P}(L)$ defined for every $X \in \mathcal{P}(L)$ by

$$\underline{Apr}(X) = \{a \in L : [a]_\theta \subseteq X\}.$$

Then $Apr(X) = (\underline{Apr}(X), \overline{Apr}(X))$ is called a *rough subset* in (L, θ) if $\underline{Apr}(X) \neq \overline{Apr}(X)$. The following proposition is well known and easily seen.

Proposition 2.1. *Let (L, θ) be an approximation space. For every subsets $X, Y \subseteq L$, we have*

1. $\underline{Apr}(X) \subseteq X \subseteq \overline{Apr}(X)$.

2. *If $X \subseteq Y$, then $\underline{Apr}(X) \subseteq \underline{Apr}(Y)$ and $\overline{Apr}(X) \subseteq \overline{Apr}(Y)$.*

3. $\overline{Apr}(X \cup Y) = \overline{Apr}(X) \cup \overline{Apr}(Y)$ *and* $\overline{Apr}(X \cap Y) \subseteq \overline{Apr}(X) \cap \overline{Apr}(Y)$.

4. $\underline{Apr}(X \cap Y) = \underline{Apr}(X) \cap \underline{Apr}(Y)$ *and* $\underline{Apr}(X \cup Y) \supseteq \underline{Apr}(X) \cup \underline{Apr}(Y)$.

5. $\underline{Apr}\left(\underline{Apr}(X)\right) = \underline{Apr}(X)$ *and* $\overline{Apr}\left(\overline{Apr}(X)\right) = \overline{Apr}(X)$.

Proof. See [13]. □

Proposition 2.2. [12] *Let $(L, \vee, \wedge)$ be a hyperlattice.Then, for each pair $(a, b) \in L \times L$ there exist $a_1, b_1 \in a \vee b$, such that $a \leq a_1$ and $b \leq b_1$.*

Definition 2.3. [19] A nonempty subset J of L is called an *ideal* of L if for all $x, y \in L$

1. $x, y \in J$ implies $x \vee y \subseteq J$.

2. If $x \in J$, then $\downarrow x \subseteq J$.

Definition 2.4. [19] A nonempty subset F of L is called a *filter* of L if for all $x, y \in L$

1. $x, y \in F$ implies $x \wedge y \in F$.

2. If $x \in F$ and $x \leq y$, then $y \in F$.

Given a hyperlattice L and a set $X \subseteq L$, let $\mathrm{I}(X)$ denote the least ideal containing X, called the *ideal generated* by X.

A *fuzzy subset* of X is any function from X into $[0, 1]$. Let $\mathcal{F}(L)$ be the set of all fuzzy subsets of L. For $\mu, \lambda \in F(X)$, we say $\mu \subseteq \lambda$ if and only if $\mu(x) \leq \lambda(x)$ for all $x \in X$.

Definition 2.5. [12] Let μ be a fuzzy subset of L. Then

1. μ is a *hyper fuzzy ideal* of L if, for all $x, y \in L$,

   (a) $\bigwedge_{a \in x \vee y} \mu(a) \geq \mu(x) \wedge \mu(y)$.

   (b) $x \leq y \Rightarrow \mu(x) \geq \mu(y)$.

2. μ is a *hyper fuzzy filter* of L if, for all $x, y \in L$,

   (a) $\mu(x \wedge y) \geq \mu(x) \wedge \mu(y)$.

   (b) $x \leq y \Rightarrow \mu(x) \leq \mu(y)$.

Definition 2.6. [12] Let μ be a proper hyper fuzzy ideal of L.

1. μ is called a *hyper fuzzy prime ideal*, if $\mu(x \wedge y) \leq \mu(x) \vee \mu(y)$, for all $x, y \in L$.

2. μ is called a *hyper fuzzy prime filter*, if $\bigwedge_{a \in x \vee y} \mu(a) \leq \mu(x) \vee \mu(y)$, for all $x, y \in L$.

Definition 2.7. [6] Let θ be an equivalence relation on L and μ a fuzzy subset of L. Then we define the fuzzy subsets $\overline{\theta}(\mu)$ and $\underline{\theta}(\mu)$ as follows:

$$\overline{\theta}(\mu)(x) = \bigvee_{a \in [x]_\theta} \mu(a) \quad \text{and} \quad \underline{\theta}(\mu)(x) = \bigwedge_{a \in [x]_\theta} \mu(a).$$

The fuzzy subsets $\overline{\theta}(\mu)$ and $\underline{\theta}(\mu)$ are, respectively, called the θ-*upper* and θ-*lower approximation* of the fuzzy subset μ. Then $\theta(\mu) = (\underline{\theta}(\mu), \overline{\theta}(\mu))$ is called a *rough fuzzy subset* with respect to μ if $\underline{\theta}(\mu) \neq \overline{\theta}(\mu)$.

Proposition 2.8. [6] *Let θ be an equivalence relation on L and $\mu, \lambda \in \mathcal{F}(L)$. Then*

1. $\underline{\theta}(\mu) \leq \mu \leq \overline{\theta}(\mu)$.

2. *If $\mu \subseteq \lambda$, then $\overline{\theta}(\mu) \leq \overline{\theta}(\lambda)$ and $\underline{\theta}(\mu) \leq \underline{\theta}(\lambda)$.*

3. $\overline{\theta}\,\overline{\theta}(\mu) = \overline{\theta}(\mu)$ *and* $\underline{\theta}\,\underline{\theta}(\mu) = \underline{\theta}(\mu)$.

4. $\underline{\theta}(\mu)(x) = \underline{\theta}(\mu)(a)$ *and* $\overline{\theta}(\mu)(x) = \overline{\theta}(\mu)(a)$, *for all $x \in L$ and $a \in [x]_\theta$.*

5. $\underline{\theta}\overline{\theta}(\mu) = \overline{\theta}(\mu)$ *and* $\overline{\theta}\underline{\theta}(\mu) = \underline{\theta}(\mu)$.

The proofs of the following propositions are straightforward.

Proposition 2.9. *Let θ be an equivalence relation on set A. Then the following statements hold:*

1. *For each $X \in \mathcal{P}(A)$,*

$$\overline{Apr}(X) = \bigcap \left\{ Y \in \mathcal{P}(A) : X \subseteq \underline{Apr}(Y) \right\} = \mathit{Min} \left\{ Y \in \mathcal{P}(A) : X \subseteq \underline{Apr}(Y) \right\}.$$

2. *For each $X \in \mathcal{P}(A)$,*

$$\underline{Apr}(X) = \bigcup \left\{Y \in \mathcal{P}(L) : \overline{Apr}(Y) \subseteq X\right\} = Max \left\{Y \in \mathcal{P}(L) : \overline{Apr}(Y) \subseteq X\right\}.$$

Proposition 2.10. *Let θ be an equivalence relation on set A. Then the following statements hold:*

1. *For each $\mu \in \mathcal{F}(A)$,*

$$\overline{\theta}(\mu) = \bigwedge \{\lambda \in \mathcal{F}(A) : \underline{\theta}(\lambda) \geq \mu\} = Min \{\lambda \in \mathcal{F}(A) : \underline{\theta}(\lambda) \geq \mu\}.$$

2. *For each $\mu \in \mathcal{F}(A)$,*

$$\underline{\theta}(\mu) = \bigvee \left\{\lambda \in \mathcal{F}(L) : \overline{\theta}(\lambda) \leq \mu\right\} = Max \left\{\lambda \in \mathcal{F}(L) : \overline{\theta}(\lambda) \leq \mu\right\}.$$

3 Upper approximations of a fuzzy subset

In this section we give some important properties of $\overline{\theta}$ with many examples, starting with the following definition.

Definition 3.1. [16, 20] Let θ be an equivalence relation on a hyperlattice L. Then θ is called a *hyper congruence relation* if $(a, b) \in \theta$ implies that $(a \vee x) \times (b \vee x) \subseteq \theta$ and $(a \wedge x, b \wedge x) \in \theta$ for all $x \in L$.

It is clear that if L is a hyperlattice L and $\theta = L \times L$, then θ is a hyper congruence relation. Also, in Example 3.9, we'll provide a non-trivial example.

Lemma 3.2. *Let θ be a hyper congruence relation on L. Then, for every $a, b, c, d \in L$,*

1. *If $(a, b) \in \theta$ and $(c, d) \in \theta$, then $(a \wedge c, b \wedge d) \in \theta$ and $(a \vee c) \times (b \vee d) \subseteq \theta$.*
2. $[a]_\theta \vee [b]_\theta \subseteq [a \vee b]_\theta$.
3. $[a]_\theta \wedge [b]_\theta \subseteq [a \wedge b]_\theta$.

Proof. Evident.

□

Proposition 3.3. *Let θ be an equivalence relation on L and $X \subseteq L$. If $\mu \in \mathcal{F}(L)$ is a hyper fuzzy ideal of L, then*

1. $\mu(\downarrow X) \subseteq \uparrow \mu(X)$.

2. $\mu(\uparrow X) \subseteq \downarrow \mu(X)$.
3. $\mu(\overline{Apr}(\downarrow X)) \subseteq \uparrow \mu(\overline{Apr}(X))$.
4. $\mu(\overline{Apr}(\uparrow X)) \subseteq \downarrow \mu(\overline{Apr}(X))$.
5. $\overline{\theta}(\mu)(\overline{Apr}(\downarrow X)) \subseteq \uparrow \overline{\theta}(\mu)(X)$.

Proof. (1) Let $a \in \downarrow X$. Then there exists $x \in X$ such that $a \leq x$. Since μ is a hyper fuzzy ideal, we conclude that $\mu(x) \leq \mu(a)$ which implies that $\mu(a) \in \uparrow \mu(X)$ and the proof is now complete.

(2) The proof is similar to the proof of (1).

(3) For each $X \subseteq L$, since $\downarrow \overline{Apr}(X) = \overline{Apr}(\downarrow X) = \downarrow \overline{Apr}(\downarrow X)$, we can then conclude from (1) that $\mu(\overline{Apr}(\downarrow X)) \subseteq \uparrow \mu(\overline{Apr}(X))$.

(4) For each $X \subseteq L$, we have $\uparrow \overline{Apr}(X) = \overline{Apr}(\uparrow X) = \uparrow \overline{Apr}(\uparrow X)$. By (2), $\mu(\overline{Apr}(\uparrow X)) \subseteq \downarrow \mu(\overline{Apr}(X))$.

(5) Since $\overline{\theta}(\mu)(\overline{Apr}(\downarrow X)) = \overline{\theta}(\mu)(\downarrow X)$, we can then conclude from (1) that $\overline{\theta}(\mu)(\overline{Apr}(\downarrow X)) \subseteq \uparrow \overline{\theta}(\mu)(X)$. □

Proposition 3.4. *Let θ be a hyper congruence relation on L and $x, y \in L$. If $\mu \in \mathcal{F}(L)$ is a hyper fuzzy ideal of L, then*

$$\bigvee_{a \in [x]_\theta, b \in [y]_\theta} \bigvee \mu(a \vee b) \leq \bigvee \overline{\theta}(\mu)(x \vee y).$$

Proof. By Lemma 3.2,

$$\begin{aligned} \bigvee_{a \in [x]_\theta, b \in [y]_\theta} \bigvee \mu(a \vee b) &\leq \bigvee_{z \in [x \vee y]_\theta} \mu(z) \\ &= \bigvee_{z \in \bigcup_{a \in x \vee y} [a]_\theta} \mu(z) \\ &= \bigvee_{a \in x \vee y} \bigvee_{z \in [a]_\theta} \mu(z) \\ &= \bigvee_{a \in x \vee y} \overline{\theta}(\mu)(a) \\ &= \bigvee \overline{\theta}(\mu)(x \vee y). \end{aligned}$$

□

Lemma 3.5. *Let θ be a hyper congruence relation on L and $x, y \in L$. If $\mu \in \mathcal{F}(L)$ is a hyper fuzzy filter of L, then $\overline{\theta}(\mu)(x \wedge y) = \bigvee_{a \in [x]_\theta, b \in [y]_\theta} \mu(a \wedge b)$.*

Proof. By Lemma 3.2, $\bigvee_{a \in [x]_\theta, b \in [y]_\theta} \mu(a \wedge b) \leq \bigvee_{z \in [x \wedge y]_\theta} \mu(z) = \overline{\theta}(\mu)(x \wedge y)$. Now, assume that $z \in [x \wedge y]_\theta$. By Lemma 3.2, $(x \vee z) \times \{x\} \subseteq (x \vee z) \times (x \vee (x \wedge y)) \subseteq \theta$ and $(y \vee z) \times \{y\} \subseteq (y \vee z) \times (y \vee (x \wedge y)) \subseteq \theta$. Also, by Proposition 2.2, there exist $z_x \in x \vee z$ and $z_y \in y \vee z$ such that $z \leq z_x$ and $z \leq z_y$. Since $z \leq z_x \wedge z_y$ and μ is a hyper fuzzy filter of L, we conclude that $\mu(z) \leq \mu(z_x \wedge z_y)$. Therefore, $\overline{\theta}(\mu)(x \wedge y) = \bigvee_{z \in [x \wedge y]_\theta} \mu(z) \leq \bigvee_{a \in [x]_\theta, b \in [y]_\theta} \mu(a \wedge b)$ and the proof is now complete. □

Definition 3.6. Let θ be an equivalence relation on L and $\mu \in \mathcal{F}(L)$. Then, μ is called an *upper rough fuzzy* (*prime*) *filter* if $\overline{\theta}(\mu)$ is a hyper fuzzy (prime) filter of L.

Proposition 3.7. *Let θ be a hyper congruence relation on L. If $\mu \in \mathcal{F}(L)$ is a hyper fuzzy filter, then μ is an upper rough fuzzy filter.*

Proof. Let $x, y \in L$ and $x \leq y$. If $z \in [x]_\theta$, then $(z, x) \in \theta$ and $(z \vee y) \times (x \vee y) \subseteq \theta$. Since $y \in x \vee y$, we conclude that $(z \vee y) \times \{y\} \subseteq \theta$. Also, by Proposition 2.2, there exists $z_1 \in z \vee y$ such that $z \leq z_1$, and so $\mu(z) \leq \mu(z_1) \leq \bigvee \mu(z \vee y)$. Therefore,

$$\begin{aligned} \overline{\theta}(\mu)(x) &= \textstyle\bigvee_{z\in[x]_\theta} \mu(z) \\ &\leq \textstyle\bigvee_{z\in[x]_\theta} \bigvee \mu(z \vee y) \\ &\leq \textstyle\bigvee_{t\in[y]_\theta} \mu(t) \\ &= \overline{\theta}(\mu)(y). \end{aligned}$$

Now, If $x, y \in L$, then

$$\begin{aligned} \overline{\theta}(\mu)(x \wedge y) &= \textstyle\bigvee_{a\in[x]_\theta, b\in[y]_\theta} \mu(a \wedge b) && \text{by Lemma 3.5} \\ &\geq \textstyle\bigvee_{a\in[x]_\theta, b\in[y]_\theta} \mu(a) \wedge \mu(b) && \mu \text{ is a hyper fuzzy filter} \\ &= \textstyle\bigvee_{a\in[x]_\theta} \mu(a) \wedge \bigvee_{b\in[y]_\theta} \mu(b) \\ &= \overline{\theta}(\mu)(x) \wedge \overline{\theta}(\mu)(y). \end{aligned}$$

Hence $\overline{\theta}(\mu)$ is a hyper fuzzy filter. □

Example 3.8. Let $L = \{0, a, b, c, d, 1\}$ and define $\wedge$ and $\vee$ by the following Cayley tables:

$\wedge$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	a	a	a	a
b	0	a	b	a	b	b
c	0	a	a	c	c	c
d	0	a	b	c	d	d
1	0	a	b	c	d	1

$\vee$	0	a	b	c	d	1
0	$\{0\}$	$\{a\}$	$\{b\}$	$\{c\}$	$\{d\}$	$\{1\}$
a	$\{a\}$	$\{a\}$	$\{b\}$	$\{c\}$	$\{d\}$	$\{1\}$
b	$\{b\}$	$\{b\}$	$\{b\}$	$\{d\}$	$\{d\}$	$\{1\}$
c	$\{c\}$	$\{c\}$	$\{d\}$	$\{c\}$	$\{d\}$	$\{1\}$
d	$\{d\}$	$\{d\}$	$\{d\}$	$\{d\}$	$\{d\}$	$\{1\}$
1	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$	$\{d, 1\}$

It is easy to see that the operations $\wedge$ and $\vee$ on L are well-defined and L is a hyperlattice. Let θ be an equivalence relation on the lattice L with the following equivalence classes: $[0]_\theta = \{0, a, b\}$; $[d]_\theta = \{d\}$; $[c]_\theta = \{c\}$; $[1]_\theta = \{1\}$. It is clear that θ is not a hyper congruence relation on the lattice L. If

$$\mu = \begin{pmatrix} 0 & a & b & c & d & 1 \\ 0.1 & 0.2 & 0.7 & 0.2 & 0.7 & 0.9 \end{pmatrix},$$

then μ is a hyper fuzzy filter and

$$\overline{\theta}(\mu) = \begin{pmatrix} 0 & a & b & c & d & 1 \\ 0.7 & 0.7 & 0.7 & 0.2 & 0.7 & 0.9 \end{pmatrix}.$$

Since $0 \leq c$ and $\overline{\theta}(\mu)(0) = 0.7 \not\leq 0.2 = \overline{\theta}(\mu)(c)$, we conclude that $\overline{\theta}(\mu)$ is not a hyper fuzzy filter.

Example 3.9. Let the hyperlattice L be as in example 3.8. Let θ be an equivalence relation on the lattice L with the following equivalence classes: $[0]_\theta = \{0\}$ and $[1]_\theta = \{a, b, c, d, 1\}$. It is clear that θ is a hyper congruence relation on the lattice L.

Lemma 3.10. *Let θ be a hyper congruence relation on L and $x, y \in L$.Then*

1. *If $a \in [x]_\theta$ and $b \in [y]_\theta$, then $[\alpha]_\theta = [\beta]_\theta$ for every $\alpha \in [x \vee y]_\theta$ and $\beta \in [a \vee b]_\theta$.*

2. *If μ is a fuzzy subset of L, then $\bigvee_{a\in[x]_\theta, b\in[y]_\theta} \bigwedge_{z\in a\vee b} \mu(z) \leq \bigwedge_{z\in x\vee y} \overline{\theta}(\mu)(z)$.*

Proof. (1) Let $\alpha \in [x \vee y]_\theta$ and $\beta \in [a \vee b]_\theta$. Then, by Lemma 3.2, $(\alpha, \beta) \in (a \vee b) \times (x \vee y) \subseteq \theta$, it follows that $[\alpha]_\theta = [\beta]_\theta$.

(2) By statement (1), we have $\mu(z) \leq \bigvee_{d\in[z]_\theta} \mu(d) = \bigvee_{d\in[z']_\theta} \mu(d)$ for every $z \in a \vee b$ and $z^{'} \in x \vee y$. Hence $\mu(z) \leq \bigwedge_{z'\in x\vee y} \bigvee_{d\in[z']_\theta} \mu(d)$ for every $z \in a \vee b$. Therefore $\bigwedge_{z\in a\vee b} \mu(z) \leq \bigwedge_{z'\in x\vee y} \bigvee_{d\in[z']_\theta} \mu(d)$, which follows that $\bigvee_{a\in[x]_\theta, b\in[y]_\theta} \bigwedge_{z\in a\vee b} \mu(z) \leq \bigwedge_{z\in x\vee y} \overline{\theta}(\mu)(z)$. □

Lemma 3.11. *Let θ be a hyper congruence relation on L and $x, y \in L$. If μ is a hyper fuzzy ideal of L and $x \leq y$, then $\overline{\theta}(\mu)(x) = \bigvee_{a\in[x]_\theta, b\in[y]_\theta} \mu(a \wedge b)$.*

Proof. It is clear that $\{a \wedge b : a \in [x]_\theta, b \in [y]_\theta\} \subseteq [x]_\theta$. Therefore,

$$\bigvee_{a\in[x]_\theta, b\in[y]_\theta} \mu(a \wedge b) \leq \bigvee_{a\in[x]_\theta} \mu(a).$$

Now, we suppose that $a \in [x]_\theta$. Then $a \wedge y \in [x]_\theta$ and since μ is a hyper fuzzy ideal of L, we conclude that $\mu(a) \leq \mu(a \wedge y)$. Therefore

$$\begin{aligned} \overline{\theta}(\mu)(x) &= \textstyle\bigvee_{a\in[x]_\theta} \mu(a) \\ &\leq \textstyle\bigvee_{a\in[x]_\theta} \mu(a \wedge y) \\ &\leq \textstyle\bigvee_{a\in[x]_\theta, b\in[y]_\theta} \mu(a \wedge b). \end{aligned}$$

Hence $\overline{\theta}(\mu)(x) = \bigvee_{a\in[x]_\theta, b\in[y]_\theta} \mu(a \wedge b)$. □

Definition 3.12. Let θ be an equivalence relation on L and $\mu \in \mathcal{F}(L)$. Then, μ is called an *upper rough fuzzy* (*prime*) *ideal* if $\overline{\theta}(\mu)$ is a hyper fuzzy (prime) ideal of L.

Proposition 3.13. *Let θ be a hyper congruence relation on L. If μ is a hyper fuzzy ideal of L, then μ is an upper rough fuzzy ideal.*

Proof. Let $x, y \in L$. Then

$$\begin{array}{rcll} \overline{\theta}(\mu)(x) \wedge \overline{\theta}(\mu)(y) & = & \bigvee_{a\in[x]_\theta} \mu(a) \wedge \bigvee_{b\in[y]_\theta} \mu(b) & \\ & = & \bigvee_{a\in[x]_\theta, b\in[y]_\theta} \mu(a) \wedge \mu(b) & \\ & \leq & \bigvee_{a\in[x]_\theta, b\in[y]_\theta} \bigwedge_{z\in a\vee b} \mu(z) & \mu \text{ is a hyper fuzzy ideal of } L \\ & \leq & \bigwedge_{z\in x\vee y} \overline{\theta}(\mu)(z) & \text{by Lemma 3.10.} \end{array}$$

Now, we suppose that $x, y \in L$ and $x \leq y$. Hence

$$\begin{array}{rcll} \overline{\theta}(\mu)(x) & = & \bigvee_{a\in[x]_\theta, b\in[y]_\theta} \mu(a \wedge b) & \text{by Lemma 3.11} \\ & \geq & \bigvee_{a\in[x]_\theta, b\in[y]_\theta} \mu(b) & \mu \text{ is a hyper fuzzy ideal of } L \\ & = & \overline{\theta}(\mu)(y). & \end{array}$$

□

Example 3.14. Let the hyperlattice L and the equivalence relation θ on L be as in example 3.8. If

$$\mu = \begin{pmatrix} 0 & a & b & c & d & 1 \\ 0.3 & 0.3 & 0.2 & 0.3 & 0.2 & 0.1 \end{pmatrix},$$

then μ is a hyper fuzzy ideal and

$$\overline{\theta}(\mu) = \begin{pmatrix} 0 & a & b & c & d & 1 \\ 0.3 & 0.3 & 0.3 & 0.3 & 0.2 & 0.1 \end{pmatrix}.$$

Since $\bigwedge_{x\in b\vee c} \overline{\theta}(\mu)(x) = 0.2 \ngeq 0.3 = \overline{\theta}(\mu)(b) \wedge \overline{\theta}(\mu)(c)$, we conclude that $\overline{\theta}(\mu)$ is not a hyper fuzzy ideal.

Definition 3.15. Let θ be a hyper congruence relation on L. Then θ is called $\vee$-*complete* if $[a \vee b]_\theta = [a]_\theta \vee [b]_\theta$ for all $a, b \in L$. Likewise, θ is called $\wedge-$*complete* if $[a \wedge b]_\theta = [a]_\theta \wedge [b]_\theta$ for all $a, b \in L$. A hyper congruence relation on L which is both $\vee$-complete and $\wedge-$complete is called *complete*.

Proposition 3.16. *Let θ be a $\wedge-$complete on L. If $\mu \in \mathcal{F}(L)$ is a hyper fuzzy prime ideal of L such that $\overline{\theta}(\mu)$ is a proper fuzzy subset of L, then μ is an upper rough fuzzy prime ideal.*

Proof. If $x, y \in L$, then

$$\begin{array}{rcll} \overline{\theta}(\mu)(x \wedge y) & = & \bigvee_{a \in [x \wedge y]_\theta} \mu(a) & \\ & = & \bigvee_{a \in [x]_\theta, b \in [y]_\theta} \mu(a \wedge b) & \theta \text{ is } \wedge\text{-complete} \\ & \leq & \bigvee_{a \in [x]_\theta, b \in [y]_\theta} \mu(a) \vee \mu(b) & \mu \text{ is a hyper fuzzy prime ideal} \\ & = & \bigvee_{a \in [x]_\theta} \mu(a) \vee \bigvee_{b \in [y]_\theta} \mu(b) & \\ & = & \overline{\theta}(\mu)(x) \vee \overline{\theta}(\mu)(y). & \end{array}$$

Now, by Proposition 3.13, the proof is complete. □

Example 3.17. Let the lattice L be as in example 3.8. Let θ be a hyper congruence relation on the lattice L with the following equivalence classes: $[0]_\theta = \{0, a\}$; $[b]_\theta = \{b\}$; $[c]_\theta = \{c\}$; $[1]_\theta = \{1, d\}$. Since

$$[b \wedge c]_\theta = [a]_\theta = \{0, a\} \neq \{a\} = [b]_\theta \wedge [c]_\theta$$

we conclude that θ is not $\wedge-$ complete. If

$$\mu = \begin{pmatrix} 0 & a & b & c & d & 1 \\ 0.9 & 0.8 & 0.8 & 0.7 & 0.7 & 0.2 \end{pmatrix},$$

then μ is a hyper fuzzy prime ideal and

$$\overline{\theta}(\mu) = \begin{pmatrix} 0 & a & b & c & d & 1 \\ 0.9 & 0.9 & 0.8 & 0.7 & 0.7 & 0.7 \end{pmatrix}.$$

Also, the ideal $\overline{\theta}(\mu)$ is not hyper fuzzy prime, because

$$\overline{\theta}(\mu)(b \wedge c) = 0.9 \nleq 0.8 = \overline{\theta}(\mu)(b) \vee \overline{\theta}(\mu)(c).$$

Definition 3.18. Let μ be a fuzzy subset of L. The least hyper fuzzy ideal of L containing μ is called a hyper fuzzy ideal of L *induced* by μ and is denoted by $< \mu >$.

By Remark 2.6 in [12], if μ is a fuzzy subset of L, then there exits $< \mu >$.

Definition 3.19. For every $\mu \in \mathcal{F}(L)$, we define

$$\mu^*(x) = sup\{\alpha \in [0, 1] : x \in I(\mu_\alpha)\}$$

for all $x \in L$.

With the following example, we prove that Theorem 2.15 in [12] is incorrect.

Example 3.20. Let the hyperlattice L be as in example 3.8. If

$$\mu = \begin{pmatrix} 0 & a & b & c & d & 1 \\ 0.5 & 0.8 & 0.4 & 0.5 & 0.7 & 0.6 \end{pmatrix},$$

then $\mu \in \mathcal{F}(L)$ and

$$\mu^* = \begin{pmatrix} 0 & a & b & c & d & 1 \\ 1 & 0.8 & 0.7 & 0.7 & 0.7 & 0.6 \end{pmatrix}.$$

If

$$\nu = \begin{pmatrix} 0 & a & b & c & d & 1 \\ 0.9 & 0.8 & 0.7 & 0.7 & 0.7 & 0.6 \end{pmatrix},$$

then ν is the hyper fuzzy ideal of L, $\mu \leq \nu$ and $\mu^* \not\leq \nu$. Therefore, μ^* is not the hyper fuzzy ideal induced by μ. Hence, Theorem 2.15 in [12] is incorrect.

Also, if $\lambda_n = 0.7 - \frac{1}{n}$, for every $n \in \mathbb{N}$, then $b \in \bigcap_{n\in\mathbb{N}} I(\mu_{\lambda_n}) =\downarrow d$, but $b \notin \mu_{\lambda_n}$ for every $n \geq 10$. Hence the last paragraph of the proof of Theorem 2.15 in [12] is incorrect.

Now we give the correct version of Theorem 2.15 in [12].

Proposition 3.21. *Let μ be a fuzzy subset of L. Then the fuzzy subset μ^* of L is the hyper fuzzy ideal of L and*

1. $\mu \leq \mu^*$.
2. $\mu^* = \bigwedge\{\lambda \in \mathcal{FI}(L) | \mu \leq \lambda \text{ and } \lambda(0) = 1\}$.

Proof. For $\lambda \in Im(\mu^*)$, let $\lambda_n = \lambda - \frac{1}{n}$, for $n \in \mathbb{N}$, and let $x \in \mu^*_\lambda$. Then $\mu^*(x) \geq \lambda$, which implies that $\mu^*(x) > \lambda_n$. Hence there exists $\beta \in \{\alpha \in [0,1] | x \in \mathrm{I}(\mu_\alpha)\}$ such that $\beta > \lambda_n$. Thus $\mu_\beta \subseteq \mu_{\lambda_n}$ and so $x \in \mathrm{I}(\mu_\beta) \subseteq \mathrm{I}(\mu_{\lambda_n})$ for all $n \in \mathbb{N}$. Therefore, $x \in \bigcap_{n\in\mathbb{N}} \mathrm{I}(\mu_{\lambda_n})$. Conversely, if $x \in \bigcap_{n\in\mathbb{N}} \mathrm{I}(\mu_{\lambda_n})$, $\lambda_n \in \{\alpha \in [0,1] : x \in \mathrm{I}(\mu_\alpha)\}$, for $n \in \mathbb{N}$. Therefore, $\lambda_n = \lambda - \frac{1}{n} \leq \bigvee\{\alpha \in [0,1] : x \in \mathrm{I}(\mu_\alpha)\} = \mu^*(x)$. Hence $\mu^*(x) \geq \lambda$, so that $x \in \mu^*_\lambda$. Then we have $\mu^*_\lambda = \bigcap_{n\in\mathbb{N}} \mathrm{I}(\mu_{\lambda_n})$ which is an ideal of L.

For $x \in L$, let $\beta \in \{\alpha \in [0,1] : x \in \mu_\alpha\}$. Then $x \in \mu_\beta$, and so $x \in \mathrm{I}(\mu_\beta)$. Thus $\beta \in \{\alpha \in [0,1] : x \in \mathrm{I}(\mu_\alpha)\}$, which implies that $\mu(x) = \bigvee\{\alpha \in [0,1] : x \in \mu_\alpha\} \leq \bigvee\{\alpha \in [0,1] : x \in \mathrm{I}(\mu_\alpha)\} = \mu^*(x)$. Therefore, $\mu \leq \mu^*$ (see [11, 12]).

Now, let ν be a hyper fuzzy ideal of L containing μ such that $\nu(0) = 1$. Then for every $\alpha \in [0,1]$, since $\nu_\alpha \neq \emptyset$, we conclude that $I(\mu_\alpha) \leq I(\nu_\alpha) = \nu_\alpha$. Hence

$$\mu^*(x) = \bigvee\{\alpha \in [0,1] : x \in I(\mu_\alpha)\} \leq \bigvee\{\alpha \in [0,1] : x \in \nu_\alpha\} = \nu(x)$$

for every $x \in L$. Also, for every $\alpha \in [0,1]$, $0 \in \mathrm{I}(\mu_\alpha)$ and we infer that $\mu^*(0) = \bigvee\{\alpha \in [0,1] : 0 \in \mathrm{I}(\mu_\alpha)\} = 1$. Therefore, $\mu^* \in \{\lambda \in \mathcal{FI}(L) | \mu \leq \lambda \; and \; \lambda(0) = 1\}$. Finally, we have

$$\mu^* = \bigwedge\{\lambda \in \mathcal{FI}(L) | \mu \leq \lambda \; and \; \lambda(0) = 1\}.$$

□

Proposition 3.22. *Let θ be a hyper congruence relation on L and $\mu \in \mathcal{F}(L)$. Then $\overline{\theta}(<\mu>) = \overline{\theta}(<\overline{\theta}(\mu)>)$ and $\overline{\theta}(\mu^*) = \overline{\theta}((\overline{\theta}(\mu))^*)$.*

Proof. Since $\mu \leq <\mu> \leq \mu^*$, we conclude from Proposition 2.8 that

$$\overline{\theta}(\mu) \leq \overline{\theta}(<\mu>) \leq \overline{\theta}(\mu^*).$$

It is clear that $\overline{\theta}(\mu^*)(0) = 1$ and by Propositions 3.13 and 3.21, we have

$$<\overline{\theta}(\mu)> \leq \overline{\theta}(<\mu>) \text{ and } (\overline{\theta}(\mu))^* \leq \overline{\theta}(\mu^*).$$

Again, by Proposition 2.8,

$$\overline{\theta}(<\overline{\theta}(\mu)>) \leq \overline{\theta}(<\mu>) \text{ and } \overline{\theta}((\overline{\theta}(\mu))^*) \leq \overline{\theta}(\mu^*).$$

Since $\mu \leq \overline{\theta}(\mu)$, we conclude that

$$<\mu> \leq <\overline{\theta}(\mu)> \text{ and } \mu^* \leq (\overline{\theta}(\mu))^*$$

and by Proposition 2.8,

$$\overline{\theta}(<\mu>) \leq \overline{\theta}(<\overline{\theta}(\mu)>) \text{ and } \overline{\theta}(\mu^*) \leq \overline{\theta}((\overline{\theta}(\mu))^*).$$

Finally, we have

$$\overline{\theta}(<\mu>) = \overline{\theta}(<\overline{\theta}(\mu)>) \text{ and } \overline{\theta}(\mu^*) = \overline{\theta}((\overline{\theta}(\mu))^*).$$

□

By the following example, we prove that the condition for an equivalence relation on L does not imply $\overline{\theta}((\overline{\theta}(\mu))^*) = \overline{\theta}(\mu^*)$.

Example 3.23. Let the hyperlattice L and the equivalence relation θ on L be as in example 3.8. If

$$\mu = \begin{pmatrix} 0 & a & b & c & d & 1 \\ 0.5 & 0.8 & 0.4 & 0.7 & 0.5 & 0.6 \end{pmatrix},$$

then

$$\overline{\theta}(\mu^*) = \begin{pmatrix} 0 & a & b & c & d & 1 \\ 1 & 1 & 1 & 0.7 & 0.6 & 0.6 \end{pmatrix},$$

and

$$\overline{\theta}((\overline{\theta}(\mu))^{*}) = \begin{pmatrix} 0 & a & b & c & d & 1 \\ 1 & 1 & 1 & 0.7 & 0.7 & 0.6 \end{pmatrix}.$$

Hence $\overline{\theta}(\mu^{*}) \neq \overline{\theta}((\overline{\theta}(\mu))^{*})$. Therefore, in general $\overline{\theta}(\mu^{*}) = \overline{\theta}((\overline{\theta}(\mu))^{*})$ doesn't hold.

4 Lower approximations of a fuzzy subset

In this section we give some important properties of $\underline{\theta}$ with many examples.

Lemma 4.1. *Let θ be a hyper congruence relation on L and $x, y \in L$. If μ is a hyper fuzzy filter of L and $x \leq y$, then $\underline{\theta}(\mu)(x) = \bigwedge_{a\in[x]_\theta, b\in[y]_\theta} \mu(a \wedge b)$.*

Proof. Since $x \leq y$, we can then conclude from Lemma 3.2 that $\{\mu(a \wedge b) : a \in [x]_\theta, b \in [y]_\theta\} \subseteq \{\mu(z) : z \in [x]_\theta\}$. Hence $\underline{\theta}(\mu)(x) \leq \bigwedge_{a\in[x]_\theta, b\in[y]_\theta} \mu(a\wedge b)$. Also, since μ is a hyper fuzzy filter of L, we conclude that $\mu(a \wedge b) \leq \mu(a)$ for every $a \in [x]_\theta$ and $b \in [y]_\theta$, which follows that $\underline{\theta}(\mu)(x) \geq \bigwedge_{a\in[x]_\theta, b\in[y]_\theta} \mu(a \wedge b)$ and the proof is now complete. $\square$

Definition 4.2. Let θ be an equivalence relation on L and $\mu \in \mathcal{F}(L)$. Then, μ is called a *lower rough fuzzy* (*prime*) *ideal* if $\underline{\theta}(\mu)$ is a hyper fuzzy (prime) ideal of L.

Proposition 4.3. *Let θ be a $\vee$-complete congruence relation on L. If $\mu \in \mathcal{F}(L)$ is a hyper fuzzy ideal, then μ is a lower rough fuzzy ideal.*

Proof. Let $x, y \in L$. Since $\bigwedge_{t\in x\vee y} \mu(t) \in \{\bigwedge_{t\in a\vee b} \mu(t) : a \in [x]_\theta, b \in [y]_\theta\}$, we conclude that

$$\begin{array}{rcll} \underline{\theta}(\mu)(x) \wedge \underline{\theta}(\mu)(y) & = & \bigwedge_{a\in[x]_\theta} \mu(a) \wedge \bigwedge_{b\in[y]_\theta} \mu(b) & \\ & = & \bigwedge_{a\in[x]_\theta, b\in[y]_\theta} \mu(a) \wedge \mu(b) & \\ & \leq & \bigwedge_{a\in[x]_\theta, b\in[y]_\theta} \bigwedge_{t\in a\vee b} \mu(t) & \mu \text{ is a hyper fuzzy ideal} \\ & = & \bigwedge_{t\in[x]_\theta\vee[y]_\theta} \mu(t) & \\ & = & \bigwedge_{t\in[x\vee y]_\theta} \mu(t) & \theta \text{ is } \vee\text{-complete} \\ & = & \bigwedge_{z\in x\vee y} \bigwedge_{t\in[z]_\theta} \mu(t) & \\ & = & \bigwedge_{z\in x\vee y} \underline{\theta}(\mu)(z). & \end{array}$$

Let $x, y \in L$ and $x \leq y$. Hence

$$\begin{array}{rcll} \underline{\theta}(\mu)(x) & = & \bigwedge_{a\in[x]_\theta, b\in[y]_\theta} \mu(a \wedge b) & \text{by Lemma 4.1} \\ & \geq & \bigwedge_{b\in[y]_\theta} \mu(b) & \mu \text{ is a hyper fuzzy ideal} \\ & = & \underline{\theta}(\mu)(y). & \end{array}$$

$\square$

Example 4.4. Let the hyperlattice L and the hyper congruence relation θ on L be as in example 3.17. Let

$$\mu = \begin{pmatrix} 0 & a & b & c & d & 1 \\ 1 & 0.8 & 0.6 & 0.4 & 0.4 & 0 \end{pmatrix}.$$

It is clear that μ is a hyper fuzzy ideal on L and

$$\underline{\theta}(\mu) = \begin{pmatrix} 0 & a & b & c & d & 1 \\ 0.8 & 0.8 & 0.6 & 0.4 & 0 & 0 \end{pmatrix}.$$

It is easy to see that θ is not $\vee$ - complete, because

$$[b \vee c]_\theta = \{1.d\} \neq \{d\} = [b]_\theta \vee [c]_\theta.$$

Also, since

$$\underline{\theta}(\mu)(b) \wedge \underline{\theta}(\mu)(c) = 0.4 \not\leq 0 = \bigwedge_{d \in b \vee c} \underline{\theta}(\mu)(d)$$

we conclude that $\underline{\theta}(\mu)$ is not a hyper fuzzy ideal.

Definition 4.5. Let θ be an equivalence relation on L and $\mu \in \mathcal{F}(L)$. Then, μ is called a *lower rough fuzzy (prime) filter* if $\underline{\theta}(\mu)$ is a hyper fuzzy (prime) filter of L.

Proposition 4.6. *Let θ be a $\wedge-$complete congruence relation on L. If $\mu \in \mathcal{F}(L)$ is a hyper fuzzy filter, then μ is a lower rough fuzzy filter.*

Proof. Let $x, y \in L$.

$$\begin{array}{rcll} \underline{\theta}(\mu)(x \wedge y) & = & \bigwedge_{a \in [x \wedge y]_\theta} \mu(a) & \\ & = & \bigwedge_{a \in [x]_\theta, b \in [y]_\theta} \mu(a \wedge b) & \theta \text{ is } \wedge-\text{complete} \\ & \geq & \bigwedge_{a \in [x]_\theta, b \in [y]_\theta} \mu(a) \wedge \mu(b) & \mu \text{ is a hyper fuzzy filter} \\ & = & \bigwedge_{a \in [x]_\theta} \mu(a) \wedge \bigwedge_{b \in [y]_\theta} \mu(b) & \\ & = & \underline{\theta}(\mu)(x) \wedge \underline{\theta}(\mu)(y). & \end{array}$$

Let $x, y \in L$ and $x \leq y$. Hence

$$\begin{array}{rcll} \underline{\theta}(\mu)(x) & = & \bigwedge_{a \in [x]_\theta, b \in [y]_\theta} \mu(a \wedge b) & \text{by Lemma 4.1} \\ & \leq & \bigwedge_{b \in [y]_\theta} \mu(b) & \mu \text{ is a hyper fuzzy filter} \\ & = & \underline{\theta}(\mu)(y). & \end{array}$$

□

Example 4.7. Let the hyperlattice L and the hyper congruence relation θ on L be as in example 3.17. If

$$\mu = \begin{pmatrix} 0 & a & b & c & d & 1 \\ 0.1 & 0.6 & 0.6 & 0.7 & 0.7 & 0.9 \end{pmatrix},$$

then μ is a hyper fuzzy filter and

$$\underline{\theta}(\mu) = \begin{pmatrix} 0 & a & b & c & d & 1 \\ 0.1 & 0.1 & 0.6 & 0.7 & 0.7 & 0.7 \end{pmatrix}.$$

Since

$$[b \wedge c]_\theta = \{0, a\} \neq \{a\} = [b]_\theta \wedge [c]_\theta,$$

we conclude that θ is not $\wedge-$ complete. Also $\underline{\theta}(\nu)$ is not a hyper fuzzy filter, because

$$\underline{\theta}(\nu)(b \wedge c) = \underline{\theta}(\nu)(a) = 0.1 \not\geq 0.6 = \underline{\theta}(\nu)(b) \wedge \underline{\theta}(\nu)(c).$$

Proposition 4.8. *Let θ be a $\vee$-complete on L. If $\mu \in \mathcal{F}(L)$ is a hyper fuzzy prime ideal such that $\underline{\theta}(\mu)$ is a proper fuzzy subset of L, then μ is a lower rough fuzzy prime ideal.*

Proof. Let $x, y \in L$.

$$\begin{array}{rcll} \underline{\theta}(\mu)(x \wedge y) & = & \bigwedge_{z \in [x \wedge y]_\theta} \mu(z) & \\ & \leq & \bigwedge_{a \in [x]_\theta, b \in [y]_\theta} \mu(a \wedge b) & \text{by Lemma 3.2} \\ & \leq & \bigwedge_{a \in [x]_\theta, b \in [y]_\theta} \mu(a) \vee \mu(b) & \mu \text{ is a hyper fuzzy prime ideal} \\ & = & \bigwedge_{a \in [x]_\theta} \mu(a) \vee \bigwedge_{b \in [y]_\theta} \mu(b) & \\ & = & \underline{\theta}(\mu)(x) \vee \underline{\theta}(\mu)(y). & \end{array}$$

By Proposition 4.3, the proof is now complete. □

Example 4.9. Let the lattice L and the hyper congruence relation θ on L be as in example 3.17. If

$$\mu = \begin{pmatrix} 0 & a & b & c & d & 1 \\ 0.9 & 0.8 & 0.8 & 0.7 & 0.7 & 0.2 \end{pmatrix},$$

then μ is a hyper fuzzy prime ideal and

$$\underline{\theta}(\mu) = \begin{pmatrix} 0 & a & b & c & d & 1 \\ 0.8 & 0.8 & 0.8 & 0.7 & 0.2 & 0.2 \end{pmatrix}.$$

Since

$$[b \vee c]_\theta = \{1.d\} \neq \{d\} = [b]_\theta \vee [c]_\theta,$$

we conclude that θ is not $\vee$ - complete. Also $\underline{\theta}(\mu)$ is not hyper fuzzy ideal, because

$$\underline{\theta}(\mu)(b) \wedge \underline{\theta}(\mu)(c) = 0.7 \not\leq 0.2 = \bigwedge_{d \in (b \vee c)} \underline{\theta}(\mu)(d).$$

Proposition 4.10. *Let θ be a complete congruence relation on L. If $\mu \in \mathcal{F}(L)$ is a hyper fuzzy prime filter such that $\underline{\theta}(\mu)$ is a proper fuzzy subset of L, then μ is a lower rough fuzzy prime filter.*

Proof. Let $x, y \in L$.

$$\begin{array}{rcll} \underline{\theta}(\mu)(x) \vee \underline{\theta}(\mu)(y) & = & \bigwedge_{a\in[x]_\theta} \mu(a) \vee \bigwedge_{b\in[y]_\theta} \mu(b) & \\ & = & \bigwedge_{a\in[x]_\theta, b\in[y]_\theta} \mu(a) \vee \mu(b) & \\ & \geq & \bigwedge_{a\in[x]_\theta, b\in[y]_\theta} \bigwedge_{t\in a\vee b} \mu(t) & \mu \text{ is a hyper fuzzy prime filter} \\ & = & \bigwedge_{z\in x\vee y} \bigwedge_{t\in[z]_\theta} \mu(t) & \theta \text{ is } \vee\text{-complete} \\ & = & \bigwedge_{z\in x\vee y} \underline{\theta}(\mu)(z). & \end{array}$$

By Proposition 4.6, the proof is now complete. □

Example 4.11. Let the hyperlattice L and the hyper congruence relation θ on L be as in example 3.17. It is clear that

$$\mu = \begin{pmatrix} 0 & a & b & c & d & 1 \\ 0.1 & 0.3 & 0.7 & 0.3 & 0.7 & 0.9 \end{pmatrix}$$

is a hyper fuzzy prime filter and

$$\underline{\theta}(\mu) = \begin{pmatrix} 0 & a & b & c & d & 1 \\ 0.1 & 0.1 & 0.7 & 0.3 & 0.7 & 0.7 \end{pmatrix}.$$

Since

$$\underline{\theta}(\mu)(b) \wedge \underline{\theta}(\mu)(c) = 0.3 \not\leq 0.1 = \underline{\theta}(\mu)(b \wedge c),$$

we conclude that $\underline{\theta}(\mu)$ is not a hyper fuzzy ideal. Also, as we have seen in Example 4.4, θ is not $\vee$ - complete.

5 Conclusion

Rough set, fuzzy set and hyperlattice are different aspects of set theory. Combining the three theories, one gets the rough concept fuzzy hyperlattice of a given context. We introduced the concepts of upper and lower rough hyper fuzzy ideals (filters) in a hyperlattice and its basic properties have been discussed. Also, we discussed the relations between hyper fuzzy (prime) ideal and hyper fuzzy (prime) filter with their upper and lower approximations, respectively. In addition, by an example we show that Theorem 2.15 in [12] is incorrect (see Example 3.20) and a corrected version is considered, Proposition 3.21.

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