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Some applications of linear difference equations in finance with wolfram|alpha and maple

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Abstract

The principle objective of this paper is to show how linear difference equations can be applied to solve some issues of financial mathematics. We focus on the area of compound interest and annuities. In both cases we determine appropriate recursive rules, which constitute the first order linear difference equations with constant coefficients, and derive formulas required for calculating examples. Finally, we present possibilities of application of two selected computer algebra systems Wolfram|Alpha and Maple in this mathematical area.

Key words: linear difference equation, compound interest, future value of an annuity, periodic payment, computer algebra systems.

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1 Introduction

The values of most economic variables are given as a sequence of values observed at discrete time intervals or periods. These sequences are often specified by recursion with some initial elements. But it is preferable to know a rule in the form of an equation for the n-th element to calculate the values of sequence elements. The recursive rule of a sequence represents a difference equation and the functional notation for the n-th element can be obtained by solving this difference equation (see [7]).

Many formulas used in financial mathematics can be derived from the recursive rules between two consecutive elements which constitute difference equations of the first order. This includes for example simple and compound interest calculation, the present and future value of an annuity and loan amortization.

2 Compound Interest

2.1 Derivation of Formula

A sum of money deposited in a bank earns interest which is added to the principal at regular intervals and the new amount is used for calculating the interest for the next conversion period. We shall develop a formula for the total amount of money that is accumulated by a given principal after a certain number of conversion periods, see also [3], [4].

Let r stand for the annual interest rate and k denote the number of conversion periods in a year. Let n be equal to the number of conversion periods in the term of the deposit. Let y_n represent the amount on deposit at the end of n conversion periods and P the initial sum deposited (i.e. principal). We obtain the following recursive rule

$$y_{n+1} = y_n + \frac{r}{k}\,y_n = \left(1 + \frac{r}{k}\right) y_n, \quad n = 0, 1, 2, \ldots$$

with $y_0 = P$, where the fraction $\frac{r}{k}$ stands for the interest rate per conversion period and $\frac{r}{k}y_n$ is the interest generated during $(n+1)$st period. The previous formula represents the first order homogeneous linear difference equation with constant coefficients

$$y_{n+1} - \left(1 + \frac{r}{k}\right) y_n = 0 \tag{1}$$

with initial condition

$$y_0 = P. \tag{2}$$

The above problem (1), (2) can be solved by using the properties of a geometric sequence, see [1] and [8]. But our approach will be different due to the use of a difference equation. The characteristic equation of (1) takes form

$$z - \left(1 + \frac{r}{k}\right) = 0$$

with real root

$$z = 1 + \frac{r}{k}.$$

According to [5], the general solution is a geometric sequence

$$y_n = C\left(1+\frac{r}{k}\right)^n, \quad C \in \mathbb{R}.$$

A constant C can be specified from the initial condition (2) for period $n = 0$, hence

$$C = P.$$

Thus the general solution is given by

$$y_n = P\left(1+\frac{r}{k}\right)^n \tag{3}$$

and represents the compound interest formula. This formula gives the amount y_n into which principal P grows when it earns compound interest for n conversion periods at an interest rate of $\frac{r}{k}$ per conversion period.

2.2 Illustrative Examples

Example 2.1. *An amount of EUR* $1,000$ *is deposited into a savings account at an annual interest rate of* 2.5%*, compounded yearly. What will the value of the account be worth after 20 years?*

To find the amount we use formula (3). We have principal $P = 1,000$, annual interest rate $r = 0.025$, number of conversion periods per year $k = 1$ and total number of conversion periods $n = 20$. After plugging those figures into the formula, we get

$$y_{20} = 1000\left(1+\frac{0.025}{1}\right)^{20} \doteq 1638.62$$

Example 2.2. *Find the number of years required for a given sum of money to double itself if the interest rate is* 3%*, compounded quarterly.*

Substituting $y_n = 2P$ in the compound interest formula (3), we have

$$2P = P\left(1+\frac{r}{k}\right)^n$$

which implies

$$2 = \left(1+\frac{r}{k}\right)^n. \tag{4}$$

Taking natural logarithms on both sides and using properties of logarithms gives

$$n = \frac{\ln 2}{\ln\left(1+\frac{r}{k}\right)}.$$

To calculate the number of years N we have to divide the total number of conversion periods n by their number in a year k

$$N = \frac{1}{k}\,\frac{\ln 2}{\ln\left(1+\frac{r}{k}\right)}. \tag{5}$$

Setting $r = 0.03$, $k = 4$ we get the required number of years

$$N = \frac{1}{4}\,\frac{\ln 2}{\ln\left(1+\frac{0.03}{4}\right)} \doteq 23.19$$

3 Future Value of an Annuity

3.1 Derivation of Formula

An annuity is essentially a sequence of periodic payments, usually equal in amount, payable at equal intervals of time over the course of a fixed time period. The future value of an annuity is the total value of its periodic payments enhanced at interest rate for given number of conversion periods. It is defined as the sum of the amounts of all payments and the total compound interest earned on these payments to the time of the last payment. See for example [1], [4].

Suppose the constant sum R is deposited at the end of each conversion period in a bank which credits interest at the annual rate r. The deposits are made k times each year over n conversion periods. Let y_n denote the total amount in the account at the end of n conversion periods. We shall find the total worth of an annuity after n deposits.

The recursive rule for the future value of an annuity can be written as

$$y_{n+1} = y_n + \frac{r}{k}y_n + R = \left(1+\frac{r}{k}\right)y_n + R, \quad n = 0,1,2,\ldots$$

with $y_0 = 0$, where $\frac{r}{k}$ is the interest rate per conversion period.

This equation constitutes the first order nonhomogeneous linear difference equation with constant coefficients

$$y_{n+1} - \left(1+\frac{r}{k}\right)y_n = R \tag{6}$$

with initial condition

$$y_0 = 0. \tag{7}$$

In financial mathematics, the above problem (6), (7) is solved by using the properties of a geometric sequence. But we will proceed by means of difference equations like in the case of compound interest. To solve nonhomogeneous difference equation (6) we consider the corresponding homogeneous difference equation

$$y_{n+1} - \left(1 + \frac{r}{k}\right) y_n = 0 \tag{8}$$

which is the same as (1) in the case of compound interest. Hence the general solution $\bar{y}_n$ of this homogeneous difference equation is given by

$$\bar{y}_n = C\left(1 + \frac{r}{k}\right)^n, \quad C \in \mathbb{R}. \tag{9}$$

The right-hand side of the nonhomogeneous difference equation (6) is a constant R which is a polynomial of degree zero. Thus a particular solution Y_n can be estimated by

$$Y_n = b, \quad b \in \mathbb{R}.$$

For more details see [5]. Using the method of undetermined coefficients (see [2]) we substitute the above estimate into (6). We get

$$b - \left(1 + \frac{r}{k}\right) b = R$$

and solving for b we obtain

$$b = -R\,\frac{k}{r}.$$

Therefore the particular solution of (6) takes the form

$$Y_n = -R\,\frac{k}{r}. \tag{10}$$

Using (9), (10) according to the superposition principle (see [6]), the general solution of the nonhomogeneous linear difference equation (6) is the sum

$$y_n = Y_n + \bar{y}_n = -R\,\frac{k}{r} + C\left(1 + \frac{r}{k}\right)^n, \quad C \in \mathbb{R}.$$

A constant C can be specified from the initial condition (7) for period $n = 0$. Hence we obtain

$$C = R\,\frac{k}{r}.$$

Consequently, the general solution of (6) takes the form

$$y_n = -R\,\frac{k}{r} + R\,\frac{k}{r}\left(1 + \frac{r}{k}\right)^n$$

which can be written as

$$y_n = R\,\frac{\left(1+\frac{r}{k}\right)^n - 1}{\frac{r}{k}}. \tag{11}$$

The above relation represents the future value of an annuity formula which gives the amount of an annuity of n payments of R at the compound rate $\frac{r}{k}$ per conversion period under the assumption that the payment interval equals the conversion period. The future value of an annuity formula is used to calculate what value at a future date would be for a series of periodic payments.

In financial mathematics, it is common to use the following form of the formula (11) setting $i = \frac{r}{k}$ where i represents the interest rate per compounding interval (see [8] and [10])

$$y_n = R\,\frac{(1+i)^n - 1}{i}. \tag{12}$$

3.2 Illustrative Examples

Example 3.1. *Suppose EUR 500 is deposited at the end of every six-month period in a bank, whose annual rate is* 3.4%*, compounded semiannually. How much will this account be worth after 7 years?*

We get the solution using (11), where $R = 500$, $r = 0.034$, $k = 2$, $n = 14$. Then we obtain

$$y_{14} = 500\,\frac{\left(1+\frac{0.034}{2}\right)^{14} - 1}{\frac{0.034}{2}} \doteq 7828.64$$

Example 3.2. *Find the payment amount that you should deposit at the end of each month in a bank so that EUR* 35,000 *will be available after* 10 *years if the interest rate is* 1.6%*, compounded monthly after each deposit.*

Solving for R from the future value of an ordinary annuity formula (11) we get

$$R = y_n\,\frac{\frac{r}{k}}{\left(1+\frac{r}{k}\right)^n - 1}.$$

In our case we have $r = 0.016$, $k = 12$, $n = 120$, $y_{120} = 35,000$. Hence the monthly payment is calculated as follows

$$R = 35,000\,\frac{\frac{0.016}{12}}{\left(1+\frac{0.016}{12}\right)^{120} - 1} \doteq 269.149$$

4 Solving with Computer Algebra Systems

In mathematics of finance, Excel is commonly used for calculations. In this paper the quoted calculations of compound interest and annuity are completed by computational tool Wolfram|Alpha and mathematical software Maple, respectively.

4.1 Wolfram|Alpha

We will demonstrate the computation of compound interest (3) and Example 2.2 through the free online service Wolfram|Alpha, which is available via any web browser at http://wolframalpha.com. This tool provides mathematical computations based on software Mathematica and accepts completely free-form input, commands are specified by the name of operation in English.

To solve the difference equation (1) with the initial condition (2) we type both equations together separated by comma into an input field writing indexes in parentheses. The provided general solution (3) is shown in Figure 1.

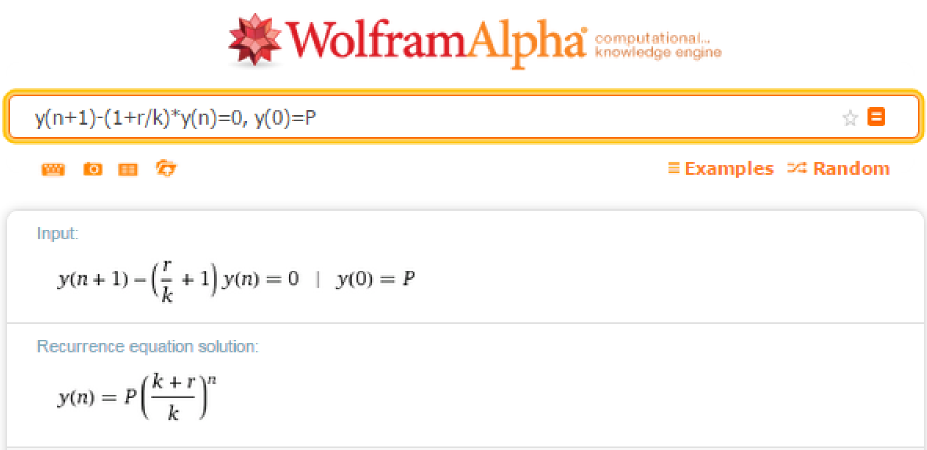

Fig. 1. Compound interest formula

Further, in Figure 2 you can see calculation of number of years from Example 2.2 using derived formula (5). We get the same result by solving equation (4) and using the command `solve` and the reserved word `for` as shown in Figure 3.

Fig. 2. Calculation of number of years by using derived formula

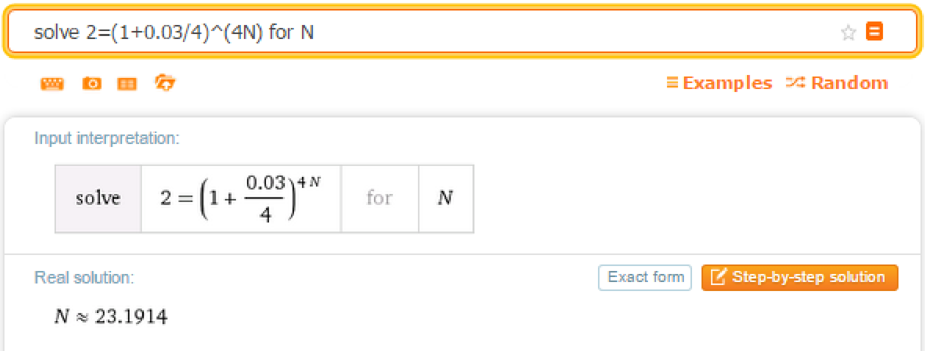

Fig. 3. Calculation of number of years by solving equation

4.2 Maple

Now we show the computation of the future value of an annuity (11) and illustrative Example 3.2.

We assign the recurrence relation (6) to the name `REq`.

```
> REq:=y(n+1)-y(n)*(1+r/k)=R;
```

Maple returns the output:

$$REq := y(n+1) - y(n)\left(1 + \frac{r}{k}\right) = R$$

Then we make the assignment of the initial condition (7) to the name `IC`.

```
> IC:=y(0)=0:
```

To solve the given difference equation we execute the command `rsolve` which solves among others the first order linear difference equations. A single recurrence relation and a boundary condition are the first argument, the second argument indicates the function that is solved for. Indexes are written in parentheses.

```
> rsolve({REq,IC},y(n));
```

$$\frac{Rk\left(\frac{k+r}{k}\right)^n}{r} - \frac{kR}{r}$$

The above obtained expression corresponds to the future value of an annuity (11).

For determining the payment R we type the following command, where FV equals to the total amount y_n in the account upon the last deposit (i.e the future value of an annuity).

```
> isolate(%=FV,R):simplify(%);
```

$$R = \frac{FV\,r}{k\left(\left(\frac{k+r}{k}\right)^n - 1\right)}$$

To make the calculation of Example 3.2 we use command `subs`.

```
> subs(k=12,r=0.016,FV=35000,n=120,%);
```

$$R = 269.1493510$$

4.3 Comparison of Used Systems

The professional Maple is very powerful tool which enables to make new procedures and modules, save and read them or together with other data store in a library. On the other hand, it requires certain programming skills.

In comparison with Maple, Wolfram|Alpha does not allow to save and reload the results of computations and make own procedures, also its performance is rather slow. But its significant advantage is that it is free online and very simple to use. Moreover, Wolfram|Alpha provides a variety of computations from other fields, for example from money and finance.

5 Conclusion

This paper has discussed linear difference equations and their applications in economics (see also [9]). These equations are frequently used especially in financial mathematics and some of their typical applications have been presented here.

Our main aim was to show relationship between some formulas of financial topics and mathematical knowledge which is required for their deriving. We have focused on derivation of the compound interest and the future value of an annuity formula by means of solution of difference equations. The simultaneous application of mathematical software has been demonstrated, the supplementary computations have been performed through Maple and Wolfram|Alpha.

Finally, the paper emphasizes the need for mathematics in economic subjects. The presented approach can be used in teaching of mathematics at economic universities and helps to provide students with the opportunities to apply their mathematics in relevant economics contexts.

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