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The views of primary education teachers on the verification of multiplication

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Abstract

Learning and using the four mathematical operations -addition, subtraction, multiplication and division- are very important in the primary school syllabus curriculum.
The verifications for the correctness of the operations are simple since they can be justified with the use of basic mathematical properties. However, this is not the case for one of the verifications of multiplication which seems to be preferred by most of the elementary school teachers in their practice. With this verification, the control of multiplication's correctness is only a necessary but not sufficient condition and it is based on the Numbers' Theory.
In this paper we present the findings of a study on the views of a group consisted of twenty four elementary school teachers using activities related to the operation of the multiplication and its verification.

Key words: multiplication and its verifications; elementary school teachers.

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1 Introduction

University education and continuous training of primary school teachers must include the essential scientific and technological knowledge which will enable them to contribute to education promotion taking into account the added value of their pedagogical role and practices. The scientific knowledge of the cognitive objects taught in elementary school and of the way that teaching is performed are essential prerequisites in order for the teachers to be successful in their work.

During their continuous education practicing teachers can face a variety of issues such as completion of their basic education, introduction of new methods of teaching or even reforms of the educational system.

In the framework of the continuous education in Mathematics of a team consisted of twenty four primary school teachers, we introduced a curriculum which included three hours of teaching multiplication and its verifications.

In the beginning we analyzed the conceptual field of multiplicative structures.

As it is well known the theory of conceptual fields according to Vergnaud [1], has two aims: "to describe and analyze the progressive complexity of competences that students develop in Mathematics inside and outside the school and to establish better connections between the operational form of knowledge, which consists in action in the physical and social world, and the predicative form of knowledge, which consists in the linguistic and symbolic expressions of this knowledge."

The conceptual field is "a set of problems and situations for the treatment of which concepts, procedures, and representations of different but narrowly interconnected types are necessary" [2]. The multiplication structures are "a conceptual field of multiplicative type, as a system of different but interrelated concepts, operations, and problems such as multiplication, division, fractions, ratios, similarity" [2] . "A single concept does not refer to only one type of situation and a single situation cannot be analyzed with only one concept" [3]. In addition, conceptual field is "a set of situations, the mastering of which requires mastery of several concepts of different natures" [3]. "Concepts-in-action serve to categorize and select information whereas theorems-in-action serve to infer appropriate goals and rules from the available and relevant information"[4].

Even the most complicated concepts, in order to be meaningful and functional should be placed in a framework and be explained via examples. Thus a concept is simultaneously a set of situations, a set of operational constants and a set of linguistic and symbolic representations. The use of the framework of conceptual fields is necessary for the analysis of the continuities and the discontinuities of development in Mathematics and for the invention of situations that will prompt and help students to move along the multifaceted complexity of conceptual field" [1] .

The multiplicative structures constitute a part of the field of the additive structures, if multiplication is considered as repeated addition. However, due to the fact that multiplication has its own internal structure and organization, it can also be considered as an independent operation.

The approach of multiplication as a repeated addition has a lot of limitations. Some of them are that "it does not easily generalize to rationals, it does not demonstrate commutativity and it emphasizes grouping over sharing approaches to division" [5]. The difficulties in learning multiplicative reasoning are often due to the different ways in which students think about multiplication problems and the models they use. The standard algorithm for teaching the multiplication of larger numbers requires memorization of the basic multiplication facts. However, a wide variety of efficient, alternative algorithms exists based on the history of Mathematics, such as finger multiplication, multiplication's area model, lattice multiplication, line, circle/radius, paper strip, egyptian, russian peasant, etc. [6].

Despite the fact that the development of procedural techniques for multiplication in the Greek school textbooks is performed mainly via the "grid method", the area model does address some of the limitations of repeated addition, even if it does not easily relate to rate. However, most applications of multiplicative reasoning include the rates, therefore, certain researchers propose the use of double number line [5].

The main types of multiplicative structures are:

1. Isomorphism of measures
2. Multiplication factor, or an area of measures
3. Product of measures or Cartesian product
4. Multiple proportion [2]

For the multiplication's problems most researchers identify four different categories of multiplicative structures. Two of them, the equal groups (repeated addition) and the multiplicative comparison, are the most prevalent in the elementary school. The two others, combinations (Cartesian products) and problems with product of measures (length on width equal acreage), are used less frequently [7].

The difference of multiplicative problems from the problems of addition or abstraction is due to the fact that their numbers represent different types of things. A number or a factor counts how many sets, groups, or parts of equal size are involved (multiplier) and the other tells the size of each set or part (multiplicand) while the third number is the whole or the total of all the parts [7].

Kindergarten and first-grade children can solve multiplication and divisions problems, even if division involves remainders. The strategies they follow are not reflection of multiplicative reasoning, but their involvement in all four operations improves their level of understanding and guides them early enough to the development of multiplicative strategies.

Strategies used for the algorithm of multiplication are more complex than these for addition and subtraction. In addition the ability to break numbers apart in flexible ways is even more important in multiplication [7].

2 Brief discussion on the verifications of the operations

During our group discussions a teacher proposed to also refer to the other operations and theirs verifications. In particular, the teachers suggested that in the set of natural numbers the verifications of the operations of addition, subtraction and division are based on their simple properties.

Thus, for the addition $\alpha+\beta=\gamma$ the verification is $\beta+\alpha=\gamma$, meaning the use of the commutative property. Certainly we can also check the correctness of the addition through subtraction, if from the sum we subtract one of the two addends, so we will find the other, that is to say $\alpha+\beta=\gamma \Leftrightarrow \gamma-\alpha=\beta$ or $\gamma-\beta=\alpha$. The second method, as it was pointed out by the teachers, can be used only if the students have already been taught subtraction.

For the subtraction $\alpha-\beta=\gamma$ the verification is $\beta+\gamma=\alpha$. Moreover the correctness of the subtraction can also be checked using the relation $\alpha/\gamma=\beta$. For the division $\Delta:\delta$ where $\frac{\Delta}{\delta}=\Pi+\frac{\nu}{\delta}$ the verification is $\Delta=\delta\cdot\Pi+\nu$ with $0\leq\nu<\delta$.

In each one of the previous cases verification is a necessary and sufficient cond for an operation to be correct.

3 The discussion on multiplication and its verification

In order to study the multiplication's verification the teachers were assigned the following activity: The numbers 4789 and 635 were given and they were asked to:

1. Find their product.

2. Perform the verification of multiplication and interpret it.

Before moving to algorithms of multiplication we mentioned various useful representations (material pieces of decimal base, area model, etc), which subsequently were excluded because the numbers used were large.

One teacher (Teacher 1) was then asked to perform the multiplication using the known traditional algorithm which is also considered as the most difficult (Figure 1). If the students fail to understand it, they place -as it was reported by the teachers- the numbers in error columns, they add the carries before they multiply and in this way they make a lot of errors.

$$\begin{array}{r} 4789 \\ \times\ 635 \\ \hline 23945 \\ 14367 \\ +\ 28734 \\ \hline 3041015 \end{array} \tag{1}$$

After extensive discussion the following ways of multiplication's performance were presented such as:

1. (Teacher 2) Analyze (635) to (600+30+5) and then multiply the multiplicand with 5, 30 and 600 meaning: perform three multiplications and then add their products (Figure 2).(Teacher 3) This method is correct and it is based on the distributive property of multiplication in regard to addition , which is an important concept for multiplication. (Teacher 1) In this method the final products are the partial products of the initial multiplication.

$$\begin{array}{r} 4789 \\ \times\ 5 \\ \hline 23945 \end{array} \qquad \begin{array}{r} 4789 \\ \times\ 30 \\ \hline 143670 \end{array} \qquad \begin{array}{r} 4789 \\ \times\ 600 \\ \hline 2873400 \end{array}$$

$$\begin{array}{r} 23945 \\ 143670 \\ +\ 2873400 \\ \hline 3041015 \end{array} \tag{2}$$

2. (Teacher 4) Another method is to change the position of the multiplicand by the multiplier, for example to perform the multiplication. This method is based on the use of the commutative property. The resulting partial products are different from the partial products of the multiplication(initial case).(Figure 3)

$$\begin{array}{r} 635 \\ \times\quad 4789 \\ \hline 5715 \\ 5080 \\ 4445 \\ +\ 2540 \\ \hline 3041015 \end{array} \tag{3}$$

3. (Teacher 5) We could also perform the multiplication without using carries, as shown below, but in this way we would have 12 products and a "great" addition afterwards. (Figure 4) In that case, emphasis should be given to the proper placement of the obtained products and to the following addition.

$$\begin{array}{r} 4789 \\ \times\quad 635 \\ \hline 45 \\ 40 \\ 35 \\ 20 \\ 27 \\ 24 \\ 21 \\ 12 \\ 54 \\ 48 \\ 42 \\ +\quad 24 \\ \hline 3041015 \end{array} \tag{4}$$

4. (Teacher 6) Because the multiplication can be performed as follows and has twelve partial products (Figure 5).

$$\begin{array}{rr} & 4789 \\ \times & 635 \\ \hline & 2400000 \\ & 420000 \\ & 48000 \\ & 5400 \\ & 120000 \\ & 21000 \\ & 270 \\ & 20000 \\ & 3500 \\ & 400 \\ + & 45 \\ \hline & 3041015 \end{array} \tag{5}$$

5. Other types of strategies for multiplication's performance were also mentioned but were later excluded by the same teachers because the numbers were large. In particular the teachers reported the following: 1) strategies without breaking numbers into parts (they usually use successive additions in different ways (Teacher 7), 2) partitioning strategies (breaking the numbers in a variety of ways and subsequently use the distributive property Teacher 7) and 3) compensation strategies (breaking the numbers in a variety of ways so that the calculations are easier and result to partial products which are then added, Teacher 8) [7].

The other issue that was discussed included whether the previous ways constitute verification of the standardized multiplication's algorithm. The view of most schoolteachers was that they represent a different way of finding the product by which the correctness of the result of standardized algorithm can be checked, without them constituting verification.

As it was found by the discussion that followed, the verification that teachers chose in their practice wasn't based on the usual properties but followed a special method, that one of the cross [8].

A detailed report of this method was presented and followed by an attempt to highlight the teachers' views regarding its validity.

Educator (E): Who would want to perform the multiplication's verification?

Teacher (T1): (He made the cross and began to supplement it explaining every step that he followed).

$$\begin{array}{c|c} 1 & 5 \\ \hline 5 & 5 \end{array}$$

Top left corner: We add the digits of the first factor until one-digit number results: 4+7+8+9=29, 2+8=10, 1+0=1. We write this number on this corner.
Top right corner: We add the digits of the second factor until one-digit number results: 6+3+5=13 and 1+4=5. We write number 5 on this corner.
Bottom left corner: we calculate the product of the two numbers and we find $1 \cdot 5 = 5$. We write it in the bottom left corner.
Bottom right corner: we calculate the sum of the digits of the two numbers, we find 3+0+4+1+0+1+5=14 and 1+4=5 . We write it on the bottom right corner.
Educator (E): Is the multiplication correct?
Teacher (T10): Yes!
Educator (E): When a multiplication- if checked with this method- is correct?
Teacher (T11): The multiplication is correct if the numbers on the second line of the cross are the same.
Educator (E): Is this always true?
The following discussion took place:
Teacher (T12): This method does not always ensure that the multiplication is correct.
Educator (E): Why?
Teachers (T): For many reasons most of the teachers answered simultaneously.
Educator (E): Who would like to discuss some and then try to analyze them?
Teacher (T13): One possibility is that the digits that are presented are not placed in the correct position. That is, instead of the number 3041015 that is the correct result, the number 3041015 is written which results by reversing two of its digits.
Teacher (T14): Another possibility is that the digits that are presented in the last product are different (due to an error in the addition of the partial sums) resulting in the same sum. That is to say, instead of the number 3041015 which expresses the correct result and number 5 being the final one-digit sum of digits, the number written is 3041915 which has also the same sum of digits.
Teacher (T15): Another case is when an additional 0 is interposed between the digits of the correct number, meaning instead of the number 3041015, the number 30410015 is written.
Teacher (T16): Or a 0 is added at the end of the number. Thus, instead of the number 3041015, the number 30410150 is written.
Teacher (T17): A 0 is skipped either between the digits of the number or before its end, so for example instead of the number 3041015 we have number

304115.
Teacher (T18): A badly written 0 can be considered as 9 or vice versa, for example instead of the number 3041015 we have number 3941015.
Teacher (T2): With this method many errors can occur.
Educator (E): Based on the above cases or others, by this verification the multiplication seems "correct", but it is not. So, what do you think about this verification, does it always show if the multiplication is correct or not?
Teacher (T9): If the multiplication is correctly performed this method verifies it. If, however, the multiplication is not correctly performed, it is not certain that this will be shown by this verification.
Teacher (T9): Why does this happen and how is it explained?
Educator (E): Do you think that this verification is similar to the verifications of the other three operations?
Teacher (T9): Does this verification is a "necessary condition" for the multiplication to be correct but not sufficient?
Educator (E): A condition can only be necessary as it happens with the cross verification.
Afterwards the educator presented the basics from the equal remainder numbers' theory. He proved the method of the particular verification of multiplication and it was applied to the example that was previously discussed.
If two numbers $x, y \in R$, then

$$S_{x.y} \equiv S_x \cdot S_y (mod 9)$$

Proof
We know that: $x \equiv S_x(\text{mod}9)$ and $y \equiv S_y(\text{mod}9)$, therefore $x \cdot y \equiv S_x \cdot S_y(\text{mod}9)$. However $x \cdot y \equiv S_{x.y}(mod9)$, therefore $S_{x.y} \equiv S_x \cdot S_y(\text{mod}9)$.
Example of the above proof:
Consider the numbers x = 4789 and y = 635. Then $S_x = 4+7+8+9 = 28 \equiv 1$ (mod9) and $S_y = 6+3+5 = 14 \equiv 5$ (mod9), therefore $S_{x.y} \equiv 1.5(mod9) \equiv 5(mod9)$.
However $x.y = 4789.635 = 3041015$, $S_{x.y} = 3+0+4+1+0+1+5 = 14 \equiv 5$.
Hence, $S_{x.y} \equiv S_x \cdot S_y(mod9)$.
With this proposal it is proved that the necessary but not the sufficient condition exists in order for the multiplication to be correct. This method, as it was shown from the discussion, is used by all the teachers in their practice, without being as easy and understandable as the verifications of the other three operations. This resulted from teachers' comments during the interpretation of the cross method, which monopolized the discussion. Regarding the clarification of the phrase 'necessary and sufficient condition'

it was shown that this was not always clear to teachers. However this was not the subject of this study.

4 Conclusion

The standard algorithm for the multiplication of large numbers is brought to Europe by the Arabic-speaking people of Africa and requires the memorization of the basic multiplication facts. Multiplication is an important tool not only for constructing a firm foundation for proportional reasoning and the algebraic thinking, but also for solving real-life problems [6].

Teachers' mathematical knowledge includes not only Mathematics but also their teaching. The framework of the conceptual field may help them organize appropriate didactic situations and interventions [1]. It is essential that the improvement of the teaching of Mathematics regarding teachers' explanations, the representations and the examples they use and also of the method with which all the above are developed in addition to the way they themselves interact with their students, something that is achieved with their continuous training [9].

In the framework of this particular training it was found that schoolteachers use different ways in order to verify if a multiplication is correct. In conclusion: A. they reverse the multiplier with the multiplicand using the commutative property, B. they calculate partial products and they sum them up afterwards, c. they perform multiplication without using held, d. they calculate the partial products writing analytical numbers in thousands, hundreds, tens and ones using the distributive property, e. they use various informal forms for the execution of multiplication and g. they use the cross method.

The justification of the first four ways of verification was complete and understandable by most of the teachers; however, the third way was not used as much as the others.

For the verification of the cross method the results showed that all teachers use it in their daily practice at school, even if it is not included in the school textbooks. Most of the teachers that participated in this study they used but didn't empirically consider it reliable since they could not explain it adequately.

It is probable that the use of this method of multiplication's verification is related to teachers' age, which in our case teachers had at least ten years of professional experience. Further research which will include candidate schoolteachers with different curriculum of undergraduate studies would be of special interest.

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