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Valuation of Barrier Options with the Binomial Pricing Model

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Abstract

Derivatives are products of different nature which are becoming increasingly common in financial markets. In certain cases, determining the assessment criteria can sometimes be a difficult task. Specifically, this paper focuses on one type of exotic option: the barrier option. This option has to satisfy some conceptual conditions which are specified at the time of its purchase and define its characteristics. In order to analyze this type of option more deeply, in this paper we choose a specific one, the so-called barrier option cap, whose value is going to be derived by the binomial pricing model.

Keywords: barrier options; exotic options; barrier option cap; binomial model.

2010 AMS subject classification: 62P05, 91G20, 97M30, 05A10.

1. Introduction

In the last years, the interest rate has reached historic low levels. As a consequence, the investment habits have changed and investors are interested in new and more profitable products. For this reason, derivatives have been selected as an alternative to traditional investment products. These are financial products whose price does not only vary according to parameters such as risk, but also depends on the market price of another asset, called the underlying asset (stock, foreign exchange, stock market index, etc.) (Carr, 1998). The option holder is committed to the evolution, up or down, of a certain underlying asset in the securities market. There are different products: options, warrant, futures, etc. The main difference is the way in which the price is derived and the nature of the transaction to which this instrument gives rise, that is to say, how and when the delivery of the asset takes place.

A derivative is a forward contract whose characteristics are established at the agreement moment, whilst the money exchange occurs at a future moment. Derivatives, like financial options, are products with higher profits since the premium is lower than the corresponding to the underlying asset, whereby the results can be multiplied, either in the positive or negative sense, in relation to the premium. Hence they are highly risky products. In order to make them more attractive, exotics options, specifically the barrier option, arise in order to allow taking more control of the operational risk by employing covertures.

Barrier options are very popular but there is a scarce economic research given its novelty and complexity (Rich, 1994). We start with an analysis of the product from a theoretical point of view and by studying the analogies and differences of this type of exotic options with financial standard options (plain vanilla).

They are options whose exercise will depend on whether the underlying asset reaches a pre-set barrier level during a certain period of time. If this occurs, the conditional option becomes a simple call or put option (knock-in options) or, on the other hand, it may cease to exist from the moment that the barrier level is reached (knock-out options).

Once this financial product has been introduced as an alternative to traditional investment products, we present this paper organization. In Section 2, barrier options are described and studied from a mathematical point of view. In Section 3, we focus on the barrier option cap which is a specific barrier option. Then, in Section 4, the financial analysis to derive the value of this type of financial option is presented. Finally, Section 5 summarizes and concludes.

2. Barrier options

Barrier options are derivatives which can be canceled or activated depending on the prices reached by the corresponding underlying asset (Soltes and Rusnakova, 2013). They are available for a predetermined period of time if, during this period, the underlying asset reaches a certain level, the conditional option is converted from that moment into a simple option (knock-in options) or, in another case, if the option already exists, it is canceled from that moment (knock-out options).

These options are similar to a call or a put option with a specified barrier (called *B*). It ensures that the option has a fixed value (called *L*) if the maximum or minimum of the underlying asset price (called *S*) do not touch the barrier (Rubinstein and Reiner, 1991). Below the different types of barrier options are explained.

2.1. Knock-in options

These options only arise if the underlying asset price reaches a certain level, known as barrier level (Fernández, 1996). They can be classified into two types:

1) Up-and-in options: The right to exercise the option is activated when the underlying asset price is above a certain level (B) during the option's life. Its price at maturity, if the strike price is denoted by K, is:

-Call up-and-in option
$$\begin{cases} \max(0; S_t - K), & \text{if } \max(S, S_1, \dots, S_t) = B \\ 0, & \text{if } \max(S, S_1, \dots, S_t) < B \end{cases}$$

-Put up-and-in option
$$\begin{cases} \max(0; K - S_t), & \text{if } \max(S, S_1, \dots, S_t) = B \\ 0, & \text{if } \max(S, S_1, \dots, S_t) < B \end{cases}$$

2) Down-and-in options: The right to exercise the option at maturity appears if the underlying asset price falls below the pre-determined barrier (*B*). In this way, we can distinguish between:

-Call down-and-in option
$$\begin{cases} \max(0; S_t - K), & \text{if } \min(S, S_1, \ldots, S_t) = B \\ 0, & \text{if } \min(S, S_1, \ldots, S_t) > B \end{cases}$$

-Put down-and-in option
$$\begin{cases} \max(0; K - S_t), & \text{if } \min(S, S_1, \ldots, S_t) = B \\ 0, & \text{if } \min(S, S_1, \ldots, S_t) > B \end{cases}$$

2.2. Knock-out options

These options only may be exercised if the underlying asset price does not reach the barrier, that is to say, the right to be exercised disappears if the underlying asset price intersects the barrier at any time of the option's life; at this moment, the option acquires a fix price (L) (Fernández, 1996). They can be classified into two types:

1) Up-and-out options: They only make sense if the underlying asset price is above a pre-determined value during the option's life:

-Call up-and-out option
$$\begin{cases} \max(0; S_t - K), & \text{if } \max(S, S_1, \ldots, S_t) = B \\ L, & \text{if } \max(S, S_1, \ldots, S_t) > B \end{cases}$$

-Put up-and-out option
$$\begin{cases} \max(0; K - S_t), & \text{if } \max(S, S_1, \ldots, S_t) = B \\ L, & \text{if } \max(S, S_1, \ldots, S_t) > B \end{cases}$$

2) Down-and-out options: The right to exercise the option disappears if the underlying asset price is below the level established by the barrier.

-Call down-and-out option
$$\begin{cases} \max(0; S_t - K), & \text{if } \min(S, S_1, \ldots, S_t) = B \\ L, & \text{if } \min(S, S_1, \ldots, S_t) < B \end{cases}$$

-Put down-and-out option
$$\begin{cases} \max(0; K - S_t), & \text{if } \min(S, S_1, \ldots, S_t) = B \\ L, & \text{if } \min(S, S_1, \ldots, S_t) > B \end{cases}$$

There is another type of option called "double barrier option" which disappears if the underlying asset does not stay within a certain interval (Kunitomo and Ikeda, 1992 and Fernández and Somalo, 2006).
The main advantage of using barrier options is its lower price, compared to a vanilla equivalent option. The saving of using barrier options versus simple options depends on:

-The proximity of the barrier to the current price of the underlying asset (with greater proximity to savings of the "out" type) and, conversely, to greater distance (in the "in" type).

-The option's life (the longer the time to maturity, the greater the probability of reaching the barrier and therefore the greater the savings in the "out" and inversely in the "in" type).

-The greater the volatility (greater probability of touching the barrier and, therefore, greater savings in the "out" type, and inversely in the "in" type).

Barrier options can be very useful in hedging commodities providing protection at a lower price than traditional options for coverage of risks (Crespo, 2001).

3. Barrier option cap

In this section, we are going to study a specific barrier option, the so-called barrier option cap. It guarantees a certain profitability called "option level" at maturity, i.e. it guarantees a final sale price independently of the share price, with the only condition that during the option's life the underlying asset does not reach a certain lower level, called the "barrier". This product was issue by PNB Paribas Bank in Spain with the name "bonus cap". It has been marketed for a short time since they were first issued in Spain on June 16, 2010.

The barrier and option level are given by the issuer bank and they are known from the beginning, that is to say, from the issue date and during the barrier option cap life.
In case that the underlying asset reaches the barrier, this does not imply that the option disappears but simply loses the guaranteed price at maturity (the "option level"), for which the barrier option cap will continue being traded with normality being able to give profits if the share has an upward tendency which allows the holder to sell above the level of purchase.
The barrier option cap profit is limited to the "option level" so it should be clarified that in case that the underlying asset quotes above the "option level" at maturity, the holder will receive at most the profitability previously fixed corresponding to the "option level". On the other hand, if the barrier option cap reaches the "option level" before maturity, the holder can get rid of his/her investment since it does not make sense to keep the investment when the highest allowed profitability has been already achieved. In this way, we would have achieved the maximum expected return without waiting to maturity.
Therefore, it can be said that the barrier option cap limits the profits which can be obtained, in exchange for ensuring a known profit provided that the underlying asset price is higher than the barrier value.

3.1. Analogies and differences of barrier options cap with other derivatives

-A **future** contract is an agreement whereby two persons (physical or legal) undertake to sell and to buy, respectively, an asset, called the underlying asset, at a price and at a future date according to the conditions fixed in advance by both parties.
However, the holder of a barrier option cap will never be the owner of the underlying asset; he/she will receive the cash corresponding to the price of the underlying asset.
The future is a compromise, whilst the purchase of a barrier option cap is an option to buy.
-An **option** is an agreement granting the buyer, in return for payment of a price (premium), the right (not the obligation) to buy or sell an underlying asset at a price (strike price) and at a future date, in accordance with the conditions set forth in advance by both parties.
As for the sale of the barrier option cap, it is a liquid product and can be sold at any time, so we could say that it keeps more similarities with the American options since it is not necessary to wait for the expiration to exercise the sale.

Barrier options cap present the following differences with respect to other derivative products which make them a new banking product:

-They present a well-known and bounded return from the moment of contracting, as long as the underlying asset does not reach the barrier during the life of the barrier option cap.
-The underlying asset is not acquired at any time.
-At maturity, the option owner will receive in cash the traded price of the underlying asset in case it reaches the barrier and never exceeding the "option level".
-It is not necessary to wait until maturity to obtain liquidity.

4. Assessment of a barrier option cap

The methodology we are going to use in this paper is the binomial model, introduced by Cox, Ross and Rubinstein (1973) to value financial options. It is a discrete-time model based on the binomial tree, with different possible trajectories. A barrier option cap is a derivative over an underlying asset (usually a share) which is defined by the following elements:
- B: barrier.
- L: option level.
- S_k : price of the underlying asset at moment k ($k = 1, 2, \ldots, n$).

The possible performances of the barrier option cap are the following ones:
-If, at any time, the underlying asset is traded between the barrier and the option level, the barrier option cap guarantees the payment of the option level.

Figure 1. Underlying asset between the barrier and the option level.

Source: Own elaboration from BNP Paribas Bank data.

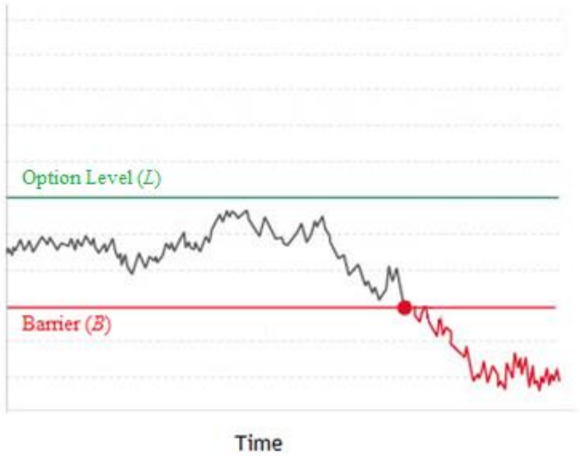
-If, at any moment, the underlying asset quotes above the option level, the option can be sold at that time obtaining, in advance, the maximum amount that could be reached with the option, i.e. the option level.

Figure 2. Underlying asset above option level.

Source: Own elaboration from BNP Paribas Bank data.

-If, at maturity, the asset quotes below the pre-set barrier level, the holder of the option will receive at maturity the price of the share at that time, with limit the level of the option.

Figure 3. Underlying asset below the barrier level.



Source: Own elaboration from BNP Paribas Bank data.

Taking into account the given definition of the barrier option cap, the option price *P* at moment 0 is given by the mathematical expectation of the following random variable (*BOC*) that represents the possible values of the option at that instant:

$$BOC = \begin{cases} L(1+r_f)^{-n}, & \text{if} \quad B < S_k < L, \quad \text{for all } k \\ L(1+r_f)^{-k}, & \text{if} \quad S_k \geq L, \quad \text{for some } k \\ \min\{S_n, L\}(1+r_f)^{-n}, & \text{if} \quad S_k \leq B, \quad \text{for some } k \end{cases}$$

where r_f is the risk-free interest rate. Therefore, $P = E[BOC]$.

In Figure 4, we are going to describe a methodology to calculate the price of a barrier option cap assuming that the underlying asset follows a binomial process with a rising factor *u* and a downward factor *d*, starting from the price volatility of the underlying asset at time 0 (S_0). To do this, we start from an example in which the option maturity is after five periods.

In this case, the value of the barrier option cap is:

$$BOC = \begin{cases} N(1+r_f)^{-5}, & \text{with probability } 1 - p^4 - q^3 - 3pq^4 \\ N(1+r_f)^{-3}, & \text{with probability } p^4 \\ S_0 u^2 d^3 (1+r_f)^{-5}, & \text{with probability } p^2 q^3 \\ S_0 u d^4 (1+r_f)^{-5}, & \text{with probability } 2pq^4 + 3pq^4 \\ S_0 d^5 (1+r_f)^{-5}, & \text{with probability } q^5 \end{cases}$$

As previously indicated, $P = E[BOC]$.

5. Conclusions

Taken into account the wide offer of financial products with different risks, profitability and liquidity, an accurate analysis of their characteristics and real values is completely necessary.

Despite barrier options increase the covertures of risks, they are cheaper than the equivalent standard option. Specifically, a barrier option cap is a derivative with a given profitability provided that a certain condition is satisfied. In this way, a barrier option cap limits the benefit which can be obtained, in exchange of ensuring a known profit (the option level) if the price of the underlying asset is higher than the barrier value. So, this paper aims to analyze the assessment of this option by employing the binomial options pricing model.

Figure 4: Value of the barrier option cap within 5 periods (Instants).

In0	**In1**	**In2**	**In3**	**In4**	**In5**	**Prob**	
					S_0u^5 →	p^5	
				S_0u^4 ↗ ↘			L
			S_0u^3 ↗ ↘		S_0u^4d →	$5p^4q$	
		S_0u^2 ↗ ↘		S_0u^3d ↗ ↘			
	S_0u ↗ ↘		S_0u^2d ↗ ↘		$S_0u^3d^2$ →	$10p^3q^2$	
S_0 ↗ ↘		S_0ud ↗ ↘		$S_0u^2d^2$ ↗ ↘			
	S_0d ↗ ↘		S_0ud^2 ↗ ↘		$S_0u^2d^3$ →	$10p^2q^3$	
		S_0d^2 ↗ ↘		S_0ud^3 ↗ ↘			
			S_0d^3 ↗ ↘		S_0ud^4 →	$5pq^4$	B
				S_0d^4 ↗ ↘			
					S_0d^5 →	q^5	

Source: Own elaboration.

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