AN EXAMPLE OF A JOIN SPACE ASSOCIATED WITH A RELATION

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Abstract In this paper a join spaces associated with a binary relation is presented.

Keywords Join spaces, Relations

First of all, let us recall what a join space is.
Let H be a nonemptyset and $o : H \times H \to P^* (H)$, where $P^* (H)$ is the set of nonempty subsets of H.
If $A \subset H$, $B \subset H$, then we set $A \text{ o } B = \cup_{a\in A} \cup_{b\in B}\, a \text{ o } b$.
We denote $A \approx B$ if $A \cap B \neq \emptyset$.
If the hyperoperation "o" is associative and $\forall\, a \in H$, we have a o H = H = H o a, then (H,o) is a **hypergroup.**
Denote $a / b = \{x \in H \mid a \in b \text{ o } x\}$, for any $(a,b) \in H^2$.
A hypergroup (H,o) is called a join space if "o" is commutative and

$$a / b \approx c / d \Rightarrow a \text{ o } d \approx b \text{ o } c.$$

Join spaces have been introduced by W. Prenowitz and used by himself and J. Jantosciak in order to rebuild some branches of non-Euclidian geometries. Afterwards, join spaces have also been used in the study of other topics (Graphs and Hypergraphs, Lattices, Binary Relations and so on).
Here, a connection between join spaces and reflexive and symmetric relations is presented.
First, we give an example:
Let $f : H \to U$ be an onto map.
We define on H the following hyperoperation:

$$\forall (x,y) \in H^2,\ x \text{ o } x = f^{-1}(f(x)),\ x \text{ o } y = x \text{ o } x \cup y\text{o}y$$

(where $\forall\, Y \subset U$, $f^{-1}(Y) = \{x \in H \mid f(x) \in Y\}$).

Proposition (H,o) is a join space.

Proof. For any $(x,y,z) \in H^3$, we have
$(x \text{ o } y)\text{o } z = x \text{ o } (y \text{ o } z) = f^{-1}(f(x)) \cup f^{-1}(f(y)) \cup f^{-1}(f(z))$
and $x \text{ o } H = \cup_{a\in H}\, x \text{ o } a = \cup_{a\in H}\, f^{-1}(f(x)) \cup f^{-1}(f(a)) = H$,
since f is onto.
So, (H, o) is a commutative hypergroup.

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Moreover, any $x \in H$ is an identity of H (since $\forall\ y \in H,\quad y \in x\ o\ y$) and for any $(x,y)\in H^2$, x is an inverse of y.
Let us check now that $a / b \approx c / d \Rightarrow a\ o\ d \approx b\ o\ c$.
Let $x \in a / b \cap c / d$ that is $a \in f^{-1}(f(x)) \cup f^{-1}(f(y))$ and $c \in f^{-1}(f(x)) \cup f^{-1}(f(d))$.
It follows $f(a) \in \{f(x), f(b)\}$ and $f(c) \in \{f(x), f(d)\}$.
We must prove that there is $y \in H$, such that $y\in a\ o\ d\ \cap b\ o\ c$, that is
$f(y) \in \{f(a), f(d)\} \cap \{f(b), f(c)\}$.
We have the following situations:

1) if $f(a) = f(x) = f(c)$ then we can choose $y = a$;
2) if $f(a) = f(x)$ and $f(c) = f(d)$, then we choose $y = d$;
3) if $f(a) = f(b)$, then we choose $y = a$.

Therefore, (H, o) is a join space.

Now, let us consider R a reflexive and symmetric relation on H.
Let us consider the following hyperoperation on H

$$\forall(x,y) \in H^2,\ x\ o_R\ x = \{z \mid (z,x) \in R\},\ x\ o_R\ y = x\ o_R\ x \cup y\ o_R\ y.$$

Theorem (H, o_R) is a join space.

Proof. The associativity is immediate and $\forall\ x \in H$, we have
$x\ o_R H = x\ o_R x \cup \cup_{a\in H}\ a\ o_R a = H$, since
$\cup_{a\in H}\ a\ o_R a = \cup_{a\in H} \{z \mid (z,a) \in R\} = H$ (R is reflexive).
So, (H, o_R) is a commutative hypergroup.
Notice that $a\in a\ o_R a \Leftrightarrow (a,a)\in R$.
Let us check now that $a / b \approx c / d \Rightarrow a \in a\ o_R d \approx b\ o_R c$.
Let $x\in H$, such that $a \in x\ o_R\ b$ and $c \in x\ o_R\ d$.
We have $a \in \{t \mid (t,x) \in R \text{ or } (t,b) \in R\}$, whence $(a,x)\in R$ or $(a,b)\in R$.
Similarly, $(c,x)\in R$ or $(c,d)\in R$.
We must prove that there is $y\in H$, such that $y\in a\ o_R d$ and $y\in b\ o_R\ c$, that is
$[(y,a) \in R$ or $(y,d) \in R]$ and $[(y,b) \in R$ or $(y,c) \in R]$,
or equivalently, $[(y,a) \in R$ and $(y,b) \in R]$ or $[(y,a) \in R$ and $(y,c) \in R]$
or $[(y,d)\in R$ and $(y,b)\in R]$ or $[(y,d) \in R$ and $(y,c)\in R]$.
We have the following situations:

1) $(a,x) \in R$ and $(c,x) \in R$. Since R is symmetric, it follows $(x,a) \in R$ and $(x,c) \in R$. In this case, we can choose $y=x$.
2) $(a.x) \in R$ and $(c,d) \in R$. We take $y=c$ and so, $(y,d)=(c,d) \in R$ and $(y,c)=(c,c) \in R$.
3) $(a,b) \in R$ and $[(c,x) \in R$ or $(c,d) \in R]$.
4) We take $y=a$, so $(y,b)=(a,b) \in R$ and $(y,a)=(a,a) \in R$.

Therefore, (H, o_R) is a join space.

Remark. If R is the relation defined as follows:
$(x,y) \in R \Leftrightarrow f(x)=f(y)$
where $f : H \rightarrow U$, then (H, o_R) is the join space presented at the beginning.

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