Cooperative Games, Finite Geometries and Hyperstructures

Antonio Maturo
Università di Chieti - Pescara

Abstract In this paper some relations between finite geometric spaces and cooperative games are considered. In particular by some recent results on blocking sets we have new results on blocking coalitions. Finally we introduce a new research field on the possible relations between quasihypergroups and cooperative games.

Keywords Cooperative Games, Finite Geometries, Blocking sets, Quasihypergoups

1. Cooperative games

Let M = {1, 2, …, n} be a finite non-empty set, called the *set of players*.
A function v: $\wp$(M) $\to$ R such that:

(C1) v($\emptyset$) = 0;
(C2) (*superadditivity*) $\forall$A, B$\in \wp$(M), (A$\cap$B = $\emptyset$) $\Rightarrow$ v(A$\cup$B) $\geq$ v(A) + v(B);

is called *characteristic function* on M.
The pair (M, v) is called *cooperative game with n players* and the subsets of M are called *coalitions*.

For every A$\in \wp$(M) the number v(A) is the total gain that the players of A can have certainly forming a coalition, independently on the actions of the players not belonging to A. We assume the condition of "*side payment*", that is in every coalition A any player can transfer an amount of his gain to another player belonging to A and so it is important only the total gain of the coalition.
The condition (C2) is a consequence of the fact that the total gain obtained with an alliance between two disjoint coalitions is not inferior to the one without cooperation.

We write v(i) to denote v({i}). By (C2) it follows that in a cooperative game (M, v) we have v(M) $\geq \Sigma_{i\in M}$ v(i). If v(M) > $\Sigma_{i\in M}$ v(i) the game (M, v) is said to be *essential*, if the equality holds (M, v) is *inessential*.
It is easy to prove that a cooperative game is inessential if and only if:

(AD) (*additivity*) $\forall$A, B$\in \wp$(M), (A$\cap$B = $\emptyset$) $\Rightarrow$ (v(A$\cup$B) = v(A) + v(B))

and so there are no advantages by the cooperation.

Two cooperative games (M, v) and (M, v'), with the same set of n players, are called *strategically equivalent*, we write $(M, v) \approx (M, v')$ if there exist n+1 real numbers $k>0$ and $c_1, c_2,\ldots, c_n$ such that:

(SE) $\forall A\in \wp(M),\ v'(A) = k\, v(A) + \Sigma_{i\in A}\, c_i$.

We obtain the game (M, v') by the game (M, v) with an initial payment c_r to any player r and by multiplying the total gain of any coalition by the scale factor k. Then we can assume the same strategies to solve (M, v) or (M, v').

Proposition 1.1. Let (M, v) a cooperative game. The system with n+1 equations and n+1 unknowns $k > 0$ and $c_1, c_2,\ldots, c_n$:

(ESI) $k\, v(M) + \Sigma_{i\in M}\, c_i = 1, \qquad k\, v(i) + c_i = 0,\ i\in M$

has determinant $v(M) - \Sigma_{i\in M}\, v(i)$ and so has solutions if and only if (M, v) is essential. In this case $k = 1/(v(M) - \Sigma_{i\in M}\, v(i))$ and so $k > 0$.
The system:

(NES) $k\, v(M) + \Sigma_{i\in M}\, c_i = 0, \qquad k\, v(i) + c_i = 0,\ i\in M$

has not trivial solution if and only if $v(M) - \Sigma_{i\in M}\, v(i) = 0$.

The relation $\approx$ is an equivalence relation among the cooperative games with the same set of players M. By proposition 1.1 we have that, for any equivalence class K with respect to $\approx$, we have a unique cooperative game $(M, v)\in K$, called *normal element* of K or *normal form* of the elements of K, such that $v(i) = 0,\ \forall i\in M$ and $v(M)\in\{0, 1\}$. Precisely, $v(M) = 1$ if the game is essential and $v(M) = 0$ if it is inessential. The inessential games are in the same equivalence class and the normal form is such that $v(A) = 0,\ \forall A\in \wp(M)$. On the contrary, for $n > 2$, the essential games are in different classes.

2. Simple cooperative games and projective spaces

Let (M, v) be an essential cooperative game in normal form. Then $v(i) = 0,\ \forall i\in M$, and $v(M) = 1$. We say that (M, v) is a *simple game* if, $\forall A\in \wp(M),\ v(A)\in\{0, 1\}$.
By (C2), for any coalition A, if $A^c = M-A$, we have three possibility:

(a) $v(A) = 1$ and $v(A^c) = 0$;
(b) $v(A) = 0$ and $v(A^c) = 1$;
(c) $v(A) = 0$ and $v(A^c) = 0$.

The set A is called *winning coalition* if (a) holds and *losing coalition* if (b) holds. It is evident that M is a winning coalition and the complement of a winning coalition is a losing coalition. So the number of winning coalitions is equal to the number of losing coalitions. The set A is said to be a *blocking coalition* if (c) holds. If A is a blocking coalition then also A^c is a blocking coalition. So, if there exist blocking coalitions, their number is even.
We have the following:

Proposition 2.1 Let W be a subset of a set $\wp(M)$, with M set of players. Then W is the set of the winning coalitions of a simple cooperative game (M, v) if and only if satisfy the following properties, called the "*axioms of Shapley*" (see [32], [34]):

(W1) $M \in W$;
(W2) $\forall A, B \in \wp(M)$, $(A \in W, A \subseteq B) \Rightarrow B \in W$;
(W3) $\forall A \in \wp(M)$, $A \in W \Rightarrow A^c \notin W$.

Proof. Let (M, v) be a simple cooperative game in normal form and let W be the set of winning coalitions. Then (W1) and (W2) are trivial and (W3) follows by (C2).
On the converse, let W be a subset of $\wp(M)$ satisfying the axioms of Shapley. We put, for any $A \in \wp(M)$, $v(A) = 1$ if $A \in W$ and $v(A) = 0$ otherwise. The pair (M, v) is a simple cooperative game and W is the set of winning coalitions.

By previous proposition, in the sequel we consider a simple cooperative game indifferently as the pair (M, v) or the pair (M, W).
All the properties of the losing coalitions are obtained from the ones of the winning coalitions by replacing any coalition A with A^c and $\subseteq$ with $\supseteq$ and vice versa.
We can prove the following:

Proposition 2.2 Let (M, W) be a simple cooperative game. A family Θ of subsets of M is the set of blocking coalitions of (M, W) if and only if:

(BC) $\forall X \in \Theta, \forall A \in W, \; X \cap A \neq \emptyset, \; X^c \cap A \neq \emptyset.$

Now we introduce some useful definitions.

Definition 2.1 Let M be a non empty set and let $\Im$ be a family of subsets of M.
We say that $\Im$ has the "*intersection property*" if we have:

(IP) $\forall A, B \in \Im, \; A \cap B \neq \emptyset.$

We say that $\Im$ has the "*non-inclusion property*" if we have:

(NI) $\forall A, B \in \Im, \; A \cap B^c \neq \emptyset$ and $A^c \cap B \neq \emptyset.$

Definition 2.2 Let M be a non empty set and let Φ and $\Im$ be two families of subsets of M. We say that "$\Im$ *is a generator of* Φ" or "Φ *is the closure* of $\Im$", and we write $\Phi = K(\Im)$, if

(GK) $\quad \Phi = \{A \in \wp(M) : \exists B \in \Im \,/\, B \subseteq A\}$.

We say that "$\Im$ *is a minimal generator of* Φ" if $\Im$ has the *non-inclusion property* and is a *generator* of Φ.

By (W3) it follows that, in a simple cooperative game (M, W), W satisfies the intersection property. The following proposition shows that any family $\Im$ of subsets of M that has the intersection property generates the winning coalitions of a simple cooperative game.

Proposition 2. 3 Let M be a n-set, whose elements are called *players*, and let $\Im$ be a family of subsets non-void of M satisfying the intersection property. Then if W is the closure of $\Im$, the pair (M, W) is a simple cooperative game with W set of winning coalitions, called "*the game generated by* (M, $\Im$)".

Proof. Let W be the closure of $\Im$. Then (W1) and (W2) are evident. If $A \in W$, then there exists $B \in \Im$ such that $A \supseteq B$ and so $A^c \cap B = \emptyset$. Then, $\forall C \in \Im$, A^c don't contains C. Otherwise A^c must contain $C \cap B \neq \emptyset$, a contradiction. It follows that $A^c \notin W$.

Now we examine some relations between the cooperative simple games and the geometric spaces.

Definition 2. 3 A geometric space is a pair (M, Δ), with M a non-empty set, called *the support* and Δ a non-empty family of subsets of M. The elements of M are called *points* and the ones of Δ are called *blocks*. If any block has *at least two points* and any two blocks have *at most one point in common* (M, Δ) is called "*space of lines*" and the blocks are called also *lines*.
(M, Δ) is *non-degenerate* if there are at least two blocks.

Definition 2.4 A *projective space* is a geometric space (M, Δ) such that (see [7]):

(PS1) $\forall P, Q \in M$, $P \neq Q$, there is exactly one block containing {P, Q}, called *the line* PQ;
(PS2) (*Veblen-Young axiom*) Let A, B, C, D four distinct points such that AB intersects CD. Then AC intersects BD.
(PS3) Any line contains at least three points.

A non-degenerate projective space is a *projective plane* (or projective space with dimension 2) if the axiom (PS2) is replaced by the stronger axiom:

(PS2S) Two lines have at least a point in common.

If a non-degenerate projective space (M, Δ) is not a projective plane, for any A, B, $C \in M$, distinct and such that C not belongs to the line AB, we define "*plane* ABC", or "2-*dimensional subspace* ABC" of M, the union of the lines CX, with $X \in AB$.
We say that (M, Δ) has dimension 3 if:

(PSD3) A line and a plane have at least a point in common.

For recurrence we can consider projective spaces and subspaces with greater dimensions.

If (M, Δ) is a projective plane we have that Δ satisfies both the *intersection property* and the *non-inclusion property*. So, by proposition 2.3, we have the following:

Proposition 2.4 Let (M, Δ) be a finite projective plane and let W be the closure of Δ. Then (M, W) is a simple cooperative game, with W set of winner coalitions, and Δ is a minimal generator of W.

If (M, Δ) is a projective space of dimension 3 or 4 the planes have both the *intersection property* and the *non-inclusion property*. Then we have:

Proposition 2. 5 Let (M, Δ) be a finite projective n-dimensional space with $n \in \{3, 4\}$ and let Δ^* be the set of all the planes of M.
If W is the closure of Δ^* then (M, W) is a simple cooperative game, with W set of winner coalitions, and Δ^* is a minimal generator of W.

Definition 2.5 Let (M, Δ) be a geometric space and let $\Im$ be a family of subsets of M. A subset X of M is called a *blocking set* with respect to $\Im$ if.

(BS) $\forall A \in \Im,\ X \cap A \neq \emptyset$ and $X^c \cap A \neq \emptyset$.

If C is a subset of M containing a $A \in \Im$, by (BS) we have: $X \cap C \neq \emptyset$ and $X^c \cap C \neq \emptyset$.
Then it follows the:

Proposition 2. 6 Let (M, Δ) be a geometric space, $\Im$ be a family of subsets of M and Φ be the closure of $\Im$. Then X is a blocking set with respect to $\Im$ if and only if it is a blocking set with respect to Φ.

Some corollaries of the previous propositions are:

Proposition 2. 7 Let (M, Δ) be a geometric space such that Δ has the intersection property. Then the blocking sets with respect to Δ are the blocking coalitions of the simple cooperative game (M, W), with W closure of Δ.

Proposition 2. 8 In a finite projective plane (M, Δ) the blocking sets with respect the lines are the blocking coalitions of the simple cooperative game (M, W), with W closure of Δ.

Proposition 2. 9 In a finite 3-dimensional or 4-dimensional projective space (M, Δ) the blocking sets with respect the planes are the blocking coalitions of the simple cooperative game (M, W), with W closure of Δ^*, set of the planes.

The previous propositions show the importance of the research of blocking sets in a finite projective space.

In particular we have the fundamental problems to find:

(a) the *minimal* or *maximal* blocking sets;
(b) the *spectrum* of the minimal blocking sets, that is the set of all the possible cardinalities of the minimal blocking coalitions;
(c) the *minimal winning coalitions*;
(d) the *winning coalitions containing blocking coalitions*.

By (BS) it follows that the complement of a blocking set is also a blocking set, so to find the maximal blocking sets is equivalent to find the minimal ones.
Now we show some results in the particular case of projective planes.

It is well known that, in a non-degenerate finite projective plane, all the lines have the same number of points. If q+1 is such number, the projective plane is said to be of *order* q and is noted π_q. Moreover, the lines through a fixed point P are also q+1 and the points of π_q are $q^2 + q + 1$.
By (PS3) we have $q \geq 2$. It is well known (see [7], [17]) that there exists a Desarguesian projective plane if and only if q is a prime or a power of a prime and such plane is unique. The first value of q with non-Desarguesian planes is q = 9.

For small values of q we have:

- in π_2 there are not blocking sets;
- in π_3 there are exactly two blocking sets;
- the blocking sets on π_4 and π_5 are classified, respectively, in papers of Berardi - Eugeni ([2]) and Berardi - Innamorati ([5]);
- the blocking sets on π_7 are classified in papers of Innamorati and Maturo (see [23], [24], [25]). If k is the cardinality of a minimal blocking set on π_7 we have $12 \leq k \leq 19$. In particular there are, up to isomorphism, only two

minimal blocking sets of order 12 and there is only a minimal blocking set with 19 points.

In the general case there are the following results (Innamorati – Maturo, [23], [25]):

Proposition 2.10 Let S(q) the spectrum of the minimal blocking sets in π_q. Then, if $q \geq 4$, $S(q) \supseteq [2q-1, 3q-5] \cup \{3q-3\}$ and, if π_q is Desarguesian, $S(q) \supseteq [2q-1, 3q-3]$.

Proposition 2.11 A sufficient condition for the existence of a minimal blocking set with 3q–4 points on a non-Desarguesian plane π_q is that π_q contains a proper subplane of order two.

In [29] H. Newmann conjectured that any finite non-Desarguesian plane contains a proper subplane of order two. By previous proposition, if the conjecture is true, we have that also for the non-Desarguesian plane of order q there exists a blocking set with 3q – 4 points.

3. Cooperative games and finite geometric spaces

We introduce the following:

Definition 3.1 Let M be a non-void set and let Ψ and $\Im$ be two families of subsets of M. We say that "$\Im$ *is a intersection-generator of* Ψ" or "Ψ *is the intersection-closure* of $\Im$", and we write $\Psi = IK(\Im)$, if

(IK) $\quad\Psi = \{A \in K(\Im): \forall B \in \Im, A \cap B \neq \varnothing\}$.

Let (M, Δ) be a geometric space. If Δ has not the intersection property, and W* is the closure of Δ, the pair (M, W*) is not a simple cooperative game because (W3) is not valid. But we have the following proposition, that generalizes proposition 2.3:

Proposition 3.1 Let (M, Δ) be a finite geometric space and let W be the intersection-closure of Δ. Then (M, W) is a simple cooperative game, called "*the game generated by* (M, Δ)".

Proof. (W1) is evident. If $A \in W$ and $A \subseteq B \subseteq M$, then $\forall C \in \Delta$, $C \cap A \neq \varnothing \Rightarrow C \cap B \neq \varnothing$ and (W2) holds. If $A \in W$ then there exists $C \in \Delta$: $C \subseteq A$ and so $C \cap A^c = \varnothing$ and $A^c \notin W$.

The game (M, W) generated by a geometric space (M, Δ) has two types of blocking coalitions:

(T1) the blocking sets with respect Δ;

(T2) the subsets of M containing at least a block and with intersection void with at least a block.

The losing coalitions are the subsets Y of M non containing blocks and having intersection void with at least a block.

Example 3.1 Let M be a n-set, whose elements are called *players*, and let $\Im$ be a family of subsets non-void of M, called *companies*.
By an economic point of view, we assume that a player belonging to a company has *a power of veto* and a coalition containing a company has *the control* of such company.
Then a winner coalition of the game generated by the geometric space (M, $\Im$) has the control of at least a company and a right of veto on all the companies, a losing coalition don't have a power of veto on at least one company and don't control any company. Finally a blocking coalition of type (T1) has veto for any company but don't control anyone, and a blocking coalition of type (T2) control at least a company but has not veto for all the companies.

We can construct a simple cooperative game by a finite geometric space (M, Δ) also with a "geometric" procedure different from the one of proposition 3.1, by assigning the set Δ^* of minimal winner coalitions.
Precisely, we consider a set Δ^* with the following properties:

(DS1) any $A \in \Delta^*$ is a union of elements of Δ;
(DS2) any element of Δ is contained in at least an element of Δ^*;
(DS3) Δ^* has the intersection and non-inclusion properties;

and we assume W equal to the closure of Δ^*.

We have the following:

Proposition 3.2 Let (M, Δ) be a finite geometric space and let Δ^* be a family of subsets of M satisfying (DS1), (DS2) and (DS3). If W is the closure of Δ^* then:

(DW1) (M, W) is a simple cooperative game;
(DW2) the blocking sets of (M, Δ) are blocking coalitions of (M, W).

Proof. Property (DW1) follows by (DS3). Let X be a blocking set of (M, Δ). Then X and X^c intersect any block and so, by (DS1), any element of Δ^*. By proposition 2.7 it follows that X is a blocking coalition of (M, W).

Proposition 2.5 is a particular case of the proposition 3.2. Another important particular case is concerning the affine planes.

Definition 3.2 A geometric space (M, Δ) is an *affine plane* if:

(AP1) Through any two distinct points there is exactly one line;
(AP2) (*Parallel axiom*) If g is a line and P is a point outside g then there is exactly one line through P that has no points in common with g;
(AP3) There exist three points that are not on a common line.

Let (M, Δ) be a finite affine plane. It is well known that all the lines have the same number $q \geq 2$ of points. The plane is said to be *of order* q and is noted α_q. The number of elements of α_q is q^2 and the lines through a fixed point are q+1.
Let Δ^* be a set whose elements are union of two non parallel lines and such that any line of Δ is contained in at least one element of Δ^*. We say that Δ^* is "*a set of paired lines*". The set Δ^* has the intersection and non-inclusion properties and so, by proposition 3.2, we have the following:

Proposition 3.3 Let (M, Δ) be a finite affine plane and let Δ^* be a set of paired lines. If W is the closure of Δ^* then (M, W) is a simple cooperative game. Moreover, the blocking sets of (M, Δ) are blocking coalitions of (M, W).

In general we can obtain simple cooperative games from *block designs*, in particular from *Steiner systems*.

Definition 3.3 Let t, k, v be natural numbers such that $2 \leq k \leq v$. A finite geometric space (M, Δ) is a *Steiner system* with parameter t, k, v, noted S(t, k, v), if:

(SS1) Through any t distinct points there is exactly one block;
(SS2) Any block has exactly k points;
(SS3) M has v points.

It is well known that necessary conditions for the existence of a S(t, k, v) is the existence of natural positive numbers b_r, $r \in \{0, 1, \ldots, t-1\}$ such that:

$$b_r \binom{k-r}{t-r} = \binom{v-r}{t-r}, \quad r = 0, 1, \ldots, t-1. \tag{3.1}$$

For any $r \in \{0, 1, \ldots, t-1\}$, b_r is the number of blocks through r fixed points. In particular b_o is the number of all the blocks. For t = k the blocks are the subsets of M with k elements and for k = v there is only a block. We say that S(t, k, v) is *non-degenerate* if $t<k<v$.
For t=2 the blocks are called *lines*. If r is a line and P is a point not incident r, $d=b_1-k$ is the number of lines through P non intersecting r. If d = 0, S(2, k, v) is a projective plane and, if d = 1, it is an affine plane.
We have the following:

Proposition 3.4 Let (M, Δ) be a S(2, k, v) and let $d = b_1 – k$. Then:

- k divides $d-d^2$;
- if r and s are incident lines, the number of lines not incident to $r\cup s$ is

$$\alpha = d^2\text{-}d - (d^2\text{-}d)/k. \tag{3.2}$$

Proof. By (3.1) we have:

$$v = (k+d)(k-1) + 1, \qquad b_0 = k^2+(2d-1)k+(d-1)^2+(d-d^2)/k. \tag{3.3}$$

So b_0 is integer if and only if k divides d^2-d. If r and s are two incident lines for each of the k points of r pass k+d-1 lines different from r, for each of the k-1 points of s not belonging to r pass exactly d lines not intersecting r. So the number of lines intersecting $r\cup s$ is $\delta = k(k+d-1) + d(k-1) + 1= k^2 + (2d-1)k - d +1$
It follows that the lines not intersecting $r\cup s$ are $\alpha = b_0 - \delta = d^2\text{-}d + (d\text{-}d^2)/k$.

If d=0 (projective plane) or d=1 (affine plane) we have α=0. Then we assume d>1. Let I[x] be the minimum integer not inferior to x. By previous proposition, we can find a set ρ_{rs} union of at most $I[(d^2\text{-}d)(k-1)/(2k)] = I[\alpha/2]$ lines, such that any line of S(2, k, v) intersects $r\cup s\cup\rho_{rs}$. Then we have the following:

Proposition 3.5 Let (M, Δ) be a S(2, k, v) with d>1. For any pair (r, s) of incident lines let ρ_{rs} be the union of a minimal set L of lines such that L intersects all the lines not incident to $r\cup s$. Let Δ^* be the family of the sets $r\cup s\cup\rho_{rs}$, with $r, s\in\Delta$. Then Δ^* satisfies (DS1), (DS2) and (DS3) and so generates a set W such that (M, W) is a simple cooperative game. Therefore every element of Δ^* is the union of at most $I[\alpha/2] + 2$ lines.

Example 3.2 For k = 2, a S(2, k, v) is the trivial case of a graph complete with v elements and d = v-3. Any element of Δ^* is the union of exactly $I[(d^2\text{-}d)/4] + 2$ lines. For v = k a S(2, k, v) has only a line and d = 0.
Now we consider the non-degenerate Steiner systems with small values of d>1.
For d = 2, by proposition 3.4, is k =2 and so we don't have non-degenerate Steiner systems.
For d = 3, k is a divisor of 6 different from 2. If k = 3 we have a S(2, 3, 13). It is proved (see [17]) that there exists two non isomorphic S(2, 3, 13). In this case we have α = 4 and the elements of Δ^* are the union of at most 4 lines. If k = 6 we have a S(2, 6, 46) and α = 5. Then there are at most 5 lines in any element of Δ^*.

6. Cooperative games and hyperstructures

In this paragraph we introduce some ideas on the possible relations between cooperative games and some particular commutative weak associative quasi-hypergroups, called “geometric hypergroupoids”. We think that it is a very interesting argument of research.

Definition 4.1 Let M be a non-empty set and let $\wp^*(M)$ be the family of non-empty subsets of M. A *hyperoperation* on M is a function $\sigma: M\times M \rightarrow \wp^*(M)$, such that to every ordered pair (x, y) of elements of M associates a non-empty subset of M, noted $x\sigma y$. The pair (M, σ) is called *hypergroupoid* with *support* M and *hyperoperation* σ.
If A and B are non-empty subsets of M, we put $A\sigma B = \cup\{a\sigma b: a\in A, b\in B\}$.
Moreover, $\forall a, b\in M$, we put, $a\sigma B = \{a\}\sigma B$ and $A\sigma b = A\sigma\{b\}$.

Definition 4.2 A hypergroupoid (M, σ) is said to be:

(SI) a *semihypergroup*, if $\forall x, y, z\in M, x\sigma(y\sigma z) = (x\sigma y)\sigma z$ (*associativity)*;
(QI) a *quasihypergroup*, if $\forall x\in M, x\sigma M = M = M\sigma x$ (*riproducibility*);
(HY) a *hypergroup* if it is both a *semihypergroup* and a *quasihypergroup*;
(CO) *commutative*, if $\forall x, y\in M, x\sigma y = y\sigma x$;
(WA) *weak associative*, if $\forall x, y, z\in M, x\sigma(y\sigma z) \cap (x\sigma y)\sigma z \neq \emptyset$;
(CL) *closed*, if $\forall x, y\in M, \{x, y\}\subseteq x\sigma y$;
(IP) *idempotent*, if $\forall x\in M, x\sigma x=\{x\}$.

Definition 4.3 We say that a hypergroupoid (M, σ) is *geometric* if it is commutative, closed and idempotent.
A geometric hypergroupoid (M, σ) is said to be a *join space* if the following *incidence axiom* holds:

(IA) $\forall a, b, c, d\in M, (\exists x\in M: a\in b\sigma x, c\in d\sigma x) \Rightarrow (\exists y\in M: y\in a\sigma d\cap b\sigma c)$.

Definition 4.4 Let (M, σ) be a geometric hypergroupoid. A geometric space (M, Δ) is said to be “associated to (M, σ)” if Δ is the set of the hyperproducts $a\sigma b$ with $a\neq b$.

Proposition 4.1 Let (M, Δ) be a space of lines. Then there exists only a geometric hypergroupoid (M, σ) with (M, Δ) associated geometric space. Precisely we have:

(GHA) $\forall x\in M, x\sigma x = \{x\}$, $\forall x, y\in M, x\neq y, x\sigma y$ is the line containing $\{x, y\}$.

Example 4.1 Let M be the support of a projective space and, for any $x, y\in M$, with $x \neq y$, put $x\sigma x = \{x\}$ and $x\sigma y$ equal to the line xy. The hypergroupoid (M, σ) is *geometric* and is a *hypergroup*. It is also a *join space*. The incidence axiom is the Veblen-Young axiom.

Example 4.2 Let M be the support of an Euclidean space and, for any x, $y \in M$, with $x \neq y$, put $x\sigma x = \{x\}$ and $x\sigma y$ equal to the segment xy. The hypergroupoid (M, σ) is *geometric*. It is a *hypergroup* and a *join space*, but not a space of lines, and the incidence axiom is the Pasch axiom.

Example 4.3 Let M be the support of an affine space and, for any x, $y \in M$, with $x \neq y$, put $x\sigma x = \{x\}$ and $x\sigma y$ equal to the line xy. (M, σ) is a *geometric* hypergroupoid and a *space of lines* but not a hypergroup. The incidence property is not valid.

In the sequel we don't distinguish from a geometric hypergroupoid and the geometric space associated. By previous considerations we have that the concept of *geometric hypergroupoid* generalizes the one of *space of lines* and so projective spaces, affine spaces and Steiner systems S(2, k, v) are particular cases.
The space of lines that are hypergroups or join spaces, e. g. projective spaces, have very interesting properties. Also join spaces that are not spaces of lines such as the one of example 4.2, have important properties.
Let (M, σ) be a geometric hypergroupoid. We call *blocks of order* 1 the singletons $\{a\}$, $a \in M$, *blocks of order* 2 the hyperproducts $a\sigma b$ with $a \neq b$ and, for $n \geq 3$, we call *blocks of order* n the hyperproducts $H\sigma K$, with H block of order $h<n$ and K block of order n-h, that are not blocks of order less than n.
A block of order n generalizes the concept of subspace of dimension n-1 of a space of lines and so we can generalize the results of the previous paragraphs. We denote by Δ_n the set of hyperproducts of order n and by Λ_n the set of hyperproducts of order $h \leq n$. Moreover we put $\Lambda_0 = \cup_{n \in N} \Lambda_n$. Then by a geometric hypergroupoid (M, σ) we obtain the geometric spaces (M, Δ_n), with n belonging to a subset of N, finite if M is finite. We have also the geometric spaces (M, Λ_n), $n \in N_0$. In particular $\Delta = \Delta_2$.
Suppose M is a finite set of players. From each of the geometric spaces (M, Δ_n), $n>1$, we can obtain a cooperative game with particular properties dependent on the algebraic structure of (M, σ).
A possible economic interpretation of a block B of order n is as the set of the players disposed to form a coalition because influenced by the set of players $\{a_1, a_2, \ldots, a_n\}$ that generates the block. If (M, σ) is not a hypergroup such coalition depend on the process of aggregation of the n players. Another possible interpretation of the block B is as a company controlled by $\{a_1, a_2, \ldots, a_n\}$.
Finally, we think that many other economic interpretations and geometric properties (e. g. blocking coalitions) depends on the algebraic structure of (M, σ) and we intend study these questions in a very near paper.

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