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# QUASI-CONCAVITY PROPERTY OF MULTIVARIATE DISTRIBUTION FUNCTIONS

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**SUNTO** - Si individuano condizioni sufficienti per garantire le proprietà di quasi-concavità per le funzioni di ripartizione multidimensionali. Si prova che la maggior parte delle distribuzioni ellittiche hanno funzione di ripartizione quasi-concava. Attraverso la caratterizzazione della copula altre famiglie di distribuzioni sono individuate. Alcune disuguaglianze probabilistiche sono infine messe in luce.

**ABSTRACT** - Conditions under which multivariate distribution functions are quasi-concave, are explored. We prove that many elliptically contoured distributions have quasi-concave distribution functions. By some characterisations of the copula, further classes are provided. A few probability inequalities are also suggested.

**KEYWORDS** - Elliptically contoured distributions, Archimedean copula, quasi-concavity conditions.

## 1. INTRODUCTION**

Many studies have been devoted to search for concavity properties of $n$-dimensional ($n \geq 2$) distribution functions (see TONG [17], IYENGAR-

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TONG [7], MARSHALL-OLKIN [10], TIBILETTI [14]).
In this note we restrict our attention to the quasi-concavity property. It is known that not all the $n$-dimensional $(n \geq 2)$ distribution functions are quasi-concave (see TIBILETTI [14]). The aim of the present paper is to point out families of distributions which satisfy this property and to extend TIBILETTI [14]'s results.
The plan of this paper is as follows. In section 1 we prove that the distribution functions of a number of elliptical random vectors are quasi-concave. In Section 2 we consider distribution functions with elliptical and Archimedean copulas. Under some assumptions on one-dimensional marginals, their quasi-concavity is showed. In Section 3 a few probability inequalities for differentiable distribution functions are provided.

## 2. NOTATION AND PRELIMINARIES

Given the random vector $X = \{X_1, ..., X_n\}$, we denote by

$$F(x) = \Pr\left[\bigcap_{i=1}^{n}(X_i \leq x_i)\right]$$

the joint distribution function of $X$ and by

$$F_i(x_i) = \Pr(X_i \leq x_i)$$

the one-dimensional marginal associated to the random variable $X_i$.
Our purpose is to investigate whether $F$ is quasi-concave, *i.e.*,

$$F(\alpha x + (1-\alpha)y) \geq Min[F(x); F(y)] \tag{1}$$

holds for all $x, y \in \Re^n$ and $0 \leq \alpha \leq 1$.
Condition (1) is equivalent to requiring that upper-level sets of $F$

$$U_q = \{x \in \Re^n : F(x) \geq q\}$$

are convex for all real $q$.

## 3. QUASI-CONCAVITY FOR ELLIPTICAL DISTRIBUTION FUNCTIONS

Let us recall the definition of elliptically contoured distribution.

**Definition 1:** Let $\Sigma$ denote a $nxn$ positive definite matrix A random vector $X$ is said to have an *elliptically contoured distribution* if its density function is of the form (with respect to Lebesgue measure)

$$f(x) = |\Sigma|^{-1/2} g\left(x'\Sigma^{-1}x\right), x \in \Re^n \quad (2)$$

where $g$ satisfies

$$\int_0^\infty r^{n-1} g\left(r^2\right) dr < \infty$$

$g$ is also called a density generator.

Multivariate normal distributions are the best-known ellipticals. For these, $g(s) \propto \exp(-s/2)$. Many other types are also possible. For example, the multivariate student $t$ distribution with $\nu$ of freedom is characterised by $g(s) \propto (\nu + s)^{-(n+\nu)/2}$.

It is worth pointing out that any linear combination of elliptical variates is still elliptical (more details are given in FANG *et al.* [3]), and in virtue of this property the use of these distributions is very common in the Portfolio Theory (see, for example, INGERSOLL [8]).

Below we will confine our attention to the class of elliptical distributions such that

a. the support of the density $f$ is unbounded;
b. the density generator $g$ is a continuous function on $\Re$;
c. the correlation matrix exists (consequently, it is proportional to $\Sigma$).

We need of the following proposition to prove Theorem 1.

**Proposition 1.** *Let* $T = (t_{ij})$ *be the correlation matrix of X. If* $t_{ij}$ = *max correlation between* $(X_i, X_j)$ *for all* $i, j = 1,..n$*, then F is quasi-concave.*

**Proof.** Since $t_{ij}$ = max correlation $(X_i, X_j)$ for all $i, j = 1,..n$, each of $X_1, ..., X_n$ is almost surely a strictly increasing function of any of the others.

A property of distribution functions leads to

$$F(x) = Min\left[F_1(x_1),\dots,F_n(x_n)\right],$$

(for $n = 2$ the result is due to FRÉCHET [4], for $n > 2$ see SCHWEIZER-SKLAR [12]). Thus, it follows that

$$U_q = \left\{x \in \Re^n : x_1 \geq F_1^{-1}(q),\dots,x_n \geq F_n^{-1}(q)\right\},$$

where $F_i^{-1}, i = 1,..,n$ are the quantile functions. Since $U_q$ is convex for all $q$, $F$ turns out to be quasi-concave and this completes the proof.

**Theorem 1.** *Let X have an elliptically contoured distribution with assumptions* a.b.c. *Then F is quasi-concave.*

**Proof.** Define the matrix

$$S_\vartheta = (1-\vartheta)R + \vartheta\mathrm{T},\ \vartheta \in [0,1]$$

where $R = (r_{ij})$ is the positive definite correlation matrix of $F$ and $\mathrm{T} = (t_{ij})$ is a matrix with entries $t_{ij}$ = max correlation $(X_i, X_j)$ for all $i, j = 1,..n$. Construct the distribution function $F_\vartheta$ having the same density generator of $F$ and correlation matrix $S_\vartheta$. Let

$$A = \left\{\vartheta \in [0,1]: F_\vartheta \text{ is } \textit{not} \text{ quasi-concave}\right\}.$$

Suppose *ad absurdum* that $A$ be not empty. Denote

$$\vartheta^* = \sup A.$$

By Proposition 1., it results $\vartheta^* <$ . Since $F_{\vartheta^*}$ is not quasi concave, there exist $x, y \in \Re^n$, $\alpha \in (0,1)$ (depending on $\vartheta^*$),such that

$$Min\left[F_{\vartheta^*}(x); F_{\vartheta^*}(y)\right] = q. \tag{3}$$

and

$$F_{\vartheta^*}(\alpha x + (1-\alpha)y) < q.$$

Note that for all $\vartheta > \vartheta^*$, we have

$$s_{ij} \le s_{ij}^* \text{ and } s_{ij} < s_{ij}^* \text{ holds for some } i,j \tag{4}$$

where $S_\vartheta = (s_{ij})$ and $S_{\vartheta^*} = (s_{ij}^*)$. In virtue of the assumption a and inequalities (4), we can apply a generalisation of Slepian's inequality due to DAS GUPTA, *et al.* [2] (as reported in TONG [16]), which gets[1]

$$F_\vartheta > F_{\vartheta^*} \text{ for all } \vartheta > \vartheta^*,$$

so we have

$$Min\left[F_{\vartheta^*}(x); F_{\vartheta^*}(y)\right] > q, \text{ for all } \vartheta > \vartheta^*.$$

Since $F_\vartheta$ is quasi-concave, it results

$$F_\vartheta(\alpha x + (1-\alpha)y) > q, \text{ for all } \vartheta > \vartheta^* \tag{5}$$

Combining (3) and (5), it follows that $h(\vartheta) = F_\vartheta(\alpha x + (1-\alpha)y)$ is *not* a continuous function in $\vartheta^*$. Nevertheless, assumption b guarantees the continuity of $h(\vartheta)$ on $[0,1]$, which contradicts the above statement.

## 4. FURTHER CLASSES OF QUASI-CONCAVE DISTRIBUTION FUNCTIONS

In this section we propose a way to characterise other quasi-concave distributions. These results are related to the property of the copula (the notion was introduced by SKLAR [13] , for a historical overview , the reader can refer to SCHWEIZER [11]). Copulas are functions that link multivariate distributions to their one-dimensional marginals $F_i, i = 1,..,n$, then there exists a continuous copula $C$ such that

$$F(x) = C(F_1(x_1),\dots,F_n(x_n)), \text{ for all } x \in \Re^n,$$

[1] Strictly inequality is guaranteed by assumption a. (see, for example, Tong [16], Th. 4.3.6, page 74).

where $C$ is a distribution function concentrated on the unit $n$-cube $I^n$, $I=[0,1]$ with uniform marginals.

Our investigation will be carried out in two steps: (1) we point out classes of quasi-concave copulas, (2) we specify additional conditions on one-dimensional marginals to guarantee the quasi-concavity of $F$.

## 4.1 ELLIPTICAL AND ARCHIMEDEAN COPULAS

The result reached from Theorem 1 suggests a definition of a new class of copulas denoted by $\Psi$.
**Definition 2.** Let $C$ be a copula. If there exist one-dimensional marginals $F_1,\dots,F_n$ such that

$$F(x)=C\big(F_1(x_1),\dots,F_n(x_n)\big),\ \text{for all } x\in\Re^n$$

is an elliptical distribution function which satisfies *a.-b.-c.* then $C\in\Psi$.

We have the following :

**Proposition 2.** *If* $C\in\Psi$, *then C is a quasi-concave function.*
The proof is omitted , because is quite similar to that of Theorem 1 (see also TIBILETTI [15].

We consider now the Archimedean copulas. We recall their definition.
**Definition 3.** Let $C$ be continuous copula such that

$$C(u)=\varphi^{-1}\big[\varphi(u_1)+\dots+\varphi(u_n)\big]$$

for some real function $\varphi$. Then $C$ is called an *Archimedean copula.*

Note that in order that $C$ be a copula, $\varphi^{-1}$ has to be completely monotone, *i.e.,*

$$(-1)^k\frac{d^k\varphi^{-1}(t)}{d^kt}\geq 0,\ \text{for all } k\geq\ , \tag{6}$$

(see SCHWEIZER -SKLAR [12], Theorem 6.3.6).

**Proposition 3.** *If C is an Archimedean copula, then it is quasi-concave.*

**Proof.** From condition (6), it follows that $\varphi$ is convex. Thus, $H(u)=\varphi(u_1)+\dots+\varphi(u_n)$ is convex, too. Since $\varphi^{-1}$ is nonincreasing, the composite function $C(u)=\varphi^{-1}(H(u))$ turns out to be quasi-concave (MANGASARIAN [9]).

Distributions with Archimedean copulas have been thoroughly investigated in the literature and many families have been defined. For instance, we recall those of Gumel, Cook-Johnson, Frank, Plankett and Mardia (for details see GENEST [5], GENEST-MacKAY [6]).

### 4.2 ADDITIONAL CONDITIONS ON ONE-DIMENSIONAL MARGINALS

Quasi-concavity of the copula does *not* guarantee the quasi-concavity of the distribution function (for a counter-example see TIBILETTI [14]). Additional conditions on marginals have to be added. We list two of them.

**Proposition 4.** (Tibiletti [14]) *Let $F$ be a distribution function with a quasi-concave copula. If its one-dimensional marginal $F_i$ is concave on the interval $[a_i,b_i], i=1,..,n$, then $F$ is quasi-concave on $[a_1,b_1]\times\ldots\times[a_n,b_n]$.*

**Proposition 5.** *Let $F$ be a distribution function with copula $C$ and continuous marginals $F_i$, $i=1,..,n$. If there exist real functions $\Phi_i, i=1,..,n$ such that $H=C(\Phi_1,\ldots,\Phi_n)$ is an elliptical distribution function, and $\Phi_i^{-1}(F_i)$ is concave*[2] on $[a_i,b_i], i=1,..,n$ *then $F$ is quasi-concave on $[a_1,b_1]\times\ldots\times[a_n,b_n]$.*

**Proof.** Note that

$$F=C(F_1,\ldots,F_n)=C\left(\Phi_1\left(\Phi_1^{-1}F_1\right),\ldots,\Phi_n\left(\Phi_n^{-1}F_n\right)\right).$$

From Theorem 1, $H=C(\Phi_1,\ldots,\Phi_n)$ is quasi-concave. Since $\Phi_i^{-1}(F_i), i=1,..,n$ are concave, it follows that $F$ is quasi-concave (see, for example, AVRIEL *et al.* [1]).

## 5. A FEW PROBABILITY INEQUALITIES

Consider now differentiable distributions[3]. Quasi-concavity of $F$ yields some inequalities involving its marginals and conditional distributions. Denote by

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[2] We denote by $\Phi_i^{-1}$ a quasi-inverse of $\Phi_i$.

[3] Obviously, if the density is continuous, then its distribution function is once-differentiable.

$f_i$ the marginal density of $X_i$, by $f(.|x_i)$ and $F(.|x_i)$ the conditional density and distribution function of $X = (X_1, \dots, X_n)$ given $X_i = x_i$ $(i = 1, .., n)$. It is straightforward to prove the following:

**Proposition 6.** *Let $F$ be once-differentiable on the open convex set $A \subseteq \Re^n$. Then $F$ is quasi-concave iff for every $x, y \in A$*

$$F(x) \geq F(y) \text{ implies that } (y - x)\nabla F(x) \geq 0 \tag{7}$$

where $\nabla F$ is the gradient of $F$ and

$$\frac{\partial F}{\partial x_i} = f_i(x) F_i(x_1, \dots, x_{i-1}, x_{i+1}, .., x_n | x_i). \tag{8}$$

**Proof.** Condition (7) is obvious, see, for example, AVRIEL *et al.* [1].
And (8) comes out by straightforward calculations.

Condition (7) is not trivial to verify. Therefore, its use for checking whether or not $F$ is quasi-concave could not be easy. Alternatively, it could be profitable to use (7) in the reversed way. In fact, if we know that $F$ is quasi-concave then (7) may produces some not trivial inequalities.
For sake of simplicity, let $n = 2$. The right-hand side of (7) becomes

$$(y_1 - x_1) f_1(x_1) F(x_2 | x_1) + (y_2 - x_2) f_2(x_2) F(x_1 | x_2) \geq 0 \tag{9}$$

For example, if $F$ is elliptical (9) can be rewritten in an explicit form (see, for example, TONG [17] for elliptical marginal and conditional probability for $n \geq 2$). An inequality which stems from (9) is stated below.

**Proposition 7.** *Let $F$ be elliptical and satisfy* a.-b.-c. *If $X_i$ has median $\mu_i$ and variance $\sigma_i^2$, $i=1,2$, then*

$$F(\mu) \geq F(x) \text{ implies } \frac{\mu_1 - x_1}{\sigma_1} + \frac{\mu_2 - x_2}{\sigma_2} \geq 0 \tag{10}$$

*where $\mu = (\mu_1, \mu_2)$ and $x = (x_1, x_2)$.*
**Proof.** Straightforward calculations show that
$f_i(\mu_i) = g(0)/\sigma_i$ and $F(\mu_j | \mu_i) = F(\mu_i | \mu_j)$
where $g$ is the density generator of $F$. By Theorem 1, $F$ is quasi-concave, therefore (9) holds. Imposing $y = \mu$, our claim comes out.

**Remark.** Condition (10) is trivial whenever $\mu_1 \geq x_1$ and $\mu_2 \geq x_2$. On the contrary, if $\mu_1 \geq x_1$ and $\mu_2 \leq x_2$, it gives a condition on the "distance" between the vector $\mu$ of the medians and the vector $x$ (see TIBILETTI [15]).

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