Ratio Mathematica 12
(1997), 66-72

OVER THE CONSTRUCTION OF AN HYPERSTRUCTURE OF QUOTIENTS FOR A MULTIPLICATIIVE HYPERRING

R. Procesi* - R. Rota**

Summary - In this paper we construct a weak hyperfield of quotients for a class of multiplicative hyperrings.

First of all we want to recall some algebraic definitions that will be used through the paper.

An hyperring $(A, \oplus, \cdot)$ is a set A with an hyperoperation $\oplus$ and a product $\cdot$ such that the following properties hold:

i) $\forall a,b,c \in A : a \oplus (b \oplus c)=(a \oplus b) \oplus c$,

ii) $\forall a,b \in A : a \oplus b=b \oplus a$,

iii) $\exists\ 0\in A / \forall a \in A : 0 \oplus a=a \oplus 0=a$,

iv) $\forall a \in A\ \exists!\ a' \in A : a \oplus a' \ni 0, (a'=-a)$,

v) $\forall a,b,c\in A / a \in b \oplus c \Rightarrow c\in a-b\ (c\in a \oplus b')$.

vi) $\forall a,b,c\in A : (a \cdot b) \cdot c=a \cdot (b \cdot c)$,

vii) $\forall a,b,c\in A : (a \oplus b) \cdot c=a \cdot c \oplus b \cdot c$,

viii) $\forall a,b,c \in A : a \cdot (b \oplus c)=a \cdot b \oplus a \cdot c.$,

ix) $\forall a \in A : a \cdot 0=0 \cdot a=0$,

We recall that $(A, \oplus)$ satisfying i),ii),iii),iv) and v) is called canonical hypergroup.

Let us observe ([3]) that axiom v) is equivalent to v)' and also to v)":

v)' $\forall a,b\in A : -(a \oplus b)= -a - b$,

* Facoltà di Scienze M.F.N. - Università La Sapienza Roma
** Facoltà di Ingegneria - Università di Roma Tre

v)" $\forall$ a,b,c,d $\in$A : (a $\oplus$ b) $\cap$(c $\oplus$ d) $\neq\emptyset \Rightarrow$ (c – a) $\cap$(b – d) $\neq\emptyset$.

An hyperring A is called hyperfield if (A*, ·) is a group, where A*=A\{0}.

A multiplicative hyperring (A,+, •) is an abelian group (A,+) together with an hyperproduct satisfying the following properties:

i) $\forall$ a,b,c $\in$A : a •(b •c)=(a •b) •c ;
ii) $\forall$ a,b,c$\in$A : (a + b) •c $\subseteq$a •c + b •c ;
iii) $\forall$ a,b,c $\in$A : a •(b + c) $\subseteq$a• b+ a •c ;
iv) $\forall$ a,b$\in$A : (–a) •b=a •(– b)= –(a •b) .

If a multiplicative hyperring satisfies, instead of properties ii) and iii), the following ones:

ii)' $\forall$ a,b,c$\in$A : (a +b)•c=a •c + b•c
iii)' $\forall$ a,b,c$\in$A : a •(b +c)=a •b +a•c

then (A, +, •) is called strongly distributive. Moreover (A,+,•) is strongly left (right) distributive if ii)' (iii)') holds.

In [2] J. Mittas studied the possibility of immersion for an hyperring in an hyperfield and constructed the hyperfield of quotients for a particular hyperring. In this paper we want to solve the analogous problem for a multiplicative hyperring.

Let (A,+,•) be a multiplicative, strongly distributive, commutative hyperring; from now on we will write ab instead of a•b and, for any two sets X and Y, X $\approx$Y if and only if X$\cap$Y$\neq\emptyset$. We suppose that the hyperring A satisfies the following properties: i) if ab$\approx$0c, a$\neq$0 $\Rightarrow$ b=0 ; ii) $\forall$ X,Y,Z,W$\in$*P**(A)= =*P*(A)\{$\emptyset$} / XY$\approx$ZW $\Rightarrow$ ($\forall$ x $\in$X and $\forall$ w$\in$W $\exists$ y$\in$Y and z $\in$Z / xy $\approx$zw).

For this structure we can prove the following lemmas:

I.- For any a,b$\in$A if 0$\in$ab and a$\neq$0 then b=0.

Proof. - Since A is strongly distributive 0$\in$0c $\forall$ c$\in$A, thus from i) b=0 follows.

II.- For any X,Y$\in$*P(A) and for any d$\in$A\{0}, dX$\approx$dY $\Leftrightarrow$ X$\approx$Y.**

Proof. - Since dX=$\bigcup$dx, x$\in$X, and dY=$\bigcup$dy, y$\in$Y, thus, from the hypothesis, there exists z$\in$dX$\cap$dY, that is z$\in$dx and z$\in$dy, for some x$\in$X and $y \in Y$; this implies, from the strong distributivity, $0 \in d(x-y)=dx-dy$ and, since $d \neq 0$, $x=y$, thus $X \approx Y$. The inverse implication is obvious.

III.- For any $X,Y \in P^*(A)$ and for any $d_i \in A \setminus \{0\}$, $(X+Y)d_1 \bullet \bullet \ldots \bullet d_k = (Xd_1 \bullet \ldots \bullet d_k)+(Yd_1 \bullet \ldots \bullet d_k)$.

Proof. We can prove the lemma for k=1; the general case will follow as a consequence of the associative property. Now $(X+Y)d=\{z \;/\; z \in (x+y)d, x \in X, y \in Y\}=\{z \;/\; z \in xd+yd, x \in X, y \in Y\}=Xd+Yd$.

Denoting by A^* the set $A \setminus \{0\}$ and by H the set $A \times A^*=\{(a,b) \;/\; a \in A, b \in A^*\}$, we define in H the following relation $(a,b)\,\rho(c,d) \Leftrightarrow ad \approx bc$; for this relation the following properties hold:

IV.- $(a,b)\rho(0,d) \Leftrightarrow a=0$.

Proof. - Since $(a,b)\rho(0,d) \Leftrightarrow ad \approx 0b$ then, from i) and $d \neq 0$, $a=0$ follows. Vice-versa, since $0 \in 0d \cap 0b$, then $(0,b)\rho(0,d)$, $\forall\, b,d \in A^*$.

V.- For any $a,b,c,d,f \in A$, if $ad \approx bc$, then $adf \approx bcf$.

Proof. - If $z \in ad \cap bc$ then $zf \subseteq adf$ and $zf \subseteq bcf$; thus $adf \approx bcf$.

VI.- ρ is an equivalence relation.

Proof. - Reflexivity and symmetry are for ρ immediate consequence of commutativity in A. As for transitivity let $(a,b)\rho(c,d)$ and $(c,d)\rho(e,f)$; then $ad \approx bc$ and $cf \approx de$ that is there exist $z,w \in A$ such that $z \in ad \cap bc$, $w \in cf \cap de$, thus $zw \subseteq adcf \cap bcde$. So $afdc \approx bedc$ and, since $d \neq 0$, from proposition II we obtain $afc \approx bec$; now, if $c \neq 0$, again from proposition II $af \approx be$, while if $c=0$ then, from proposition IV, $a=e=0$. In both cases it results $(a,b)\rho(e,f)$ as requested.

Let now $K=H/\rho$, we want to define in K two commutative hyperoperations $\oplus$ and $\otimes$ in such a way that the following conditions hold:

1) $(K,\oplus)$ is a canonical hypergroup;
2) $\forall\, x,y,z \in K : x\otimes(y\otimes z)=(x\otimes y)\otimes z$;
3) $\forall\, x,y,z \in K : x\otimes(y\oplus z)\subseteq(x\otimes y)\oplus(x\otimes z)$;
4) $\forall\, x,y \in K : -(x\otimes y)=(-x)\otimes y=x\otimes(-y)$;
5) $\exists\, 1 \in K \;/\; \forall\, x \in K : x \in x\otimes 1$;
6) $\forall\, x \in K \setminus \{0\}\ \exists\, y \in K \setminus \{0\} \;/\; 1 \in x \otimes y$.

For any $[(a,b)]$, $[(c,d)] \in K$ let us define $[(a,b)]\oplus[(c,d)]=[(ad+bc,bd)]=\{[(s,t)] \;/\; s \in ad+bc, t \in bd\}$ and $[(a,b)]\otimes[(c,d)]=[(ac,bd)]=\{[(s,t)] \;/\; s \in ac, t \in bd\}$; first

In order to prove the existence of a zero element in $(K,\oplus)$ we first recall that, as a consequence of proposition IV, $[(0,y)]=\{(0,d)\ /\ d\in A^*\}$, moreover $[(a,b)]\oplus[(0,y)]=\{[(s,t)]\ /\ s\in ay+b0,\ t\in by\}=[(a,b)]$; in fact, since $(by)a\subseteq \subseteq ayb+b0b$, we have $bya\approx(ay+b0)b$ and, from ii), $\forall\ s\in ay+b0\ \forall\ t\in by$ it results $sb\approx ta$. This proves that $[(0,y)]$ is a zero for $\oplus$; let us now prove that it is unique. To do this let $[(x,y)]\in K$ such that $[(a,b)]\oplus[(x,y)]=[(a,b)]$, that is $\forall (s,t)\in H$ such that $s\in ay+bx$, $t\in by$ then $sb\approx ta$. From this it follows that $ayb+bxb\approx bya$, then there exists $z\in(ayb+bxb)\cap bya$, $z=z'+z''$, $z'\in ayb$ and $z''\in bxb$, $z\in bya$; that is $z''\in bya-bya$ or $(bya-bya)\cap bxb\neq\emptyset$. Thus $((a-a)y)b\approx \approx(bx)b$ and, from proposition II, $0y\approx bx$ with $b\neq 0$; because of i) $x=0$ and the uniqueness is proved.

We want now to verify the existence and uniqueness, for any $[(a,b)]\in K$, of an element $[(z,w)]\in K$ such that $[(a,b)]\oplus[(z,w)]\ni[(0,y)]$. First of all we observe that $[(a,b)]\oplus[(-a,b)]=\{[(s,t)]\ /\ s\in ab-ab,\ t\in bb\}$ and this set trivially contains $[(0,t)]$. As for the uniqueness let $[(z,w)]\in K$ such that $[(0,y)]\in[(a,b)]\oplus \oplus[(z,w)]=\{[(s,t)]\ /\ s\in aw+bz,\ t\in bw\}$; then $0\in aw+bz$ or $0=u+v$, $u\in aw$, $v\in bz$ that is $v=-u\in-(aw)=(-a)w$ which implies $(-a)w\approx bz$ and this means $[(z,w)]=[(-a,b)]$.

Finally we must prove condition v) in the definition of canonical hypergroup which is equivalent to the following condition: $-([(a,b)]\oplus[(c,d)])=(-[(a,b)])\oplus \oplus(-[(c,d)]=[(-a,b)]\oplus[(-c,d)]$.

To prove such equality, we first must prove that $-([(a,b)]\oplus[(c,d)])\subseteq[(-a,b)]\oplus \oplus[(-c,d)]$; to do this let $[(-s,t)]$ such that $s\in ad+bc$, $t\in bd$. Thus $-s\in-(ad+bc)= =-ad-bc$ that is $-s=(-z)+(-w)$, $z\in ad$, $w\in bc$; from this, since $-(ad)=(-a)d$ and $-(bc)=b(-c)$, $[(-s,t)]\in[(-a,b)]\oplus[(-c,d)]$ follows. Similarly the inverse inclusion can be proved.

From all that has been proved we obtain the requested result. Moreover we can prove the following proposition:

X.- In $(K,\oplus,\otimes)$ the following hold:

α) $\forall\ x,y,z\in K : x\otimes(y\otimes z)=(x\otimes y)\otimes z$;

β) $\exists\ 1\in K\ /\ \forall\ x\in K : x\in x\otimes 1$;

γ) $\forall\ x,y,z\in K : x\otimes(y\oplus z)\subseteq(x\otimes y)\oplus(x\otimes z)$;

δ) $\forall\ x,y\in K : -(x\otimes y)=(-x)\otimes y=x\otimes(-y)$;

ε) $\forall\ x\in K$, x different from zero, $\exists\ y\in K$, y different from zero, such that $1\in x\otimes y$.

Proof. - $\forall\ [(a,b)],[(c,d)],[(e,f)]\in K$ we have $([(a,b)]\otimes[(c,d)])\otimes[(e,f)]=\{[(s,t)]\ /\ s\in ac,\ t\in bd\}\otimes[(e,f)]=\{[(x,y)]\ /\ x\in se,\ y\in tf\}=\{[(x,y)]\ /\ x\in(ac)e,\ y\in(bd)f\}$;

of all we observe that, because of i), t is always different from zero since b and d are different from zero.

Let us now prove that ⊕ and ⊗ are well defined by proving the following two propositions.

VII-. If (a',b')∈[(a,b)] and (c',d')∈[(c,d)] then [(ad+bc,bd)]= =[(a'd'+b'c',b'd')].

Proof. - In order to prove the requested result we must prove that for any [(s',t')]∈[(a'd'+b'c',b'd')] there exists [(s,t)]∈[(ad+bc,bd)] such that (s',t')ρ(s,t) and vice-versa. If (a',b')∈[(a,b)] and (c',d')∈[(c,d)] then a'b≈b'a and c'd≈d'c from which we can have (a'd'+b'c')bd≈b'd'(ad+bc); in fact, as for proposition V, we have a'bdd'≈b'add' and c'dbb'≈d'cbb' from which the set a'bdd'+c'dbb' intersects the set b'add'+d'cbb', by proposition III we have, as requested, (a'd'+ b'c')bd≈b'd'(ad+bc). Now let (s',t')∈H such that s'∈a'd'+ b'c' and t'∈b'd'; from (a'd'+b'c')bd≈b'd'(ad+bc) and ii) there must exist (s,t)∈H with s∈ad+bc and t∈bd with the condition s't≈t's. Thus (s',t')ρ(s,t). Similarly it can be proved that for any [(s,t)]∈[(ad+bc,bd)] there exists [(s',t')]∈[(a'd'+b'c',b'd')] such that (s',t')ρ(s,t); thus ⊕ is well defined.

VIII.- If (a',b')∈[(a,b)] and (c',d')∈[(c,d)] , then [(ac,bd)]= =[(a'c',b'd')].

Proof. - As in proposition VII we prove that for any [(s',t')]∈[(a'c',b'd')] there exists [(s,t)]∈[(ac,bd)] such that (s',t')ρ(s,t) and vice-versa. If (a',b')∈[(a,b)] and (c',d')∈[(c,d)] then a'b≈b'a and c'd≈d'c that is a'bc'd≈b'ad'c or a'c'bd≈b'd'ac; from ii) the requested result follows.

At this point we can study the properties of hyperstructure (K,⊕,⊗).

IX.- (K,⊕) is a canonical hypergroup.

Proof. As for associativity we have ([(a,b)]⊕[(c,d)])⊕[(e,f)]={[(s,t)] / s∈ad+bc, t∈bd}⊕[(e,f)]={[(z,w)] / [(z,w)]∈[(s,t)]⊕[(e,f)]}={[(z,w)] / z∈sf+te, w∈tf, s∈ad+bc, t∈bd}={[(z,w)] / z∈(ad+bc)f+(bd)e, w∈(bd)f}, while [(a,b)]⊕([(c,d)]⊕[(e,f)])=[(a,b)]⊕{[(u,v)] / u∈cf +de, v∈df}={[(p,q)] / [(p,q)]∈[(a,b)]⊕[(u,v)]}={[(p,q)] / p∈av+bu, q∈bv, u∈cf+de, v∈df}={[(p,q)] / p∈a(df)+b(cf+de), q∈b(df)} and, from associativity and distributivity in A, the two sets are equal. Commutativity of ⊕ follows from commutativity of + and • in A as can be easily proved.

moreover $[(a,b)]\otimes([(c,d)]\otimes[(e,f)])=[(a,b)]\otimes\{[(u,v)] \ / \ u\in ce,\ v\in df\}=\{[(x,y)] \ / \ x\in au,\ y\in bv\}=\{[(x,y)] \ / \ x\in a(ce),\ y\in b(df)\}$.

Thus associativity for $\otimes$ follows from the analogous property in $(A,\bullet)$. Similarly commutativity in $(K,\otimes)$ follows from commutativity in $(A,\bullet)$.

As for the existence of a unity let us consider the element $[(x,x)]\in K$, $x\neq 0$; obviously $[(x,x)]=[(y,y)]$ $\forall$ $y\in A^*$. For this element and for any other element $[(a,b)]\in K$ it results: $[(a,b)]\otimes[(x,x)]=\{[(s,t)] \ / \ s\in ax,\ t\in bx\}$. At this point we observe that, since $\{a\}bx=\{b\}ax$, from ii) we have that, $\forall$ $s\in bx$, $t\in ax$ such that $as\approx bt$ and this implies that $[(a,b)]\in[(a,b)]\otimes[(x,x)]$.

Let us now prove property γ); $\forall$ $[(a,b)],[(c,d)],[(e,f)]\in K$ it results $[(a,b)]\otimes([(c,d)]\oplus[(e,f)])=[(a,b)]\otimes\{[(s,t)] \ / \ s\in cf+de,\ t\in df\}=\{[(x,y)] \ / \ x\in as,\ y\in bt\}=\{[(x,y)] \ / \ x\in a(cf+de),\ y\in b(df)\}=\{[(x,y)] \ / \ x\in acf+ade,\ y\in bdf\}$ and $([(a,b)]\otimes[(c,d)])\oplus([(a,b)]\otimes[(e,f)])=\{[(s,t)] \ / \ s\in ac,\ t\in bd\}\oplus\{[(u,v)] \ / \ u\in ae,\ v\in bf\}=\{[(z,w)] \ / \ z\in sv+tu,\ w\in tv\}=\{[(z,w)] \ / \ z\in acbf+bdae,\ w\in bdbf\}=$ $=\{[(z,w)] \ / \ z\in b(acf+ade),\ w\in b(bdf)\}$. Since $[(x,y)]\otimes[(b,b)]\ni[(x,y)]$, the previous set contains $\{[(x,y)] \ / \ x\in acf+ade,\ y\in bdf\}$, that is:

$$[(a,b)]\otimes([(c,d)]\oplus[(e,f)])\subseteq[(a,b)]\otimes[(c,d)]\oplus[(a,b)]\otimes[(e,f)] .$$

Moreover, in order to prove δ), we have $-([(a,b)]\otimes[(c,d)])= -\{[(s,t)] \ / \ s\in ac,\ t\in bd\} = \{[(-s,t)] \ / \ -s\in-(ac),\ t\in bd\} = \{[(-s,t)] \ / \ -s\in(-a)c,\ t\in bd\}=\{[(-s,t)] \ / \ -s\in a(-c),\ t\in bd\}=(-[(a,b)])\otimes[(c,d)]=[(a,b)]\otimes(-[(c,d)])$.

Finally, $\forall$ $[(a,b)]\in K$, $a\neq 0$, it results $[(a,b)]\otimes[(b,a)]=\{[(s,t)] \ / \ s\in ab,\ t\in ba\}\ni$ $\ni[(x,x)]$, $x\neq 0$, and this ends the proof.

We want now to prove, under particular hypothesis, that there exists a substructure of K which is weakly isomorphic to A . To do this let us remember that an element $1\in A^*$ is called weak unity if and only if $\forall$ $x\in A$ it results $1\bullet x\ni x$ ([4]). Then, if $(A,+,\bullet)$ is a multiplicative, strongly distributive, commutative hyperring such that: i) if $a\bullet b\approx 0\bullet c$, $a\neq 0$ $\Rightarrow$ $b=0$; ii) $\forall$ $X,Y,Z,W\in P^*(A)$ / $XY\approx ZW$ $\Rightarrow$ ($\forall$ $x\in X$ and $\forall$ $w\in W$ $\exists$ $y\in Y$ and $z\in Z$ / $x\bullet y\approx z\bullet w$); iii) $\exists$ $1\in A$ / $\forall$ $x\in A$ it results $1\bullet x\ni x$, it is possible to prove the following two results:

XI.-The map $\varphi : A \longrightarrow K$ defined as $\varphi(a)=[(a,1)]$, $\forall$ $a\in A$, is a weak monomorphysm.

Proof. - Because of proposition II φ is injective; in fact, if $[(a,1)]=[(b,1)]$, then $a\bullet 1\approx b\bullet 1$ and, from proposition II, this implies $a=b$. Moreover $\varphi(a+b)=[(a+b,1)]$ while $\varphi(a)\oplus\varphi(b)=[(a,1)]\oplus[(b,1)]=\{[(s,t)] \ / \ s\in a\bullet 1+ b\bullet 1,\ t\in 1\bullet 1\}$ and, since $a+b\in a\bullet 1+b\bullet 1$ and $1\in 1\bullet 1$, then $\varphi(a+b)\in\varphi(a)\oplus\varphi(b)$.

Finally it results $\varphi(a\bullet b)=\{\ \varphi(x)\ /\ x\in a\bullet b\}=\{[(x,1)]\ /\ x\in a\bullet b\}\subseteq\varphi(a)\otimes\varphi(b)=$ $=[(a,1)]\otimes[(b,1)]=\{[(s,t)]\ /\ s\in a\bullet b,\ t\in 1\bullet 1\}$.

XII.- Each element of K belongs to a product $x\otimes y$ where $x\in Im\varphi$ and y is such that there exists $y'\in Im\varphi$: $y\otimes y'$ contains $[(z,z)]$.

Proof. - For $[(a,b)]\in K$ it results $[(a,b)]\in[(a,1)]\otimes[(1,b)]=\{[(s,t)]\ /\ s\in a\bullet 1,$ $t\in 1\bullet b\}$.

As a consequence of what has been proved we will call $(K,\oplus,\otimes)$ the weak hyperfield of quotients for $(A,+,\bullet)$.

REFERENCES

1 P. CORSINI. *Sur les homomorphismes d'hypergroupes*, Rend. Sem. Mat. Univ. Padova 52 (1974)

2 J. MITTAS. *Sur les hyperanneaux et les hypercorps*, Math. Balk. 3 (1973)

3 J. MITTAS. *Sur les structures hypercompositionelles*, Proc. AHA (1990)

4 R. PROCESI - R. ROTA. *On some classes of hyperstructures*, (preprint)