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Cactus graphs with cycle blocks and square product labeling

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Abstract

In many real world problems, cactus graphs were considered as models from both algorithmic and theoretical point of view and this graph is a subclass of planar graph and superclass of a tree. In this article, the study has been carried out on some cactus graphs with cycle blocks for obtaining results on square product labeling.

Keywords: Square Product Labeling (SPL) and Cactus Graphs.
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1 Introduction

Graph labeling is one of the field in graph theory and it is the labeling of integers to the vertices or edges, or both under particular conditions which was introduced by A. Rosa [1967] in 1967. Graph labeling has a wide range of applications such as in X-ray crystallography, coding theory, radar, astronomy, circuit design, network theory, communication networks and database management. Nowadays, research in labeling of graph is increasingly expanding by studing more than 300 kinds of labelings. One such labeling is square sum labeling introduced by V. Ajitha et al. [2009] and they explored some results on it. J. B. Babujee and Babitha [2012] also worked on square sum labeling and obtained the results on it. Further, J.Shiama [2012] has worked on square difference labeling and proved some results. K. G. Mirajkar and Sthavarmath [2022] initiated square product labeling and obtained results for some class of graphs, cycle related graphs, Cartesian product of graphs and silicate and oxide networks. Khan et al. [2010a] have studied on cactus graphs for proving the results for $(2, 1)-$ total labeling and for $L(2, 1)-$ Labeling of cactus graphs in Khan et al. [2010b]. K. Kalaiarasi and Mahalakshmi [2022] shown the applications for cactus fuzzy labeling graphs. Further, S. Philomena, M. Pal, and K. Thirusanga Philomena et al. [2014] investigated some results for square and cube difference labeling on cactus graphs. In this article, the results are obtained for cactus graphs with cycle blocks on square product labeling and applied number theory concepts to establish the results.

All examined graphs here are finite, undirected, simple and connected. For undefined expressions and symbols refer F.Harary [1969], for number theory concepts refer Burton [2006] and for different labeling concepts we refer Gallian [2020].

2 Preliminaries

Definition 2.1. *A graph G is said to be a square product labeling (SPL), if there exists function $f : V(G) \to \{1, 2, 3, \ldots, p\}$ which is bijective, here p is counting of vertices inducing $f^* : E(G) \to N$ which is injective, where $f^*(uv) = f(u)^2 f(v)^2$ and the resulting edges are distinctly labeled.*

Definition 2.2. *Cactus graph Khan et al. [2010a] is a connected graph, in which every block is a cycle or an edge, in other words, no edge belongs to more than one cycle.*

3 Main Results

Theorem 3.1. *For two cycles C_n and C_m of any lengths n and m with a common cutvertex v admits square product labeling.*

Proof. Consider the cycles C_n and C_m of any lengths n and m having a common cutvertex v labeled by 1 with $V(C_n) = \{v_1, v_2, v_3, ..., v_n\}$ and $V(C_m) = \{v_1^{'}, v_2^{'}, v_3^{'}, ..., v_m^{'}\}$, hence the number of vertices are $(n+m)-1$ and edges are $2n$. We consider six cases to prove the result with $f : V(G) \rightarrow \{1, 2, 3, \ldots, (n+m)-1\}$

Case 1: For $n < m$

$$\begin{aligned} f(v_i) &= 2i \; for \; 1 \leq i \leq n-1, \; f(v_i^{'}) = f(v_{n-1}) + 2i \; for \; 1 \leq i \leq m-n \\ f(v_{i+1}^{'}) &= 2i+1 \; for \; m-n \leq i \leq m-2 \end{aligned}$$

Labeling of edges acquired here are distinct, as it forms quadratic sequences of the forms $(4P(P+1))^2$, $P \geqslant 1$ for $\{f^*(v_i v_{i+1}) : \; 1 \leq i \leq n-2\}$ and $(4P^2-1)^2$, $P \geqslant 1$ for $\{f^*(v_{m-n+1+(i-1)}^{'} v_{m-n+2+(i-1)}^{'}) : \; m-n \leq i \leq m-2\}$. The remaining edges are labeled as follows,

$$\begin{aligned} f^*(vv_n) &= (f(v_n))^2, \; f^*(vv_1) = 4, \; f^*(vv_1^{'}) = (f(v_1^{'}))^2 \\ f^*(vv_n^{'}) &= (f(v_n^{'}))^2, \; f^*(v_i^{'} v_{i+1}^{'}) = (f(v_n^{'}))^2 (f(v_n^{'}))^2 \; for \; 1 \leq i \leq m-n \end{aligned}$$

Case 2: For $n > m$

$$\begin{aligned} f(v_i) &= 2i \; for \; 1 \leq i \leq n-2, \; f(v_{n-1}) = 3 \\ f(v_i^{'}) &= f(v_{n-1}) + 2i \; for \; 1 \leq i \leq m-1 \end{aligned}$$

Labeling of edges acquired here are distinct, as it forms quadratic sequences of the forms $(4P(P+1))^2$, $P \geqslant 1$ for $\{f^*(v_i v_{i+1}) : \; 1 \leq i \leq n-3\}$ and $(4P^2 + 16P + 15)^2$, $P \geqslant 1$ for $\{f^*(v_i^{'} v_{i+1}^{'}) : \; m-n \leq i \leq m-2\}$. The remaining edges are labeled as follows,

$$\begin{aligned} f^*(vv_n) &= (f(v_n))^2, \; f^*(vv_1) = 4, \; f^*(vv_1^{'}) = (f(v_1^{'}))^2 \\ f^*(v_{n-1} v_n) &= (f(v_{n-1}))^2 \; (f(v_n))^2 \end{aligned}$$

Case 3: For $n = m$

$$\begin{aligned} f(v_i) &= 2i, \; 1 \leq i \leq (n-1), \; f(v_i^{'}) = 2i-1, \; 1 \leq i \leq m-1 \\ f(v_i^{'}) &= f(v_{n-1}) + 2i, \; 1 \leq i \leq m-1 \end{aligned}$$

Labeling of edges acquired here are distinct, as it forms quadratic sequences of the forms $(4P(P+1))^2$, $P \geqslant 1$ for $\{f^*(v_i v_{i+1}) : \; 1 \leq i \leq n-2\}$ and $(4P^2-1)^2$,

$P \geqslant 1$ for $\{f^*(v_i^{'}v_{i+1}^{'}) : \ m-n \leq i \leq m-2\}$. The remaining edges are labeled as follows,

$$\begin{aligned} f^*(vv_n) &= (f(v_n))^2, \ f^*(vv_1) = 4, \ f^*(vv_1^{'}) = (f(v_1^{'}))^2 \\ f^*(v_m^{'}v) &= (f(v^{'}))^2 \end{aligned}$$

Case 4: For $n = 2P(P+1), \ P \geqslant 1$ and $n < m$

$$\begin{aligned} For \ n = 2P(P+1), \ P = 1 & \\ f(v_1) &= 2, \ f(v_2) = 6, \ f(v_3) = 4, \ f(v_1^{'}) = f(v_2) + 2 \\ f(v_{m-1}^{'} - (i-1)) &= 2i+1 \ for \ 1 \leq i \leq m-n \\ if \ n \geqslant 4, \ n = 2P(P+1), \ P > 1 & \\ f(v_i) &= 2i \ for \ 1 \leq i \leq n-1, \ f(v_1^{'}) = 3 \\ f(v_{i+1}^{'}) &= 2i+1 \ for \ 2 \leq i \leq m-3 \\ f(v_{i+1}^{'}) &= f(v_{n-1}) + 2 + 2(i-1) \ for \ 1 \leq i \leq m-n \end{aligned}$$

Labeling of edges acquired here are distinct, as it forms quadratic sequences of the forms $(4P(P+1))^2$, $P \geqslant 1$ for $\{f^*(v_iv_{i+1}) : \ 1 \leq i \leq n-3\}$ and $(4P^2-1)^2$, $P \geqslant 1$ for $\{f^*(v_{(m-1)-(i-1)}^{'}v_{(m-2)-(i-1)}^{'}) : \ 1 \leq i \leq m-3\}$. The remaining edges are labeled as follows,

$$\begin{aligned} f^*(vv_n) &= (f(v_n))^2, \ f^*(vv_1) = 4, \ f^*(vv_1^{'}) = (f(v_1^{'}))^2, \ f^*(v_m^{'}v) = (f(v^{'}))^2 \\ f^*(v_{n-1}v) &= (f(v_{n-1}))^2, \ f^*(v_1v_2) = f(v_1)^2 \ f(v_2)^2 \\ f^*(v_2v_3) &= f(v_2)^2 \ f(v_3)^2, \ f^*(v_1^{'}v_2^{'}) = (f(v_1^{'}))^2 \ (f(v_2^{'}))^2 \end{aligned}$$

Case 5: For $n = 2P^2 + 2P + 1 \ m \geqslant (n-1)$

$$\begin{aligned} f(v_i) &= 2i \ for \ 1 \leq i \leq n-3, \ f(v_{n-2}) = (2n-2) \\ f(v_{n-1}) &= (2n-4) \\ f(v_1^{'}) &= 3, \ f(v_{i+1}^{'}) = f(v_{n-2}) + 2i \ for \ 1 \leq i \leq m-n \\ f(v_{(m-n+1)+(i-1)}^{'}) &= 2i+1 \ for \ m-n \leq i \leq m-2 \end{aligned}$$

Labeling of edges acquired here are distinct, as it forms quadratic sequences of the forms $(4P(P+1))^2$, $P \geqslant 1$ for $\{f^*(v_iv_{i+1}) : \ 1 \leq i \leq n-3\}$ and $(4P^2 + 16P+15)^2$, $P \geqslant 1$ for $\{f^*(v_i^{'}v_{i+1}^{'}) : \ m-n \leq i \leq m-2\}$. The remaining edges are labeled as follows,

$$\begin{aligned} f^*(vv_n) &= (f(v_n))^2, \ f^*(vv_1) = 4, \ f^*(vv_1^{'}) = (f(v_1^{'}))^2 \\ f^*(v_{n-1}v_n) &= (f(v_{n-1}))^2 \ (f(v_n))^2, \ f^*(v_1^{'}v_2^{'}) = 9 \ (f(v_{n-1}) + 2))^2 \\ if \ n = 5, \ f^*(v_2^{'}v_3^{'}) &= (f(v_{n-1}) + 2))^2 \ (f(v_3^{'}))^2 \\ f^*(v_{i+2}^{'}v_{i+3}^{'}) &= (4P^2 + 16P + 15)^2 \ for \ 1 \leq i \leq m-3 \end{aligned}$$

Case 6: For $n = 4P^2 + 8P + 3$ and $n = 4(P^2 + 2P + 1)$ number of vertices where $P \geq 1$
$v'_{n-1} = n + m - 2$ and $v'_{n-2} = n + m - 1$ and if $f(v'_i)$ is labeled with even numbers before the label 3 for $n = 4(P^2 + 2P + 1)$ then change the label with preceding vertex label. The remaining labels of both the edges and vertices are same as in the above cases. The labels of both vertices and edges are distinct in all cases, hence the result. □.

Example 3.1: The square product labeling of two cycles of any lengths n and m with a common cut vertex v is as shown in below figure.

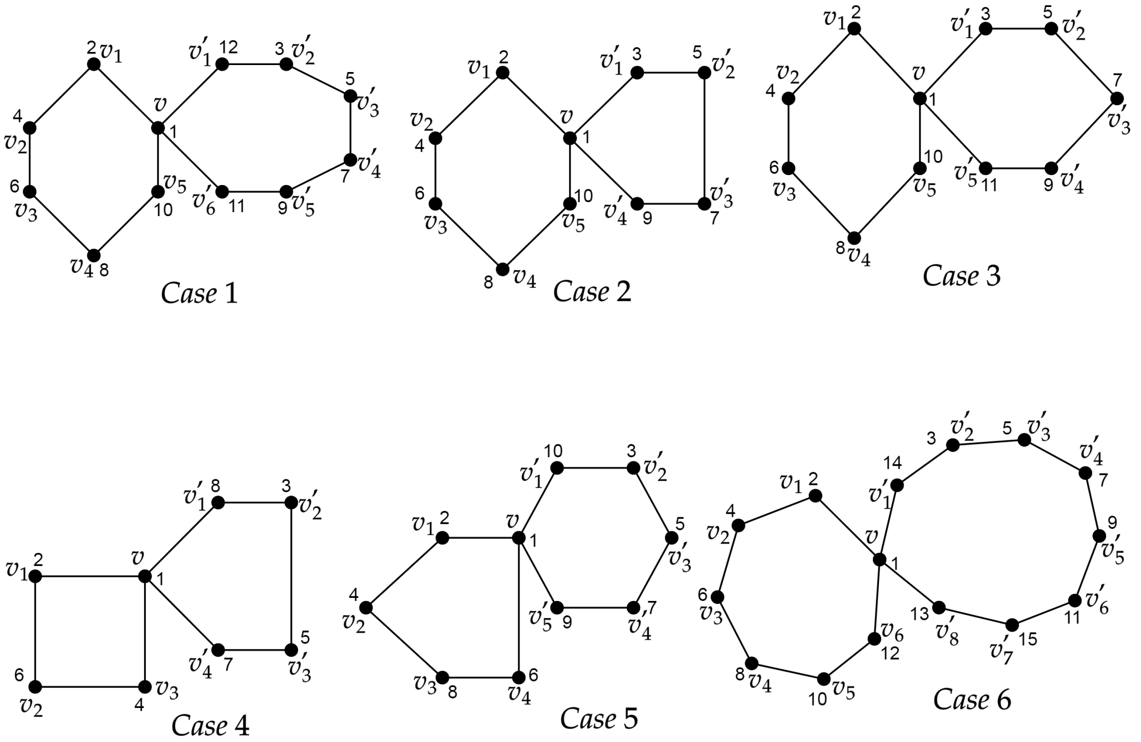

Figure 1: Square product labeling of two cycles of any lengths n and m with a common cutvertex v

Theorem 3.2. *For three cycles C_n, C_m, and C_l of any lengths n, m, and l with a common cut vertex v admits SPL.*

Proof. Consider three cycles C_n, C_m, and C_l having a common cut vertex v with fixed label as 1 and $V(C_n) = \{v_1, v_2, v_3, ..., v_n\}$, $V(C_m) = \{v'_1, v'_2, v'_3, ..., v'_m\}$, and $V(C_l) = \{v''_1, v''_2, v''_3, ..., v''_l\}$, hence the number of vertices are $(n+m+l)-2$ and edges are $3n$. We consider nine cases to prove the result and in each case the labels of vertices of cactus graph starts from even numbers among the total number of vertices of that graph later the graph G is given odd numbers. let $f : V(G) \to \{1, 2, 3, \ldots, (n + m + l) - 2\}$, thus both the vertices and edges for all nine cases are labeled as follows,

Case 1: For $n = 2P(P + 1),\ P \geq 1$ and $n < m$

$$\begin{aligned}
if\ n = 2P(P+1),\ P = 1 & \\
f(v_1) &= 2,\ f(v_2) = 6,\ f(v_3) = 4,\ f(v_i^{'}) = f(v_2) + 2 \\
i\ for\ 1 \leq i \leq m + n - l & \\
f(v_{m+n-l}^{'} + i) &= 2i + 1\ for\ 1 \leq i \leq l - m,\ if\ l > m \\
if\ n = 2P(P+1),\ P > 1 & \\
f(v_i) &= 2i,\ 1 \leq i \leq (n-2),\ f(v_i^{'}) = f(v_{n-2}) + 2i \\
for\ 1 \leq i \leq m + n - l & \\
f(v_i^{''}) &= f(v_{n-1}^{'}) + 2i \\
for\ 1 \leq i \leq n + l - m + 1 & \\
f(v_{n-2}) &= 2n,\ f(v_{n-1}) = 2n - 2
\end{aligned}$$

The edge labels acquired here are distinct, as it forms a quadratic sequence of the form $(4P(P + 1))^2$, $P \geqslant 1$ for $\{f^*(v_i v_{i+1}) :\ 1 \leq i \leq n - 3\}$. The remaining edges are labeled as follows,

$$\begin{aligned}
f^*(vv_n) &= (f(v_n))^2,\ f^*(vv_1) = 4,\ f^*(vv_1^{'}) = (f(v_1^{'}))^2 \\
f^*(v_1v_2) &= (f(v_1))^2\ (f(v_2))^2,\ f^*(v_2v_3) = (f(v_2))^2\ (f(v_3))^2 \\
f^*(vv_{n-1}^{'}) &= (f(v_{n-1}^{'}))^2,\ f^*(v_i^{'}v_{i+1}^{'}) = (f(v_{n-2}) + 2i))^2 (f(v_i^{'}) + 2i))^2, \\
for\ 1 \leq i \leq m - 3 & \\
f^*(v_{n-2}^{'}v_{n-1}^{'}) &= (f(v_{n-2}^{'}))^2 (f(v_{n-1}^{'}))^2 \\
f^*(v_i^{''}v_{i+1}^{''}) &= (f(v_{n-2}^{'}) + 2i))^2 (f(v_i^{''}) + 2i))^2\ for\ 1 \leq i \leq l + n - m
\end{aligned}$$

Case 2: For $n = 2P^2 + P + 1,\ P \geq 1$

$$\begin{aligned}
f(v_i) &= 2i\ for\ 1 \leq i \leq \frac{n-1}{2},\ f(v_{n-2}) = (2n - 2),\ f(v_{n-1}) = 2n - 4 \\
f(v_i^{'}) &= f(v_{n-2}) + 2i\ for\ 1 \leq i \leq m + n - l \\
f(v_i^{''}) &= f(v_{n-1}^{'}) + 2i\ for\ 1 \leq i \leq (n + l - m) + 1
\end{aligned}$$

The edge labels acquired here are distinct, as it forms quadratic sequences of the form $(4P(P + 1))^2$, $P \geqslant 1$ for $\{f^*(v_i v_{i+1}) :\ 1 \leq i \leq n - 3\}$, $(4P^2 + 24P + 35))^2$, $P \geqslant 1$ for $\{f^*(v_i^{''} v_{i+1}^{''}) :\ 1 \leq i \leq l - 2\}$ and $(4P^2 - 1)^2$, $P \geqslant 1$ for

$\{f^*(v'_{(n+l-m)-1+i}v'_{(n+l-m)+1+(i-1)}):\ 1 \leq i \leq m-l\}$ if $l < m$. The remaining edges are labeled as follows,

$$\begin{aligned}
f^*(vv_{n-1}) &= (f(v_{n-1}))^2,\ f^*(vv_1) = 4,\ f^*(vv'_1) = (f(v'_1))^2\\
f^*(v'_{n-2}v'_{n-1}) &= (f(v'_{n-2}))^2\ (f(v'_{n-1}))^2\\
f^*(v'_{n-3}v'_{n-2}) &= (f(v'_{n-3}))^2\ (f(v'_{n-2}))^2\\
f^*(vv'_{n-1}) &= (f(v'_{n-1}))^2,\ f^*(vv''_{n-1}) = (f(v''_{n-1}))^2\\
f^*(v'_iv'_{i+1}) &= (f(v_{n-2})+2i))^2(f(v'_i)+2i))^2\\
for\ 1 \leq i \leq (l+n-m)-2 &\\
f^*(v'_{(n+l-m)-1}v'_{(n+l-m)+1)}) &= (f(v'_{(n+l-m)-1)}))^2(f(v'_{(n+l-m)+1)}))^2
\end{aligned}$$

Case 3: For $n = m = l$

$$\begin{aligned}
f(v_i) &= 2i\ for\ 1 \leq i \leq n-1,\ f(v'_i) = f(v_{n-1}) + 2i\\
for\ 1 \leq i \leq m-4 &\\
f(v'_{(m-4)+i}) &= 2i+1\ for\ 1 \leq i \leq m-4,\ f(v''_i) = f(v'_{n-1}) + 2i\\
for\ 1 \leq i \leq l-1 &
\end{aligned}$$

The edge labels acquired here are distinct, as it forms quadratic sequences of the form $(4P(P+1))^2$, $P \geqslant 1$ for $\{f^*(v_iv_{i+1}):\ 1 \leq i \leq n-2\}$ and $(4P^2-1)^2$, $P \geqslant 1$ for $\{f^*(v_{(m-3)+(i-1)}v_{(m-2)+(i-1)}):\ 1 \leq i \leq m-5\}$. The remaining edges are labeled as follows,

$$\begin{aligned}
f^*(vv_{n-1}) &= (f(v_{n-1}))^2,\ f^*(vv_1) = 4,\ f^*(vv'_1) = (f(v'_1))^2\\
f^*(vv'_{n-1}) &= (f(v'_{n-1}))^2,\ f^*(vv''_{n-1}) = (f(v''_{n-1}))^2\\
f^*(v'_iv'_{i+1}) &= (f(v_{n-1})+2i))^2(f(v'_i)+2i))^2\ for\ 1 \leq i \leq m-5\\
f^*(v_{m-4}v_{m-3}) &= (f(v_{m-4}))^2\ (f(v_{m-3}))^2\\
f^*(v''_iv''_{i+1}) &= (f(v'_{m-1})+2i))^2(f(v'_i)+2i))^2\ for\ 1 \leq i \leq l-2
\end{aligned}$$

Case 4: For $V(G) = 4P^2+8P+3,\ P \geq 1, n < m\ and\ n \leqslant l$

$$\begin{aligned}
f(v_i) &= 2i\ for\ 1 \leq i \leq n-1,\ f(v''_{n-2}) = (n+m+l-2)\\
f(v''_{n-1}) &= (n+m+l-4)\\
f(v'_i) &= f(v_{n-1}) + 2i\ for\ 1 \leq i \leq m-l-1\\
f(v'_{m-(l+1)+i}) &= 2i+1\ for\ 1 \leq i \leq m-2\\
f(v''_i) &= f(v'_{n-1}) + 2i\ for\ 1 \leq i \leq l-3
\end{aligned}$$

The edge labels acquired here are distinct, as it forms a quadratic sequence of the form $(4P(P+1))^2$, $P \geqslant 1$ for $\{f^*(v_iv_{i+1}):\ 1 \leq i \leq n-2\}$. The remaining edges are labeled as follows,

$$\begin{aligned}
f^*(vv_{n-1}) &= (f(v_{n-1}))^2,\ f^*(vv_1) = 4,\ f^*(vv_1^{'}) = (f(v_1^{'}))^2 \\
f^*(v_{n-2}^{''}v_{n-1}^{''}) &= (f(v_{n-2}^{''}))^2,\ f^*(v_1^{'}v_2^{'}) = (f(v_1^{'}))^2\ (f(v_2^{'}))^2 \\
f^*(vv_{n-1}^{'}) &= (f(v_{n-1}^{'}))^2,\ f^*(vv_{n-1}^{''}) = (f(v_{n-1}^{''}))^2,\ f^*(vv_1^{''}) = (f(v_1^{''}))^2 \\
f^*(v_i^{'}v_{i+1}^{'}) &= (f(v_{n-2}) + 2i))^2(f(v_i^{'}) + 2i))^2\ for\ 2 \leq i \leq m-2 \\
f^*(v_i^{''}v_{(i+1)}^{''}) &= (f(v_{(n-1)}^{'} + 2i))^2(f(v_i^{'}) + 2i)^2\ for\ 1 \leq i \leq l-3
\end{aligned}$$

Case 5: For $V(G) = 4(P^2 + 2P + 1)$, $P \geq 1$

$$\begin{aligned}
f(v_i) &= 2i\ for\ 1 \leq i \leq (n-1),\ f(v_i^{'}) = f(v_{n-1}) + 2i\ for\ 1 \leq i \leq m-l \\
f(v_{(m-4)+i}^{'}) &= 2i+1\ for\ 1 \leq i \leq m-4,\ f(v_{(m-l)+1}^{'}) = f(v_{(m-l)+4}^{'}) \\
f(v_{(m-l)+2}^{'}) &= f(v_{(m-l)+2}^{'}),\ f(v_i^{''}) = f(v_{n-1}^{'}) + 2i\ for\ 1 \leq i \leq l-3 \\
f(v_{n-2}^{''}) &= f(v_{(m-l)}^{'}) + 3,\ f(v_{n-1}^{''}) = f(v_{(n-1)}^{'}) - 2
\end{aligned}$$

The edge labels acquired here are distinct, as it forms quadratic sequences of the form $(4P(P+1))^2$, $P \geqslant 1$ for $\{f^*(v_iv_{i+1}) :\ 1 \leq i \leq n-2\}$ and $(4P^2-1)^2$, $P \geqslant 1$ for $\{f^*(v_{(m-3)+(i-1)}^{'}v_{(m-2)+(i-1)}^{'}) :\ 1 \leq i \leq m-5\}$. The remaining edges are labeled as follows,

$$\begin{aligned}
f^*(vv_{n-1}) &= (f(v_{n-1}))^2,\ f^*(vv_1) = 4,\ f^*(vv_1^{'}) = (f(v_1^{'}))^2 \\
f^*(v_{(m-l)+1}^{'}v_{(m-l)+2}^{'}) &= (f(v_{(m-l)}^{'} + 4))^2\ (f(v_{(m-l)}^{'} + 2))^2, \\
f^*(v_{m-4}^{'}v_{m-3}^{'}) &= (f(v_{m-4}^{'}))^2\ (f(v_{m-3}^{'}))^2 \\
f^*(vv_{n-1}^{'}) &= (f(v_{n-1}^{'}))^2,\ f^*(vv_{n-1}^{''}) = (f(v_{n-1}^{''}))^2 \\
f^*(v_i^{'}v_{i+1}^{'}) &= (f(v_{n-1}) + 2i))^2(f(v_i^{'}) + 2i))^2\ for\ 1 \leq i \leq m-l \\
f^*(v_i^{''}v_{(i+1)}^{''}) &= (f(v_{(n-1)}^{'}) + 2i)^2\ (f(v_i^{'}) + 2i))^2\ for\ 1 \leq i \leq l-2
\end{aligned}$$

Case 6: For $V(V) = 12P^2 + 4P + 1\ and\ 12P^2 + 4P + 1,\ P \geq 1$, where $n < m$

$$\begin{aligned}
f(v_i) &= 2i\ for\ 1 \leq i \leq n-1 \\
f(v_i^{'}) &= f(v_{n-1}) + 2i\ for\ 1 \leq i \leq m-n \\
f(v_{(m-n)}^{'} + 1) &= (n+m+l-2),\ f(v_{(m-n)+2}^{'}) = (n+m+l-4) \\
f(v_{(m-n)+2}^{'} + i) &= 2i+1\ for\ 1 \leq i \leq (m-n)+2 \\
f(v_{n-2}^{''}) &= f(v_{(m-l)}^{'}) + 3,\ f(v_i^{''}) = f(v_{(n-1)}^{'}) + 2i\ for\ 1 \leq i \leq l-2
\end{aligned}$$

The edge labels acquired here are distinct, as it forms quadratic sequences of the forms $(4P(P+1))^2$, $P \geqslant 1$ for $\{f^*(v_iv_{i+1}) :\ 1 \leq i \leq n-2\}$ and $(4P^2-1)^2$,

$P \geqslant 1$ for $\{f^*(v^{'}_{(m-n)+2+i}v^{'}_{(m-n)+4+i}) : \ 1 \leq i \leq (m-n)+1\}$. The remaining edges are labeled as follows,

$$\begin{aligned} f^*(vv_{n-1}) &= (f(v_{n-1}))^2,\ f^*(vv_1) = 4,\ f^*(vv^{'}_1) = (f(v_{n-1})+2)^2 \\ f^*(vv^{''}_1) &= (f(v^{'}_{n-1})+2)^2 \\ f^*(vv^{'}_{n-1}) &= (f(v^{'}_{n-1}))^2,\ f^*(vv^{''}_{n-1}) = (f(v^{''}_{n-1}))^2 \\ f^*(v^{'}_iv^{'}_{i+1}) &= (f(v_{n-1})+2i))^2(f(v^{'}_i)+2i))^2 \ for\ 1 \leq i \leq m-n \\ f^*(v^{''}_iv^{''}_{(i+1)}) &= (f(v^{'}_{(n-1)})+2i)^2\ (f(v^{''}_i)+2i))^2\ for\ 1 \leq i \leq l-2 \\ f(v^{'}_{(m-n)+1}v^{'}_{(m-n)+2}) &= (n+m+l-2)^2\ (n+m+l-4)^2 \end{aligned}$$

Case 7: For $n+m = 2P^2+6P+5\ P \geq 1$ and $l = 2P^2+6P+4,\ P \geq 1$

$$\begin{aligned} f(v_i) &= 2i\ for\ 1 \leq i \leq (n-2),\ f(v^{'}_i) = f(v_{n-1})+2i,\ 1 \leq i \leq (m-3) \\ f(v^{'}_{(n-2)}) &= 2(n+m)-2,\ f(v^{'}_{(n-1)+2}) = 2(n+m)-4, \\ f(v^{''}_i) &= 2i+1\ for\ 1 \leq i \leq l-1 \end{aligned}$$

The edge labels acquired here are distinct, as it forms quadratic sequences of the form $(4P(P+1))^2$, $P \geqslant 1$ for $\{f^*(v_iv_{i+1}) : \ 1 \leq i \leq n-2\}$ and $(4P^2-1)^2$, $P \geqslant 1$ for $\{f^*(v^{''}_iv^{''}_{i+1}) : \ 1 \leq i \leq l-2\}$. The remaining edges are labeled as follows,

$$\begin{aligned} f^*(vv_{n-1}) &= (f(v_{n-1}))^2,\ f^*(vv_1) = 4,\ f^*(vv^{'}_1) = (f(v^{'}_1)^2, \\ f^*(vv^{''}_1) &= (f(v^{''}_1)^2 \\ f^*(vv^{'}_{n-1}) &= (f(v^{'}_{n-1}))^2,\ f^*(vv^{''}_{n-1}) = (f(v^{''}_{n-1}))^2 \\ f^*(v^{'}_iv^{'}_{i+1}) &= (f(v_{n-1})+2i))^2(f(v^{'}_i)+2i))^2\ for\ 1 \leq i \leq m-4 \\ f^*(v^{'}_{m-3}v^{'}_{(m-2)}) &= (f(v^{'}_{(m-3)}))^2\ (f(v^{'}_{m-2}))^2 \\ f(v^{'}_{(m-2)}v^{'}_{(m-1)}) &= (2(n+m)-2)^2\ (2(n+m)-4)^2 \end{aligned}$$

Case 8: For $n+m = (2P^2+6P+4),\ P \geq 1$ and $l = (2P^2+6P+5),\ P \geq 1$

$$\begin{aligned} f(v_i) &= 2i\ for\ 1 \leq i \leq (n-2), \\ f(v^{'}_i) &= f(v_{n-1})+2i\ for\ 1 \leq i \leq m-2 \\ f(v^{''}_2) &= 2(l-1),\ f(v^{''}_1) = 3,\ f(v^{''}_1) = 9 \\ f(v^{''}_1) &= 5,\ f(v^{''}_1) = 7,\ f(v^{''}_{5+i}) = 2i+9\ for\ 1 \leq i \leq l-6 \end{aligned}$$

The edge labels acquired here are distinct, as it forms quadratic sequences of the form $(4P(P+1))^2$, $P \geqslant 1$ for $\{f^*(v_iv_{i+1}) : \ 1 \leq i \leq n-2\}$ and $(2P+9)^2$,

$P \geqslant 1$ for $\{f^*(v^{''}_{5+i}v^{''}_{6+i}) : \ 1 \leq i \leq l-7\}$. The remaining edges are labeled as follows,

$$\begin{aligned}
f^*(vv_{n-1}) &= (f(v_{n-1}))^2,\ f^*(vv_1) = 4,\ f^*(vv^{'}_1) = (f(v_{(n-1)+2})^2 \\
f^*(vv^{''}_1) &= (f(v^{''}_1)^2 \\
f^*(vv^{'}_{n-1}) &= (f(v^{'}_{n-1}))^2,\ f^*(vv^{''}_{n-1}) = (f(v^{''}_{n-1}))^2 \\
f^*(v^{'}_i v^{'}_{i+1}) &= (f(v_{n-1}) + 2i))^2 (f(v^{'}_i) + 2i))^2 \ for\ 1 \leq i \leq m-2 \\
f^*(v^{''}_5 v^{''}_6) &= (f(v^{''}_5))^2\ (f(v^{''}_6))^2,\ f^*(v^{''}_1 v^{''}_2) = (f(v^{''}_1))^2,\ (f(v^{''}_2))^2 \\
f^*(v^{''}_3 v^{''}_4) &= (f(v^{''}_3))^2\ (f(v^{''}_4))^2,\ f^*(v^{''}_4 v^{''}_5) = (f(v^{''}_4))^2\ (f(v^{''}_5))^2
\end{aligned}$$

Case 9: For all n, m, and l except the above cases

$$\begin{aligned}
f(v_i) &= 2i\ for\ 1 \leq i \leq (n-1), \\
f(v^{'}_i) &= f(v_{n-1}) + 2i\ for\ 1 \leq i \leq n-m,\ if\ n > m\ and\ 1 \leq i \leq m-n \\
if\ n < m & \\
f(v^{'}_{(n-m)+i}) &= 2i+1\ for\ 1 \leq i \leq m-2,\ f(v^{''}_i) = f(v^{'}_{n-1}) + 2i\ for\ 1 \leq i \leq l-1 \\
f(v^{''}_i) &= 2i+1\ for\ 1 \leq i \leq l-1.
\end{aligned}$$

The edge labels acquired here are distinct, as it forms quadratic sequence of the forms $(4P(P+1))^2$, $P \geqslant 1$ for $\{f^*(v_iv_{i+1}) : \ 1 \leq i \leq n-2\}$ and $(4P^2-1)^2$, $P \geqslant 1$ for $\{f^*(v^{'}_{(n-m)+(i-1))}v^{'}_{(n-m+1)+(i-1)}) : \ 1 \leq i \leq m-3\}$. The remaining edges are labeled as follows,

$$\begin{aligned}
f^*(vv_{n-1}) &= (f(v_{n-1}))^2,\ f^*(vv_1) = 4,\ f^*(vv^{'}_1) = (f(v^{'}_1)^2,\ f^*(vv^{''}_1) = (f(v^{''}_1)^2 \\
f^*(vv^{'}_{n-1}) &= (f(v^{'}_{n-1}))^2,\ f^*(vv^{''}_{n-1}) = (f(v^{''}_{n-1}))^2 \\
f^*(v^{'}_i v^{'}_{i+1}) &= (f(v_{n-1}) + 2i))^2 (f(v^{'}_i) + 2i))^2\ for\ 1 \leq i \leq n-m\ if\ n > m \\
f^*(v^{''}_i v^{''}_{(i+1)}) &= (f(v^{'}_{(n-1)} + 2i))^2\ (f(v^{''}_i + 2i))^2\ for\ 1 \leq i \leq l-2
\end{aligned}$$

In all the cases, the labeling pattern of edges and vertices are distinct so, for three cycles C_n, C_m, and C_l of any lengths n, m, and l with a common cut vertex v admits SPL. □

Example 3.2: The square product labeling of three cycles of any lengths n, m, and l with a common cut vertex v is shown in the below figure.

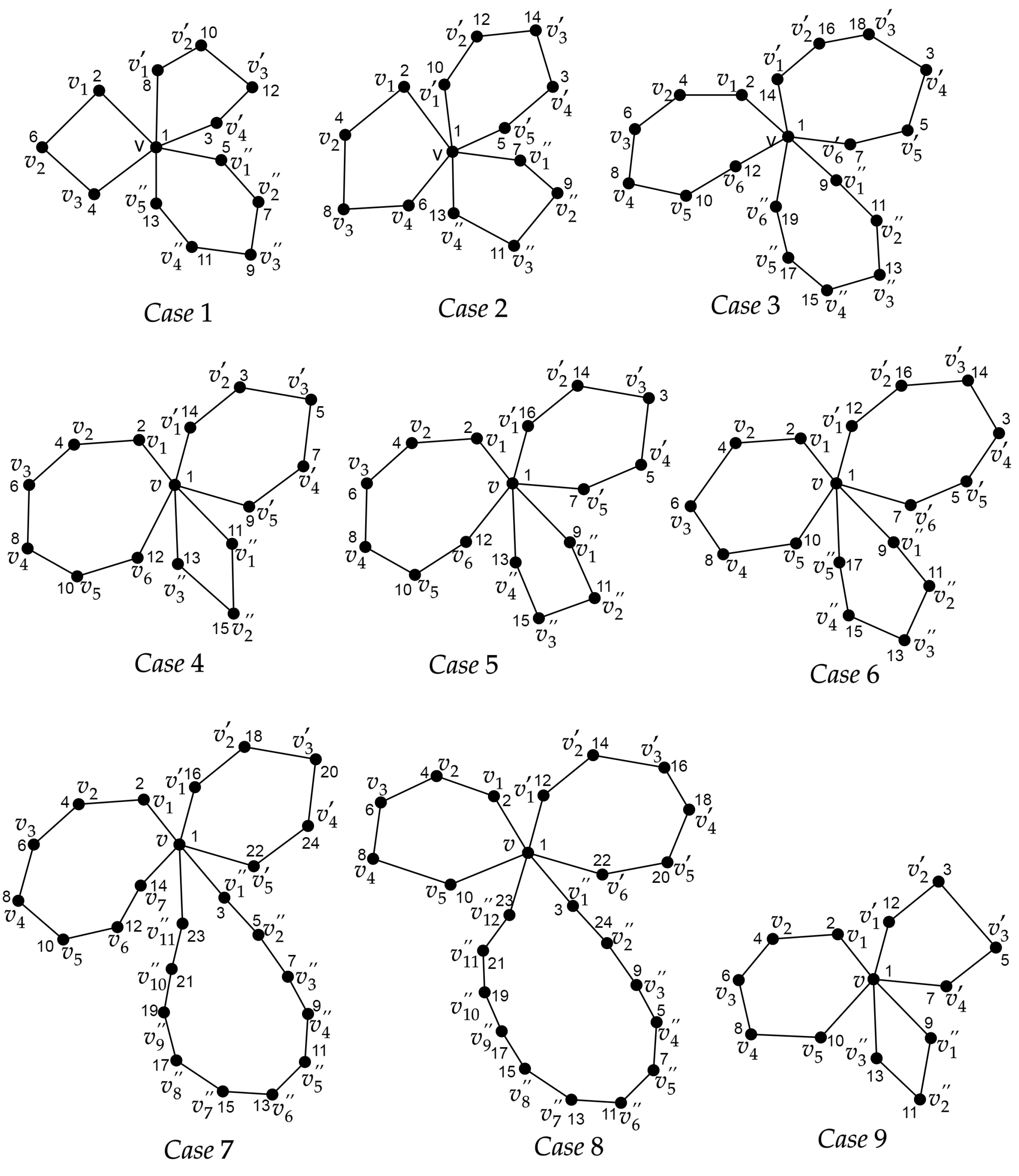

Figure 2: Three cycles of any lengths n, m, and l with a common cutvertex v

Theorem 3.3. *A graph G having r number of cycle of length n, with a common cutvertex v, except for the graph G with r number of cycle of length 3 having a common cutvertex v admits SPL.*

Proof. Consider the graph G having r number of cycles of length n with a common cutvertex v with $nr-(r-1)$ vertices and nr edges, here we consider two cases for prove the result with $f: V(G) \to \{1, 2, 3, \ldots, (nr-(r-1))\}$.
Case 1: Suppose G contains even copies of cycles of length n
In this case, the vertices of first half copies $\frac{r}{2}$ of cycle of length n are labeled by even numbers then next half copies by odd numbers. In first $\frac{r}{2}$ copies of cycle, the vertices are labeled as,

$$\begin{aligned}
f(v_1^1) &= 2,\ f(v_2^i) = f(v_1^i) + 2\ for\ 1 \le i \le \frac{r}{2} \\
f(v_{n-1}^i) &= f(v_1^i) + 4\ for\ 1 \le i \le \frac{r}{2} \\
f(v_3^i) &= f(v_{n-1}^i) + 2\ for\ 1 \le i \le \frac{r}{2},\ f(v_{3+i}^i) = f(v_3^i) + 2i\ for\ 1 \le i \le n-5 \\
f(v^(i+1)_1) &= f(v_{n-2}^i) + 2\ for\ 1 \le i \le \frac{r-2}{2}
\end{aligned}$$

In second $\frac{r}{2}$ copies of cycle, the vertices are labeled by,

$$\begin{aligned}
f\big(v_1^{\frac{r+2}{2}}\big) &= 3,\ f\big(v_2^{\frac{r+2}{2}}\big) = f\big(v_1^{\frac{r+2}{2}}\big) + 2\ for\ 1 \le i \le \frac{r}{2} \\
f\big(v_{n-1}^{\frac{r+2}{2}}\big) &= f\big(v_1^{\frac{r+2}{2}}\big) + 2\ for\ 1 \le i \le \frac{r}{2} \\
f\big(v_1^{\frac{r}{2}+(i-1)}\big) &= f\big(v_{n-2}^{\frac{r}{2}+i}\big) + 2\ for\ 1 \le i \le \frac{r-2}{2} \\
f\big(v_2^{\frac{r}{2}+i}\big) &= f\big(v_{n-1}^{\frac{r}{2}+i}\big) + 2\ for\ 1 \le i \le \frac{r}{2} \\
f\big(v_{2+i}^{\frac{r}{2}+i}\big) &= f\big(v_2^{\frac{r}{2}+i}\big) + 2\ for\ 1 \le i \le n-4
\end{aligned}$$

The labels of vertices of $f(v_1^i)$ and $f(v_{n-1}^i)$ with 1 yields distinct edge labels where $1 \le i \le r$, the remaining edge labels are as below

$$\begin{aligned}
f^*(v_1^i v_2^i) &= (f(v_1^i))^2\ (f(v_2^i))^2\ for\ 1 \le i \le \frac{r}{2}, \\
f^*(v_2^i v_3^i) &= (f(v_1^i) + 2)^2\ (f(v_{n-1}^i) + 2)^2\ for\ 1 \le i \le \frac{r}{2} \\
f^*(v_{n-2}^i v_{n-1}^i) &= (f(v_{n-2}^i))^2\ (f(v_1^i) + 4)^2\ for\ 1 \le i \le n-4 \\
f^*(v_3^i v_{3+i}^i) &= (f(v_{n-1}^i + 2))^2\ (f(v_3^i + 2i))^2\ for\ 1 \le i \le n-5
\end{aligned}$$

The labels of edges of remaining $\frac{r}{2}$ copies of G are,

$$\begin{aligned}
f^*\big(v_1^{\frac{r}{2}+i} v_2^{\frac{r}{2}+i}\big) &= \big(f(v_1^{\frac{r}{2}+i})\big)^2\ \big(f(v_{n-1}^{\frac{r}{2}+i} + 2)\big)^2\ for\ 1 \le i \le \frac{r-2}{2} \\
f^*\big(v_{n-2}^{\frac{r}{2}+i} v_{n-1}^{\frac{r}{2}+i}\big) &= \big(f(v_{n-2}^{\frac{r}{2}+i})\big)^2\ \big(f(v_1^{\frac{r}{2}+i} + 2)\big)^2\ for\ 1 \le i \le \frac{r-2}{2} \\
f^*\big(v_2^{\frac{r}{2}+i} v_{2+i}^{\frac{r}{2}+i}\big) &= \big(f(v_{n-1}^{\frac{r}{2}+i} + 2)\big)^2\ \big(f(v_2^{\frac{r}{2}+i} + 2i)\big)^2\ for\ 1 \le i \le n-4
\end{aligned}$$

Case 2: Suppose G contains odd copies of cycles of length n
In this case, the vertices of r copies of cycle of length n are first labeled by even numbers then by odd numbers as follows,

$$\begin{aligned}
f(v_1^1) &= 2,\ f(v_2^i) = f(v_1^i) + 2\ for\ 1 \le i \le \frac{r-1}{2} \\
f(v_1^{i+1}) &= f(v_{n-2}^i) + 2\ for\ 1 \le i \le \frac{r-3}{2} \\
f(v_{n-1}^i) &= f(v_1^i) + 4\ for\ 1 \le i \le \frac{r-1}{2} \\
f(v_2^i) &= f(v_1^i) + 2i\ for\ 1 \le i \le \frac{r-1}{2} \\
f(v^(i)_3) &= f(v_{n-1}^i) + 2\ for\ 1 \le i \le \frac{r-1}{2} \\
f(v^(i)_{3+i}) &= f(v_3^i) + 2i\ for\ 1 \le i \le n-5 \\
f\big(v_1^{\frac{r+1}{2}}\big) &= f(v_{n-2}^{\frac{r-1}{2}}) + 2,\ f\big(v_{n-1}^{\frac{r+1}{2}}\big) = f\big(v_1^{\frac{r+1}{2}}\big) + 4 \\
f\big(v_2^{\frac{r-1}{2}+i}\big) &= f\big(v_1^{\frac{r-1}{2}+i}\big) + 2\ for\ 1 \le i \le n-6 \\
f\big(v_{n-4}^{\frac{r-1}{2}+i}\big) &= 2i + 1\ for\ 1 \le i \le n-4 \\
f\big(v_1^{\frac{r+1}{2}+i}\big) &= f\big(v_{n-2}^{\frac{r+1}{2}+i}\big) + 2\ for\ 1 \le i \le \frac{r-1}{2} \\
f\big(v_{n-1}^{\frac{r+1}{2}+i}\big) &= f\big(v_1^{\frac{r+1}{2}+i}\big) + 2\ for\ 1 \le i \le \frac{r-1}{2} \\
f\big(v_{i+1}^{\frac{r+1}{2}+j}\big) &= f\big(v_1^{\frac{r+1}{2}+j}\big) + 4 + 2(i-1)\ for\ 1 \le i \le n-3\ and\ 1 \le j \le \frac{r-1}{2}
\end{aligned}$$

The labels of vertices of $f(v_1^i)$ and $f(v_{n-1}^i)$ with cutvertex label 1 yields distinct edge labels where $1 \le i \le r$, the remaining edge labels are as below.

$$\begin{aligned}
f^*(v_1^i v_2^i) &= (f(v_1^i))^2\ (f(v_1^i + 2))^2\ for\ 1 \le i \le \frac{r-1}{2}, \\
f^*(v_2^i v_3^i) &= (f(v_1^i + 2))^2\ (f(v_{n-1}^i + 2))^2\ for\ 1 \le i \le \frac{r-1}{2}, \\
f^*(v_{n-2}^i v_{n-1}^i) &= (f(v_{n-2}^i))^2\ (f(v_1^i + 4))^2\ for\ 1 \le i \le \frac{r-1}{2} \\
f^*(v_3^i v_{3+i}^i) &= (f(v_{n-1}^i + 2))^2\ (f(v_3^i + 2i))^2\ for\ 1 \le i \le n-4 \\
f^*\big(v_{n-2}^{\frac{r-1}{2}+i} v_{n-1}^{\frac{r-1}{2}+i}\big) &= \big(f(v_1^{\frac{r-1}{2}+2})\big)^2\ \big(f(v_1^{\frac{r-1}{2}+i} + 4)\big)^2\ for\ 1 \le i \le \frac{r-3}{2} \\
f^*\big(v_1^{\frac{r+1}{2}+i} v_2^{\frac{r+1}{2}+i}\big) &= \big(f(v_{n-2}^{\frac{r-1}{2}+i} + 2)\big)^2\ \big(f(v_{n-1}^{\frac{r-1}{2}+i} + 2)\big)^2\ for\ 1 \le i \le \frac{r-1}{2} \\
f^*\big(v_i^{\frac{r+1}{2}+i} v_{2+i}^{\frac{r+1}{2}+i}\big) &= \big(f(v_{n-1}^{\frac{r+1}{2}+i} + 2)\big)^2\ \big(f(v_2^{\frac{r+1}{2}+i} + 2i)\big)^2\ for\ 1 \le i \le n-3
\end{aligned}$$

Subcase 1: For cycle of length 4 with even cpoies

$$\begin{aligned}
f(v_1^1) &= 2,\ f(v_{n-1}^i) = f(v_1^i) + 2\ for\ 1 \le i \le \frac{r}{2} \\
f(v_1^{i+1}) &= f(v_2^i) + 2\ for\ 1 \le i \le \frac{r-2}{2} \\
f(v_2^i) &= f(v_{n-1}^i) + 2\ for\ 1 \le i \le \frac{r}{2},\ f\big(v_1^{\frac{r+2}{2}}\big) = 3 \\
f\big(v_{n-1}^{\frac{r+2}{2}}\big) &= f\big(v_1^{\frac{r}{2}+i}\big) + 2\ for\ 1 \le i \le \frac{r}{2},\ f\big(v_1^{\frac{r+2}{2}+i}\big) = f\big(v_2^{\frac{r}{2}}\big) + 2\ for\ 1 \le i \le \frac{r-1}{2} \\
f\big(v_2^{\frac{r}{2}}\big) &= f\big(v_{n-1}^{\frac{r}{2}+i}\big) + 2\ for\ 1 \le i \le \frac{r}{2}
\end{aligned}$$

The labels of vertices of $f(v_1^i)$ and $f(v_{n-1}^i)$ with 1 yields distinct edge labels where $1 \le i \le r$, the remaining edge labels are as below

$$\begin{aligned}
f^*(v_1^i v_2^i) &= (f(v_1^i))^2\ (f(v_{n-1}^i + 2))^2\ for\ 1 \le i \le \frac{r}{2} \\
f^*(v_2^i v_{n-1}^i) &= (f(v_{n-1}^i + 2))^2\ (f(v_1^i + 2))^2\ for\ 1 \le i \le \frac{r}{2} \\
f^*\big(v_1^{\frac{r}{2}+i} v_2^i\big) &= \big(f(v_1^{\frac{r}{2}+i})\big)^2\ \big(f(v_{n-1}^{\frac{r}{2}+i} + 2)\big)^2\ for\ 1 \le i \le \frac{r}{2} \\
f^*\big(v_2^{\frac{r+2}{2}+i} v_{n-1}^{\frac{r}{2}+i}\big) &= \big(f(v_{n-1}^{\frac{r}{2}+i} + 2)\big)^2\ \big(f(v_1^{\frac{r}{2}+i} + 2)\big)^2\ for\ 1 \le i \le \frac{r}{2}
\end{aligned}$$

Subcase 2: For cycle of length 5 with odd cpoies

$$\begin{aligned}
f(v_1^1) &= 2,\ f(v_2^i) = f(v_1^i) + 2\ for\ 1 \le i \le \frac{r-1}{2} \\
f(v_1^{i+1}) &= f(v_{n-2}^i) + 2\ for\ 1 \le i \le \frac{r-2}{2} \\
f(v_{n-1}^i) &= f(v_1^i) + 4\ for\ 1 \le i \le \frac{r-1}{2}, \\
f(v_2^i) &= f(v_1^i) + 2\ for\ 1 \le i \le \frac{r-1}{2} \\
f(v_3^i) &= f(v_{n-1}^i) + 2\ for\ 1 \le i \le \frac{r-1}{2} \\
f\big(v_1^{\frac{r+1}{2}}\big) &= f\big(v_3^{\frac{r-1}{2}}\big) + 2,\ f\big(v_3^{\frac{r+1}{2}}\big) = 3,\ f\big(v_{n-1}^{\frac{r+1}{2}}\big) = 7 \\
f\big(v_1^{\frac{r+1}{2}+i}\big) &= f\big(v_{n-1}^{\frac{r-1}{2}+i}\big) + 2\ for\ 1 \le i \le \frac{r-1}{2} \\
f\big(v_2^{\frac{r+3}{2}}\big) &= 9,\ f\big(v_3^{\frac{r+1}{2}+i}\big) = f\big(v_{n-1}^{\frac{r+3}{2}+i}\big) + 2i\ for\ 1 \le i \le \frac{r-1}{2} \\
f\big(v_2^{\frac{r+1}{2}+j}\big) &= f\big(v_1^{\frac{r+3}{2}+j}\big) + 2i\ for\ 1 \le i \le n-2\ and\ 1 \le j \le \frac{r-3}{2}
\end{aligned}$$

The labels of vertices of $f(v_1^i)$ and $f(v_{n-1}^i)$ with 1 yields distinct edge labels where $1 \leq i \leq r$, the remaining edge labels are as below

$$\begin{aligned}
f^*(v_1^i v_2^i) &= (f(v_1^i))^2 \ (f(v_{n-1}^i + 2))^2 \ for \ 1 \leq i \leq \frac{r-1}{2} \\
f^*(v_2^i v_3^i) &= (f(v_1^i + 2))^2 \ (f(v_{n-1}^i + 2))^2 \ for \ 1 \leq i \leq \frac{r-1}{2} \\
f^*(v_{n-2}^i v_{n-1}^i) &= (f(v_{n-2}^i))^2 \ (f(v_1^i + 4))^2 \ for \ 1 \leq i \leq \frac{r-1}{2} \\
f^*(v_3^i v_{3+i}^i) &= (f(v_{n-1}^i + 2))^2 \ (f(v_3^i + 2i))^2 \ for \ 1 \leq i \leq n-4 \\
f^*\big(v_1^{\frac{r+1}{2}} v_2^{\frac{r+1}{2}}\big) &= \big(f(v_{n-2}^{\frac{r-1}{2}} + 2)\big)^2 \ \big(f(v_1^{\frac{r+1}{2}} + 2)\big)^2 \\
f^*\big(v_2^{\frac{r+1}{2}} v_3^{\frac{r+1}{2}}\big) &= 9 \ \big(f(v_1^{\frac{r+1}{2}} + 2)\big)^2 \\
f^*\big(v_2^{\frac{r+1}{3}} v_{n-1}^{\frac{r+1}{2}}\big) &= 9 \ \big(f(v_{n-1}^{\frac{r+1}{2}})\big)^2 \\
f^*\big(v_1^{\frac{r+1}{2}} v_2^{\frac{r+1}{2}}\big) &= 25 \ \big(f(v_{n-1}^{\frac{r+1}{2}} + 2)\big)^2 \\
f^*\big(v_2^{\frac{r+3}{2}} v_3^{\frac{r+3}{2}}\big) &= \big(f(v_{n-1}^{\frac{r+1}{2}} + 2)\big)^2 \ \big(f(v_{n-1}^{\frac{r+3}{2}} + 2)\big)^2 \\
f^*\big(v_2^{\frac{r+3}{3}} v_{n-1}^{\frac{r+3}{2}}\big) &= \big(f(v_{n-1}^{\frac{r+3}{2}} + 2)\big)^2 \ \big(f(v_{n-2}^{\frac{r+3}{2}} + 2)\big)^2 \\
f^*\big(v_2^{\frac{r+3}{3}+i} v_2^{\frac{r+3}{2}+i}\big) &= \big(f(v_{n-1}^{\frac{r+1}{2}+i} + 2)\big)^2 \ \big(f(v_{n-2}^{\frac{r+3}{2}+i} + 2i)\big)^2 \ for \ 1 \leq i \leq \frac{r-3}{2}
\end{aligned}$$

For r copies of cycle of length 3 with common cut vertex v, $\mid V(G) \mid \geq (v_1^1)^2 \ (v_2^1)^2$ are not square product graphs.

Consider a graph G containing r cycles of length 3 with fixed cut vertex labeled as 1, labeling of vertices carries distinct non negative integers and $V(G)$ is bijective. The cut vertex v is adjacent with all other vertices since the graph is C_3 with r copies. Hence, while labeling the edges the product of labels of vertices with the fixed cut vertex 1 is not injective for $\mid V(G) \mid \geq (v_1)^2 \ (v_2)^2$. In the below figure, $f^*(v_1 v_2) = 36 \ and \ f^*(v v_1^2) = 36$ which is not injective.

□

Example 3.3: The square product labeling of even copies of C_6, odd copies of C_6, even copies of C_4, odd copies of C_5 and r copies of C_3 are shown in the below figures.

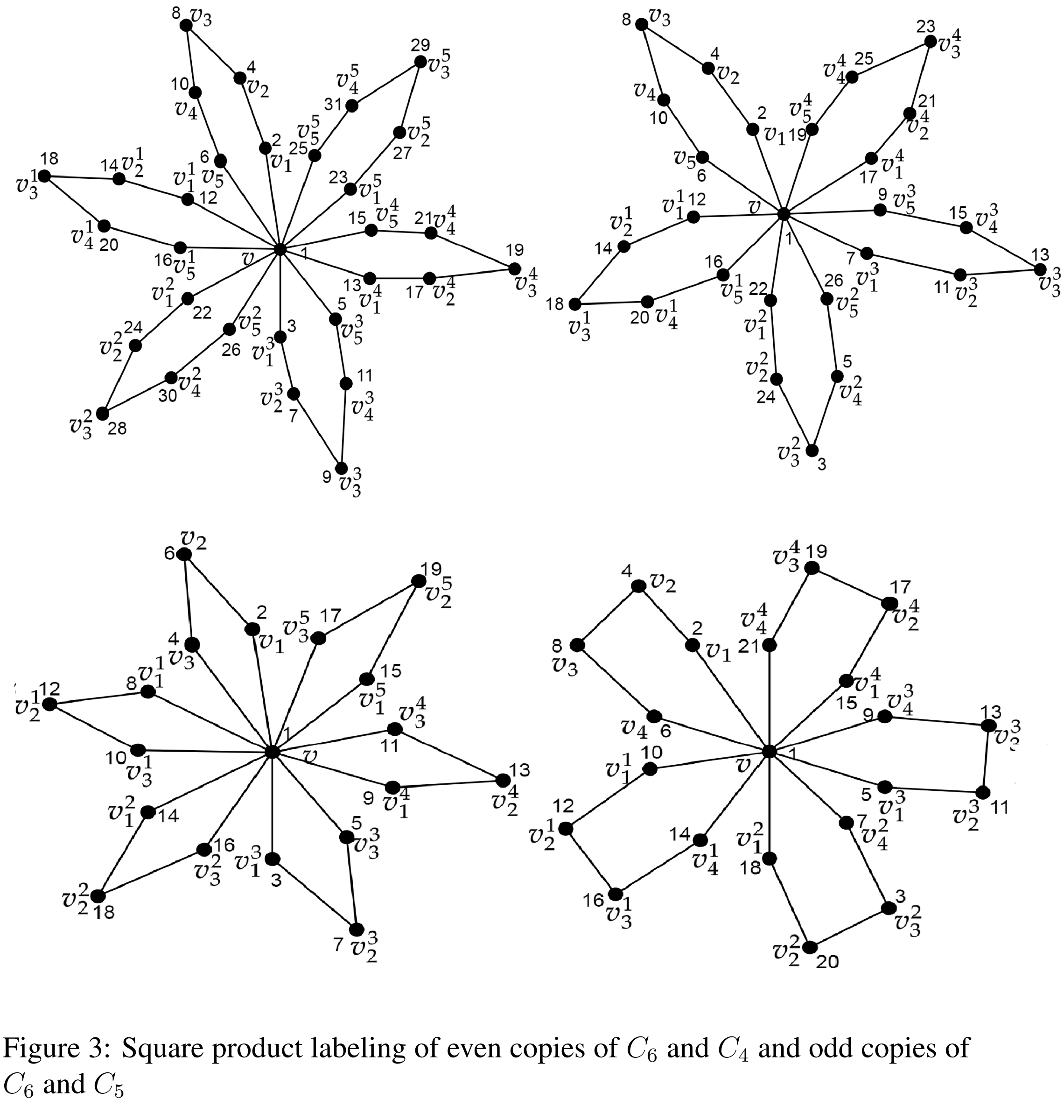

Figure 3: Square product labeling of even copies of C_6 and C_4 and odd copies of C_6 and C_5

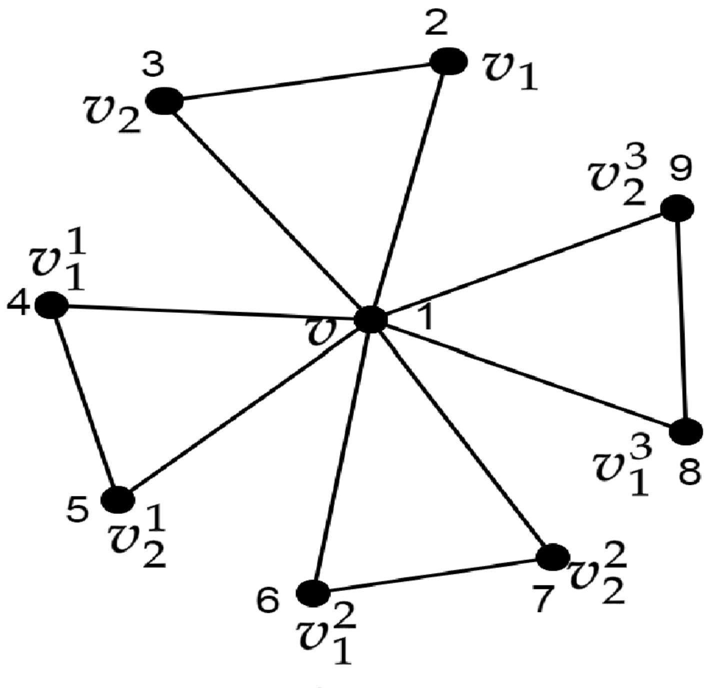

Figure 4: Square product labeling of r copies of C_3

4 Conclusion

In this article, results on square product labeling for cactus graphs with cycle blocks are established. Here the limitation is results can be established only for r copies of cycle of any length with a common cutvertex except for r copies of cycles of length 3 which are not square product graphs for $\mid V(G) \mid \geq (v_1)^2 \ (v_2)^2$. In this article, the results are established only for cactus graphs with cycle blocks. The results on square product labeling can also be extended to trees.

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