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Radio Mean Labeling of Digraphs

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Abstract

Let D be a strong digraph and let $\vec{d}(u, v)$ denote the distance between any two vertices in D. A radio mean labeling is a one-to-one mapping f from $V(D)$ to N satisfying the condition $\vec{d}(u, v) + \left\lceil \frac{f(u)+f(v)}{2} \right\rceil \geq 1 + diam(D)$ for every $u, v \in V(D)$. The span of a labeling f is the maximum integer that f maps to a vertex of D. The radio mean number of D, $rmn\ (D)$ is the lowest span taken over all radio mean labelings of the graph D. In this paper, we analyze radio mean labeling for some newly defined digraphs.

Keywords: Radio Mean, Radio Mean Number, Radio Mean Labeling, Digraphs.

AMS Subject Classification: 05C78‡.

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1. Introduction

The graph labeling problem is one of the recent developing area in graph theory. Alex Rosa first introduced this problem in 1967[10]. Radio labeling is motivated by the channel assignment problem introduced by W. K. Hale in 1980[4].In 2001, Gary Chartrand defined the concept of radio labeling of G[2].Liu and Zhu first determined the radio number in 2005[5].Ponraj et al.[8] introduced the notion of radio mean labeling of graphs and investigated radio mean number of some graphs [9].

In this paper, we introduce a new definition for radio mean labeling of digraphs and also we study radio mean number of some newly defined digraphs.

Radio Labeling is used for X-ray, crystallography, coding theory, network security, network addressing, channel assignment process, social network analysis such as connectivity, scalability, routing, computing, cell biology etc.,

The following results are used in the subsequent section.

Definition 1.1. Let D be a strong digraph and let $\vec{d}(u, v)$ denote the distance between any two vertices in$D$. A radio mean labeling is a one-to-one mapping f from $V(D)$ to N satisfying the condition $\vec{d}(u, v) + \left\lceil \frac{f(u)+f(v)}{2} \right\rceil \geq 1 + diam(D)$ for every $u, v \in V(D)$. The span of a labeling f is the maximum integer that f maps to a vertex of$D$. The radio mean number of$D$, $rmn\ (D)$ is the lowest span taken over all radio mean labelings of the graph$D$.

Definition 1.2. Consider a globe. Let u, v be the vertices of degree$n$. Orient the edges of all but one $u - v$ path of length two in same direction. Orient the left out $u - v$ path in opposite direction. It is strongly connected and is called a Diglobe. It is denoted as $\overrightarrow{Gl(n)}$.

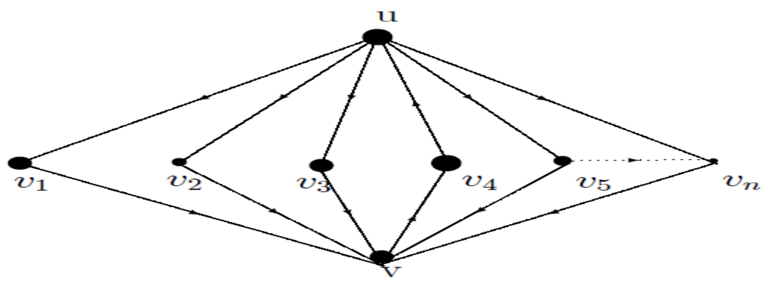

Figure 1.1. Diglobe

Definition 1.3. If the edges of all $u - v$ paths are in a single common direction it is not a strong digraph. We name it as sole diglobe ($\overrightarrow{SGl(n)}$).

Remark 1.4. If there are atleast two paths are oriented in opposite directions, the diglobe becomes strongly connected.

Remark 1.5. Orient the edges of the globe in such a way that the two edges in each $u - v$ path of length 2 get opposite directions. The resulting digraph is called alternate diglobe and is denoted as $\overrightarrow{AGl(n)}$. $\overrightarrow{AGl(n)}$ is not a strong digraph.

Definition 1.6. Consider a book with n pages sharing a common edge. The common edge is called the spine or base of the book. Orient all the edges except the spine in the one single direction and the spine in opposite direction. The resulting digraph is strongly connected and is called as directed book. The directed book is denoted as $\overrightarrow{B(m,\, n)}$.

Remark 1.7. The ditriangular book with n-pages is defined as n copies of cycle C3 sharing a common edge in a directed book. The ditriangular book is denoted as $\overrightarrow{B(3,\, n)}$.

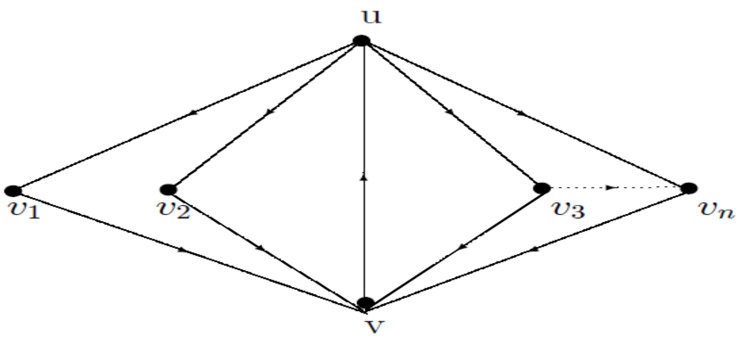

Figure 1.2. Ditriangular Book

Remark 1.8. The diquadrilateral book with n-pages is defined as n copies of cycle C_4 sharing a common edge in a directed book. The diquadrilateral book is denoted as $\overrightarrow{B(4,n)}$.

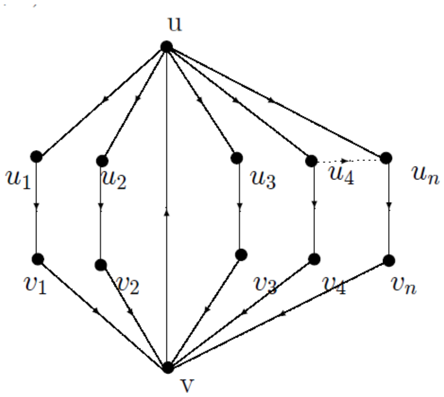

Figure 1.3. Diquadrilateral Book

2. Main Results

Theorem 2.1. The radio mean number of diglobe$\left(\overrightarrow{Gl(n)}\right)$ is less than or equal to $n+3$ for$3 \leq n \leq 4$.

Proof. Let D be a diglobe.

Let $V(D) = \{v_1, v_2, v_3, \dots \dots, v_n, u, v\}$ and

$A(D) = \{\overrightarrow{uv_i}/1 \leq i \leq \left\lfloor\frac{n}{2}\right\rfloor\} \cup \{\overrightarrow{uv_i}/\left\lfloor\frac{n}{2}\right\rfloor + 2 \leq i \leq n\} \cup \{\overrightarrow{v_{\lfloor\frac{n}{2}\rfloor+1}u}\} \cup \{\overrightarrow{v_iv}/1 \leq i \leq \left\lfloor\frac{n}{2}\right\rfloor\} \cup \{\overrightarrow{v_iv}/\left\lfloor\frac{n}{2}\right\rfloor + 2 \leq i \leq n\} \cup \{\overrightarrow{vv_{\lfloor\frac{n}{2}\rfloor+1}}\}$

The diameter of diglobe is 4.

Define $f: V(D) \to N$ by

$f(v_i) = i + 1 \;, \qquad 1 \leq i \leq n$

$f(u) = n + 2$

$f(v) = n + 3$

Claim. f is a valid radio mean labeling.

Since the diameter is 4, to prove f is a radio mean labeling, it is enough to prove that, $\vec{d}(x,y) + \left\lceil\frac{f(x)+f(y)}{2}\right\rceil \geq 5 \dots \dots \dots \dots$ (1)for every pair of vertices (x,y) where $x \neq y$.

Equivalently, it is enough to prove (1) for pair of vertices with minimum f values and minimum $\vec{d}(x,y)$values. Hence, the proof involves the following cases

Case a. Consider the pairs(u, v_i). Here, $\vec{d}(u, v_i) \geq 1$

$$\vec{d}(u, v_i) + \left\lceil\frac{f(u) + f(v_i)}{2}\right\rceil \geq 1 + \left\lceil\frac{n+2+i+1}{2}\right\rceil \geq 5$$

Case b. Consider the pairs(v_i, v). Here, $\vec{d}(v_i, v) \geq 1$

$$\vec{d}(v_i, v) + \left\lceil\frac{f(v_i) + f(v)}{2}\right\rceil \geq 1 + \left\lceil\frac{i+1+n+3}{2}\right\rceil \geq 5$$

Case c. Consider the pairs (u, v) $and\, \vec{d}(u, v) = 2$

$$\vec{d}(u, v) + \left\lceil\frac{f(u) + f(v)}{2}\right\rceil \geq 2 + \left\lceil\frac{n+2+n+3}{2}\right\rceil > 5$$

Case d. Consider the pairs (v_i, v_{i+1}) and $\vec{d}(v_i, v_{i+1}) = 2$, then,

$$\vec{d}(v_i, v_{i+1}) + \left\lceil\frac{f(v_i) + f(v_{i+1})}{2}\right\rceil \geq 2 + \left\lceil\frac{i+1+i+2}{2}\right\rceil > 5$$

Hence, by all the above cases, the radio mean condition is satisfied by f.

Further, f attains its maximum corresponding to v and therefore $rmn\left(\overrightarrow{Gl(n)}\right) \leq n+3$ for $3 \leq n \leq 4$.

Theorem 2.2. The radio mean number of diglobe$\left(\overrightarrow{Gl(n)}\right)$ is $n+2$ for n > 4.

Proof. Let D be a diglobe.

Let $V(D) = \{v_1, v_2, v_3, \dots \dots, v_n, u, v\}$ and

$A(D) = \{\overrightarrow{uv_i}/1 \leq i \leq \left\lfloor\frac{n}{2}\right\rfloor\} \cup \{\overrightarrow{uv_i}/\left\lfloor\frac{n}{2}\right\rfloor + 2 \leq i \leq n\} \cup \{\overrightarrow{v_{\lfloor\frac{n}{2}\rfloor+1}u}\} \cup \{\overrightarrow{v_iv}/1 \leq i$

$\leq \left\lfloor \frac{n}{2} \right\rfloor\} \cup \{\overrightarrow{v_i v} / \left\lfloor \frac{n}{2} \right\rfloor + 2 \leq i \leq n\} \cup \{\overrightarrow{vv_{\lfloor \frac{n}{2} \rfloor + 1}}\}$

The diameter of diglobe is 4.

Define $f: V(D) \to N$ by

$f(v_i) = i \ , \qquad 1 \leq i \leq n$

$f(u) = n + 1$

$f(v) = n + 2$

Claim. f is a valid radio mean labeling.

Since the diameter is 4, to prove f is a radio mean labeling, it is enough to prove that, $\vec{d}(x, y) + \left\lceil \frac{f(x)+f(y)}{2} \right\rceil \geq 5$ ... ... (1) for every pair of vertices (x, y) where $x \neq y$.

Equivalently, it is enough to prove (1) for pair of vertices with minimum f values and minimum $\vec{d}(x, y)$values. Hence, the proof involves the following cases

Case a. Consider the pairs (u, v_i). Here, $\vec{d}(u, v_i) \geq 1$

$$\vec{d}(u, v_i) + \left\lceil \frac{f(u) + f(v_i)}{2} \right\rceil \geq 1 + \left\lceil \frac{n + 1 + i}{2} \right\rceil \geq 5$$

Case b. Consider the pairs (v_i, v). Here, $\vec{d}(v_i, v) \geq 1$

$$\vec{d}(v_i, v) + \left\lceil \frac{f(v_i) + f(v)}{2} \right\rceil \geq 1 + \left\lceil \frac{i + n + 2}{2} \right\rceil \geq 5$$

Case c. Consider the pairs (u, v) $\ and \vec{d}(u, v) = 2$

$$\vec{d}(u, v) + \left\lceil \frac{f(u) + f(v)}{2} \right\rceil \geq 2 + \left\lceil \frac{n + 1 + n + 2}{2} \right\rceil > 5$$

Case d. Consider the pairs (v_i, v_{i+1}) and $\vec{d}(v_i, v_{i+1}) = 2$, then,

$$\vec{d}(v_i, v_{i+1}) + \left\lceil \frac{f(v_i) + f(v_{i+1})}{2} \right\rceil \geq 2 + \left\lceil \frac{i + i + 1}{2} \right\rceil > 5$$

Hence, by all the above cases, the radio mean condition is satisfied by f.

Further, f attains its maximum corresponding to v and is $n + 2$ for $n > 4$.

Since D contains only $n + 2$ vertices, $n + 2$ is the minimum of the maximum integer that could be assigned to the vertices of D.

Hence $rmn(D) = n + 2$ for$n > 4$.

Theorem 2.3. The radio mean number of ditriangular book$\left(\overrightarrow{B(3, n)}\right)$ is less than or equal to $n + 3$ for$n = 2$.

Proof. Let D be a ditriangular book.

Let $V(D) = \{v_1, v_2, v_3, \dots \dots ., v_n, u, v\}$ and

$A(D) = \{\overrightarrow{uv_i} / 1 \leq i \leq n\} \cup \{\overrightarrow{v_i v} / 1 \leq i \leq n\} \cup \{\overrightarrow{vu}\}$

The diameter of ditriangular book is 3.

Define $f: V(D) \to N$ by

$f(v_i) = i + 1 \ , \qquad 1 \leq i \leq n$

$f(u) = n + 2$

$f(v) = n + 3$

Claim. f is a valid radio mean labeling.

Since the diameter is 3, to prove f is a radio mean labeling, it is enough to prove that, $\vec{d}(x,y) + \left\lceil \frac{f(x)+f(y)}{2} \right\rceil \geq 4$ ... ... (1)for every pair of vertices (x,y) where $x \neq y$. Equivalently, it is enough to prove (1) for pair of vertices with minimum f values and minimum $\vec{d}(x,y)$values. Hence, the proof involves the following cases

Case a. Consider the pairs (u, v_i). Here, $\vec{d}(u, v_i) \geq 1$

$$\vec{d}(u,v_i) + \left\lceil \frac{f(u)+f(v_i)}{2} \right\rceil \geq 1 + \left\lceil \frac{n+2+i+1}{2} \right\rceil \geq 4$$

Case b. Consider the pairs (v_i, v). Here, $\vec{d}(v_i, v) \geq 1$

$$\vec{d}(v_i,v) + \left\lceil \frac{f(v_i)+f(v)}{2} \right\rceil \geq 1 + \left\lceil \frac{i+1+n+3}{2} \right\rceil > 4$$

Case c. Consider the pairs (u, v) $and\, \vec{d}(u, v) = 1$

$$\vec{d}(u,v) + \left\lceil \frac{f(u)+f(v)}{2} \right\rceil \geq 1 + \left\lceil \frac{n+2+n+3}{2} \right\rceil > 4$$

Case d. Consider the pairs (v_i, v_{i+1}) and $\vec{d}(v_i, v_{i+1}) = 3$, then,

$$\vec{d}(v_i,v_{i+1}) + \left\lceil \frac{f(v_i)+f(v_{i+1})}{2} \right\rceil \geq 3 + \left\lceil \frac{i+1+i+2}{2} \right\rceil > 4$$

Hence, by all the above cases, the radio mean condition is satisfied by f.

Further, f attains its maximum corresponding to v and therefore $rmn(\overrightarrow{B(3,n)}) \leq n + 3$ for $n = 2$.

Theorem 2.4. The radio mean number of ditriangular book$(\overrightarrow{B(3,n)})$ is $n + 2$ for $n > 2$.

Proof. Let D be a ditriangular book.

Let $V(D) = \{v_1, v_2, v_3, \dots \dots ., v_n, u, v\}$ and

$A(D) = \{\overrightarrow{uv_i}/1 \leq i \leq n\} \cup \{\overrightarrow{v_i v}/1 \leq i \leq n\} \cup \{\overrightarrow{vu}\}$

The diameter of ditriangular book is 3.

Define $f: V(D) \to N$ by

$f(v_i) = i$, $1 \leq i \leq n$

$f(u) = n + 1$

$f(v) = n + 2$

Claim. f is a valid radio mean labeling.

Since the diameter is 3, to prove f is a radio mean labeling, it is enough to prove that, $\vec{d}(x,y) + \left\lceil \frac{f(x)+f(y)}{2} \right\rceil \geq 4$ ... ... (1)for every pair of vertices (x,y) where $x \neq y$. Equivalently, it is enough to prove (1) for pair of vertices with minimum f values and minimum $\vec{d}(x,y)$values. Hence, the proof involves the following cases

Case a. Consider the pairs (u, v_i). Here, $\vec{d}(u, v_i) \geq 1$

$$\vec{d}(u,v_i) + \left\lceil \frac{f(u)+f(v_i)}{2} \right\rceil \geq 1 + \left\lceil \frac{n+1+i}{2} \right\rceil \geq 4$$

Case b. Consider the pairs (v_i, v). Here, $\vec{d}(v_i, v) \geq 1$

$$\vec{d}(v_i,v) + \left\lceil \frac{f(v_i)+f(v)}{2} \right\rceil \geq 1 + \left\lceil \frac{i+n+2}{2} \right\rceil \geq 4$$

Case c. Consider the pairs (u,v) $and\,\overrightarrow{d}(u,v)=1$

$$\overrightarrow{d}(u,v)+\left\lceil\frac{f(u)+f(v)}{2}\right\rceil\geq 1+\left\lceil\frac{n+1+n+2}{2}\right\rceil>4$$

Case d. Consider the pairs (v_i,v_{i+1}) and $\overrightarrow{d}(v_i,v_{i+1})=3$, then,

$$\overrightarrow{d}(v_i,v_{i+1})+\left\lceil\frac{f(v_i)+f(v_{i+1})}{2}\right\rceil\geq 3+\left\lceil\frac{i+i+1}{2}\right\rceil>4$$

Hence, by all the above cases, the radio mean condition is satisfied by$f$.
Further, f attains its maximum corresponding to v and is $n+2$ for $n>2$.
Since D contains only $n+2$ vertices, $n+2$ is the minimum of the maximum integer that could be assigned to the vertices of$D$.
Hence $rmn(D)=n+2$ for$n>2$.

Theorem 2.5. The radio mean number of diquadrilateral book$\left(\overrightarrow{B(4,n)}\right)$ is $2n+2$ for $n>3$.
Proof. Let D be a diquadrilateral book.
Let $V(D)=\{v_1,v_2,v_3,\ldots\ldots,v_n,u_1,u_2,u_3,\ldots\ldots,u_n,u,v\}$ and
$A(D)=\{\overrightarrow{uu_i}/1\leq i\leq n\}\cup\{\overrightarrow{u_iv_i}/1\leq i\leq n\}\cup\{\overrightarrow{v_iv}/1\leq i\leq n\}\cup\{\overrightarrow{vu}\}$
The diameter of diquadrilateral book is 5.
Define $f:V(D)\rightarrow N$ by
$f(u_i)=i\ ,\qquad 1\leq i\leq n$
$f(v_i)=2n-i+1\ ,\qquad 1\leq i\leq n$
$f(u)=2n+1$
$f(v)=2n+2$
Claim. f is a valid radio mean labeling.
Since the diameter is5, to prove f is a radio mean labeling, it is enough to prove that, $\vec{d}(x,y)+\left\lceil\frac{f(x)+f(y)}{2}\right\rceil\geq 6\ldots\ldots\ldots\ldots(1)$for every pair of vertices (x,y) where $x\neq y$.
Equivalently, it is enough to prove (1) for pair of vertices with minimum f values and minimum $\vec{d}(x,y)$values. Hence, the proof involves the following cases

Case a. Consider the pairs (u,u_i). Here, $\overrightarrow{d}(u,u_i)\geq 1$

$$\overrightarrow{d}(u,u_i)+\left\lceil\frac{f(u)+f(u_i)}{2}\right\rceil\geq 1+\left\lceil\frac{2n+1+i}{2}\right\rceil\geq 6$$

Case b. Consider the pairs (v_i,v). Here, $\overrightarrow{d}(v_i,v)\geq 1$

$$\overrightarrow{d}(v_i,v)+\left\lceil\frac{f(v_i)+f(v)}{2}\right\rceil\geq 1+\left\lceil\frac{2n-i+1+2n+2}{2}\right\rceil>6$$

Case c. Consider the pairs (u_i,v_i) $and\,\overrightarrow{d}(u_i,v_i)=1$

$$\overrightarrow{d}(u_i,v_i)+\left\lceil\frac{f(u_i)+f(v_i)}{2}\right\rceil\geq 1+\left\lceil\frac{i+2n-i+1}{2}\right\rceil>6$$

Case d. Consider the pairs (u_i,u_{i+1}) and $\overrightarrow{d}(u_i,u_{i+1})=4$, then,

$$\overrightarrow{d}(u_i,u_{i+1})+\left\lceil\frac{f(u_i)+f(u_{i+1})}{2}\right\rceil\geq 4+\left\lceil\frac{i+i+1}{2}\right\rceil\geq 6$$

Case e. Consider the pairs (v_i, v_{i+1}) and $\overrightarrow{d}(v_i, v_{i+1}) = 4$, then,

$$\overrightarrow{d}(v_i, v_{i+1}) + \left\lceil \frac{f(v_i) + f(v_{i+1})}{2} \right\rceil \geq 4 + \left\lceil \frac{2n - i + 1 + 2n - i}{2} \right\rceil > 6$$

Hence, by all the above cases, the radio mean condition is satisfied by f.
Further, f attains its maximum corresponding to v and is $2n + 2$ for $n > 3$.
Since D contains only $2n + 2$ vertices, $2n + 2$ is the minimum of the maximum integer that could be assigned to the vertices of D.
Hence, $rmn(D) = 2n + 2$ for $n > 3$.

3. Conclusions

In this papers we compute some types of newly defined digraphs. In future we will find that digraphs (Radio Labeling) is used for X-ray, crystallography, coding theory, network security, network addressing, channel assignment process, social network analysis such as connectivity, scalability, routing, computing, cell biology etc.,

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