Learning Mathematics for Personal Understanding and Productions: A Viewpoint

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In this paper we reflect on what makes mathematics more meaningful and more easily understood and thus enabling the learner to apply it to everyday situations in his/her life world. We identify personal – in relation to 'collective' or 'public' – mathematising as one key component towards real understanding of mathematics. We observe that today's mathematics learner is often typified by such orientations as approaching the subject with timidity and in a cookbook fashion, adopting a re-productive rather than a productive mode, and showing lack of intrinsic interest in the subject. Debilitating effects of some of these characteristics in relation to learning mathematics for personal development, include learner's failure to exploit the subject's natural features for developing own mental orientations such as algorithmic, stochastic, reflective, and creative thinking so essential in coping with modern life environments. We propose that, for inspirational effects, learners should have closer contact with and appreciation for the activities and practices of the professional mathematician. The mathematics teacher could enhance the learner's mathematical learning experience by orienting instructional designs in ways that make the learning processes and outcomes more personal to the learner.

Introduction

> Naturally, I set myself the task of constructing all these functions. I conducted a systematic siege and one after another, carried all the outworks; there was however one which still held out and whose fall would bring about that of the whole position. But all my efforts served only to make me better acquainted with the difficulty, which in itself was something. All this work was perfectly conscious. At this point I left for Mont-Valerin, where I was to discharge my military service. I had therefore very different preoccupations. One day, while crossing the boulevard, the solution of the difficulty which had stopped me appeared to me all of a sudden. I did not seek to go into it immediately, and it was only after my service that I resumed the question. I had all the elements, and had only to assemble and order them. So I wrote out my definitive memoir at one stroke and with no difficulty.[Henri Poincaré] (Bell, 1965, p. 551).

This paper takes a reflective look at the problem of personal mathematising and problem-solving applications in everyday lives of people in general, and the school learner in particular. Each person needs to develop some mathematical concepts and methods for understanding and better managing of everyday activities in our modern world (Davis & Hersh, 1980). For example, plain every language has some mathematical ideas involved. Therefore for one to communicate effectively, one needs some basic mathematical concepts. Fuller understanding of mathematical concepts and methods is achieved through personal mathematising by the learner her/himself. By personal mathematising we mean that each individual learner is involved in mathematical activities associated with the formation of a mathematical concept or method.

Some theoretical considerations

The notion of mathematising has been popularised in the last few decades as'pedagogical scaffolding' aimed at achieving learning mathematics with understanding (Freudenthal, 1973; 1983; 1991; Gellert & Jablonka, 2007). Notable proponents of this theory are the Dutch mathematician-cum-educator, Hans Freudenthal, and his followers in the Realistic Mathematics movement, whose work has contributed to an increased focus in learning processes in mathematics education (de Lange, 1996; Gravemeijer, 1999; Gravemeijer & Cobb, 2002; Treffers, 1987; 1993). According to this school of thought, mathematising is the crucial skill or knowledge required in order for one to learn mathematics with understanding, and at the heart of mathematisation lies the idea of what they call the 'reinvention principle' that is partially captured in the following statement:

> Children should repeat the learning process of mankind, not as factually took place but rather as it would have done if people in the past had known a bit more of what we know now. (Freudenthal, 1991, p. 48)

This notion of mathematisation has continued to be elaborated by other authors to include dimensions such as the epistemological in conjunction with the closely related notion of mathematical modelling, and the socio-cultural through the associated notion of demathematisation (Gellert & Jablonka, 2007).

Mathematical activities involve, at elementary level, investigations of quantity, that is number, measurement and the relationships that occur among them and of shape and its properties (Davis & Hersh, 1980). However, true mathematical activities involve the search for patterns and trends and deduction of valid conclusions from given premises or hypotheses. Thus the study of mathematics should foster an analytic mind so that given a problem in any area of human endeavor one is able to look for trends and patterns and so go a long way towards a solution of the problem. For example, very intricate problems of chance and logic have been conquered through direct application of mathematical processes (Hartsfield & Ringel, 1994; Zawaira & Hitchcock, 2009). Mathematical processes is a collective term that embraces many sub-processes involving specific and distinctive types of actions and objects – almost entirely mental – and interrelations among them (Gray & Tall, 2007).

One of the key characteristics of mathematics is that of abstraction, described by Damerow (2007, p. 22) as a 'meta-cognitive construct' that is necessary for the production of mathematical concepts, or as a deliberate effort requiring coordination of a variety of, often parallel, (mental) actions – a description that builds on Piagetian conceptualisations (Dubinsky, 1991; von Glasersfeld, 1999). Damerow further places the idea of representation or symbolic operation at the centre of abstraction whilst at the same time making linkages of these mental processes and objects to the socio-cultural life worlds of the learner. More recently Gray and Tall (2007, p. 38) have postulated that abstraction is driven by a phenomenon they have called 'compression,' saying: "We believe that the natural process of abstraction through compression of knowledge into more sophisticated thinkable concepts is the key to increasingly powerful thinking" (p. 38).

Abstraction plays a crucial role when, for instance, from a few concrete examples of an idea or method one picks out the essential points of the concept or points that makes the method work and then use these as conditions obeyed by more general systems (Courant & Hilbert, 1953). Thus movement from specific examples to more general systems or generalisation is an important characteristic of mathematics.

Let us look at two examples of generalisation.

Example 1: Addition of fractions: **Find the sum** $\frac{2}{5}+\frac{1}{3}$

To work at, say, Grade 4 level, one needs to realise that we have two fractions of different types, that is, in concrete terms, two pieces of fifths (two pieces of a whole divided into five equal parts) and one piece of thirds (one piece of a whole divided into three equal parts). How can these be combined so that we come up with something that tells us what part of the whole we end up with? One of the simplest methods is to divide each of the two fifths into a further three equal parts (thus getting six pieces of a whole divided into fifteen equal parts), and dividing the one third into five equal parts (thus getting five pieces of a whole divided into fifteen equal parts). Combining them, we then have eleven equal parts of fifteenths. Problem solved.

The first generalisation is to say that this method works for all additions of fractions.

Further generalisation of the *method* is to say the method works for the addition of algebraic fractions.

Example 2: Indices

From the fact that we can write a^m for $a \times a \times a \times \ldots \times a$ (m factors), deduce the following rules for the manipulation of indices of positive whole numbers m and n:

$$a^m \times a^n = a^{m+n}, \; a^m \div a^n = a^{m-n} \; (m > n), \; (a^m)^n = a^{mn}$$

We can only show that these rules hold for positive whole numbers.

We then generalise by assuming that the rules hold for indices of all positive whole numbers, negative whole numbers, fractions and zero.

Assuming that these rules hold for these indices, we then deduce the meanings of each of the following:

$a^0, a^{-n}, a^{1/n}, a^{m/n}$ (Skemp, 1971)

We are saying here that the generalising and abstracting thought processes must happen deliberately and consciously (though these could happen subconsciously later). In other words it must be sort of a live self-conversation, something related to the notion of meta-cognition (Gray & Tall, 2007). Our claim here is that these processes, which constitute what we have called personal and authentic mathematising, are not well developed in today's learner. Think, for example, of how the mathematician Henri Poincare had internalised the problem, and while performing different ordinary tasks, his subconscious was busy working on the problem, and hence obtained a solution in the end (refer to the quotation at the beginning of the introduction). If learners can activate the personal dimension and engage in authentic mathematising, they can more easily apply mathematical ideas and skills in solving everyday problems in their own life worlds.

As most concepts in mathematics arose naturally from the need to solve everyday problems, learning these concepts has to involve use of similar everyday problems that inspired the development of those concepts. For example, the natural numbers and counting arose most probably from the need to record one's possessions. The study of geometry might have arisen to serve the needs of builders, agriculturalists, and others (Zawaira & Hitchcock, 2009). Personal mathematising thus leads to greater understanding of mathematical concepts and methods by each learner as the learner is involved in concept or method formation – in our view a more productive rather than re-productive learning outcome.

Clearly the learning of mathematics is more effective if concepts are introduced in as many different ways as possible. The learner is able to pick out the essential elements of a concept and discard the irrelevant elements (Davis & Hersh, 1980). Consider the concept of twoness or the number two. The number occurs in two oranges, two pigs, two vegetables, two people, two things, etcetera. The learner will realise that the twoness does not involve color, quality of objects, type of objects, etcetera, but has to do with quantity (see Damerow (2007) for a very good discussion of the notion of quantity and space).

While the above discussion has centred on attempts to explicate the meaning of and mechanisms for acquiring mathematical knowledge or for doing mathematics, our interest in this paper is not so much to extend the discussion in that same vein. The main concepts, however, provide us with a theoretical context that in turn provides a language for talking about and lens for viewing personalisation of mathematics learning, which is the primary focus of the paper.

Some characteristics of a modern learner of mathematics

We believe that without robust personal mathematising, the learner shows some or all of the following characteristics that indicate lack of full understanding:

- learns in a cookbook fashion
- reproduces for the teacher or for the public and not a producer for self or the public
- is intimidated by mathematical language
- shows little interest in the subject
- is afraid of or avoids taking risks with own ideas or concepts.

When one learns a concept in a cookbook fashion, one is often unable to apply the concept to problems that appear slightly different from the ones met before. In other words, there is no transfer of learning.

Take, for example, this problem which are often encountered in undergraduate mathematics:

Show that $\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{n+n}$, $n = 1, 2, 3, \ldots$ is bounded above and below.

This problem is basically a Grade 4 problem where we show that $\frac{a}{b} > \frac{a}{c}$ if and only if $b < c$, for example, $\frac{3}{5} > \frac{3}{7}$ because $5 < 7$.

This follows clearly from understanding our notation of common fractions: $\frac{3}{7}$ means that we have three pieces of a whole (unity) which has been divided into seven equal parts, that is, the seven relates to the number of equal parts into which the whole or unity has been divided. Thus if the denominator is small, the size of the equal parts into which the whole or unity is cut is large. Now applying this Grade 4 idea to our undergraduate problem, we have:

$$\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{n+n}\leq\frac{1}{n+1}+\frac{1}{n+1}+\cdots+\frac{1}{n+1}$$

And $\frac{n}{n+n}\leq\frac{1}{n+1}+\frac{1}{n+1}+\cdots+\frac{1}{n+1}\leq\frac{n}{n+1}\leq\frac{n}{n}$

Hence $\frac{1}{2}\leq\frac{1}{n+1}+\frac{1}{n+1}+\cdots+\frac{1}{n+1}\leq\frac{n}{n+1}\leq 1$

Therefore the expression is bounded below by $\frac{1}{2}$ and above 1.

The teaching of mathematics should significantly contribute towards producing a person who has a productive and satisfying life in the modern society. It should produce learners who can exploit the facility and power of mathematics as a tool or medium in their everyday lives to enhance their capacities as

- consumers: sorting and valuing products, buying/selling/trading/exchanging, decision making (*algorithmic thinking*)
- citizens: negotiating rights and privileges, assessing dangers, and balancing acts (*statistical thinking)*
- family persons: planning, predicting, managing others, minimum/maximum problems (*stochastic thinking*)
- social/cultured beings: communicating, the connected world (the information processing society: internet, cellphones), that is, interacting with others and the material world (*reflective thinking*).
- individual beings (as opposed to an animal): intellectual satisfaction, that is, exercising imagination, beauty of form, elements of proof, logic (*imaginative thinking*)
- professionals (any profession including mathematician): enjoy profession, producing and creating products, technical/ engineering aspects of mathematics (*creative thinking*)

We suggest that it is best to think of this personalisation dimension that is being highlighted here as a general mental and affective orientation towards engagement with mathematical material. That orientation, or 'habit of the mind and heart,' is something we are saying needs to be attended to explicitly and developed effectively in learners. When well developed and functioning optimally and in concert with other complementary 'orientation-cum-skills' such as heuristic thinking (Polya, 1973; Shoenfeld, 1985a), commonsensical thinking (Kilpatrick, 2007), symbolic thinking (Damerow, 2007; Gray & Tall, 2007), visual thinking (Tall & Vinner, 1981; Vinner & Hershkowitz, 1980) and meta-cognitive thinking (Shoenfeld, 1985b; Silver, 1987), the learner should be able to gain better understanding of mathematics and use relevant aspects of the knowledge and skills productively in his/her daily life.

We are not saying that all people should be molded like professional mathematicians; rather we are saying it would be useful for every individual to have some capacity to behave like a sort of *amateur mathematician* at the very least, if you like. It is needless to remind that mathematicians are ordinary human beings too. We suspect that there may be a general problem of under-valuing mathematics as something that may contribute significantly to the overall composite well-being of individual life of every person. And so we are saying mathematics assists and contributes to the realisation of the full potential of the person.

Teaching and learning

The question of how we as educators can assist in the transformation of the current learner product (as described above) to the desirable product becomes an imperative. How can we intervene through learning and teaching? We offer some suggestions to the learner, teacher and to the public. Mathematics is commonly regarded as something to be feared, something very difficult, hence often considered the exclusive province for geniuses. To help overcome this perception, we must make the learning environment non-threatening. Furthermore we must encourage cooperation and collaboration in tackling problems in mathematics. Adequate and relevant resources for teaching and learning should be made available in as many forms of presentation as possible and through as many different types of sources as possible, for example experiential, imaginative, simulations (as in many internet sources), written text, verbal renditions, physical representations, etcetera.

Meaningful learning of mathematics should help one to develop a number of mental abilities. If there is fruitful learning of mathematics, a learner is provided with opportunities to search for relationships between things, to look for patterns and trends among given data, observe unusual phenomena, etcetera. Thus in general, learning in mathematics should develop heuristic thinking, a mental skill which is needed by all citizens.

The ability to abstract is one that is developed very effectively in the learning of mathematics. As noted earlier, from a few concrete examples one isolates the crucial elements that makes them work and then applies them to any system in which these elements are present (Courant & Hilbert, 1953). Thus, we obtain an abstract system, that is, a system that does not apply to one special kind but to many systems in which the conditions apply. The examples above of adding arithmetical fractions and of indices illustrate the point sufficiently. Another important ability is that of visualising, that is, the ability to form mental pictures of real or imagined objects.

In mathematics we constantly use symbols to represent numbers, statements, etcetera, and operate with these symbols. Thus the ability to symbolise and to operate with symbols is developed considerably in the study of mathematics. Indeed there is a calculus of logic where logic itself is operated symbolically. This is the whole province of the area of specialisation called Mathematical Logic.

Learners can appreciate the finer aspects of mathematics learning from studying the lives of creative mathematicians, in particular, about their passion for the subject and about their perseverance. We shall briefly describe here two anecdotes involving two great mathematicians: Archimedes and Abel.

Archimedes lived in the city of Syracuse about 200 BC during the time of the Punic wars between Rome and Carthage. The Romans wanted to take Syracuse but were prevented from doing so the first time by artillery designed by Archimedes using levers and pulleys. Frustrated, the Romans retreated but returned to capture the city anyway. They found Archimedes so involved in his mathematics that he did not notice that the city had been captured by the Romans. He was busy drawing mathematical diagrams on dust. He only noticed that something was amiss when the shadow of a Roman soldier fell over his diagram, to which he exclaimed “don’t disturb my circles!” (Bell, 1965, p. 34).

Then, with Abel – of the *Abelian Groups* fame:

> There is a charming picture of Abel after his mathematical genius seized him by the fireside with the others chattering and laughing in the room while he researched with one eye on his mathematics and the other on his brothers and sisters. The noise never distracted him and he joined in the badinage as he wrote. (Bell, 1965, p. 308)

Such anecdotes, whether true or exaggerated, serve to make us believe some kind of passion is a necessary condition or, at least, ingredient, for any meaningful personal production of mathematics.

Concluding remarks

Considering the changing times and the pervasiveness of modern technology in our daily lives, we see the notion of personal mathematising as both compelling and promising as a strategy to explore and exploit in the learning of mathematics. In personal mathematising the power and authenticity of mathematical procedures derives from the mathematics itself, rather than from authorities such as the teacher or the textbook, neither of which are readily available to consult with in these times we live in.

An important task for mathematics teachers becomes one of trying to link mathematical concepts and procedures with activities that are meaningful in the learner's lifeworlds, while at the same time cultivating in learners – and in themselves – a passion for doing mathematics for personal knowledge and use.

It would be valuable to reflect on what has been discussed above, consider all suggestions made, and see how practicable they could be for the reader's own personal real life situation. In doing so, the crucial question remains: "How can we get some sort of passion for learning and doing mathematics for personal use and knowledge into our learners?" We leave the question open for the reader to engage with.

References

Bell, E. T. (1965). *Men of mathematics.* New York, NY: Simon and Schuster.

Courant, R., & Hilbert, D. (1953). *Methods of mathematical physics.* London: Interscience Publishers.

Damerow, P. (2007). The material culture of calculation: A theoretical framework for a historical epistemology for the concept of number. In U. Gellert & E. Jablonka (Eds.), *Mathematisation and demathematisation: Social, philosophical and educational ramifications* (pp. 19-56). Rotterdam, The Netherlands: Sense Publishers.

Davis, P. J., & Hersh, R. (1980). *The mathematical experience.* Boston, MA: Birkhauser.

de Lange, J. (1996). Using and applying mathematics in education. In A. J. Bishop, M. A. Clements, C. Keitel, J. Kilpatrick, & C. Laborde (Eds.), *International handbook on mathematics education, Part 1* (pp. 49-97). Dordrecht, The Netherlands: Kluwer Academic Publishers.

Dubinsky, E. (1991). Reflective abstraction in advanced mathematical thinking. In D. O. Tall (Ed.), *Advanced mathematical thinking* (pp. 95-123). Dordrecht, The Netherlands: Kluwer Academic Publishers.

Freudenthal, H. (1973). *Mathematics as an educational task*. Dordrecht, The Netherlands: Riedel.

Freudenthal, H. (1983). *Didactical phenomenology of mathematical structures*. Dordrecht, The Netherlands: Riedel.

Freudenthal, H. (1991). *Revisiting mathematics education: China lectures*. Dordrecht, The Netherlands: Kluwer Academic Publishers.

Gellert, U., & Jablonka, E. (2007). Mathematisation – Demathematisation. In U. Gellert & E. Jablonka, *Mathematisation and demathematisation: Social, philosophical and educational ramifications* (pp. 1-18). Rotterdam, The Netherlands: Sense Publishers.

Gray, E., & Tall, D. (2007). Abstraction as a natural process of mental compression. *Mathematics Education Research Journal, 9*(2), 23-40. doi: 10.1007/BF03217454

Gravemeijer, K. P. E. (1999). How emergent models may foster the constitution of formal mathematics. *Mathematical Thinking and Learning*, *1*(2), 155-177. doi: 10.1207/s15327833mtl0102_4

Gravemeijer, K., & Cobb, P. (2001, April). *Designing classroom-learning environments that support mathematical learning*. Paper presented at the Annual Meeting of the American Educational Research Association. Seattle, WA.

Hartsfield, N., & Ringel, G. (1994). *Pearls in graph theory*. London: Academic Press.

Kilpatrick, J. (2007). Developing common sense in teaching mathematics. In U. Gellert & E. Jablonka (Eds.), *Mathematisation and demathematisation: Social, philosophical and educational ramifications* (pp. 161-170). Rotterdam, The Netherlands: Sense Publishers.

Polya, G. (1973). *How to solve it*. (*2*nd ed.). New York, NY: Doubleday.

Shoenfeld, A. H. (1985a). *Mathematical problem solving*. Orlando, FL: Academic Press.

Shoenfeld, A. H. (1985b). Meta-cognitive and epistemological issues in mathematical understanding. In E. A. Silver (Ed.), *Teaching and learning mathematical problem solving: Multiple research perspectives* (pp. 361-380). Hillsdale, NJ: Lawrence Erlbaum Associates.

Silver, E. A. (1987). Foundations of cognitive theory and research for mathematics problem-solving instruction. In A. H. Shoenfeld (Ed.), *Cognitive science and mathematics education* (pp. 33-60). Hillsdale, NJ: Lawrence Erlbaum Associates.

Skemp, R. R. (1971). *The psychology of learning mathematics.* London: Penguin.

Tall, D. O., & Vinner, S. (1981). Concept image and concept definition in mathematics, with special reference to limits and continuity. *Educational Studies in Mathematics*, *12*(2), 151-169. doi: 10.1007/BF00305619

Treffers, A. (1987). *Three dimensions: A model of goal and theory description in mathematics instruction*. Dordrecht, The Netherlands: Kluwer Academic Publishers.

Treffers, A. (1993). Wiskobas and Freudenthal: Realistic mathematics education, *Educational Studies in Education*, *25*(1), 89-108. doi: 10.1007/BF01274104

Vinner, S., & Hershkowitz, R. (1980). Concept images and some common cognitive paths in the development of simple geometric concepts. In R. Karplus (Ed.), *Proceedings of the 4th Conference of the International Group for the Psychology of Mathematics Education* (pp. 177-184). Berkeley, CA: Lawrence Hall of Science. (ERIC Document Reproduction Service No. ED 250 186).

von Glasersfeld, E. (1991). Abstraction, re-presentation, and reflection: An interpretation of experience and Piaget's approach. In L. P. Steffe (Ed.), *Epistemological foundations of mathematical experience* (pp. 45-65). New York, NY: Springer Verlag.

Zawaira, A., & Hitchcock, G. (2009). *A primer for mathematics competitions*. Oxford: Oxford University Press.