In *Pythagoras 63*, Margot Berger's article **Making mathematical meaning: from preconcepts to pseudoconcepts to concepts** contained errors. An electronic gremlin changed some of the mathematical signs to boxes.

The relevant page, along with introductory text from the previous page of the original article, is reprinted below with apologies to the author.

Brief demonstration

I will use the above theory to explain how a first-year mathematics major student at a South African university moves from an idiosyncratic usage of signs (using, I claim, preconceptual thinking) to a conceptual (or perhaps pseudoconceptual) usage of signs.

The activity took place during an interview which I conducted, video-taped and later transcribed and analysed in 2002 (Berger, 2002).

John had been given the following definition which he has not seen before, although he is familiar with the definite integral and the notion of a limit.

> **Definition of an improper integral with an infinite integration limit**
>
> If f is continuous on the interval [a, ∞), then
>
> $$\int_a^\infty f(x)dx = \lim_{b\to\infty}\int_a^b f(x)dx$$
>
> If $\lim_{b\to\infty}\int_a^b f(x)dx$ exists, we say that the improper integral converges. Otherwise the improper integral diverges.

This is followed by several questions each of which is presented on its own, in order (for example, John has not seen Question 4 when he first encounters, say, Question 1).

> 1. (a) Can you make up an example of an improper integral with an infinite integration limit?
>
> 1. (b) Can you make up an example of a convergent improper integral with an infinite integration limit?
>
> N
>
> 4. Determine whether $\int_1^\infty \frac{dx}{x^3}$ converges or diverges.

John's response, in part, to Question 1(a) is to generate a string of signifiers:

$$\int_0^\infty f(x)dx = \lim_{2\to\infty}\int_0^2 f(x)dx = \lim_{2\to\infty}\int_0^2 \sqrt{x}dx = \int_0^2 \sqrt{x}dx$$

Clearly what John has written is objectively meaningless and inconsistent.

But the point is that John is using the new mathematical signs in mathematical activities (incoherent as they are to the outsider).

In response to Question 1(b), he writes:

$$\lim_{2\to\infty}\int_0^2 f(x)dx = \lim_{2\to\infty}\int_0^\infty \sqrt{x}dx = \int_0^\infty \sqrt{x}dx$$

Again his response appears incoherent and confused. But once more John is using the 'new' (to him) signs in mathematical activities.

I suggest that notions of complex thinking can help the educator understand what is happening. Specifically, I suggest that John's response to both Question 1(a) and 1(b) is dominated by complex thinking. In Question 1(a) he has manipulated the template of an improper integral so that it eventually has the form of a definite integral (i.e. $\int_0^2 \sqrt{x}dx$), a form with which he is familiar. In Question 1(b), he manipulates this further to get back to the template of an improper integral (albeit it does not converge).

The point is: by using various signs in mathematical activities (a functional usage involving template-matching, associations and manipulations primarily) John is able to engage with the mathematical object on first contact, albeit in an idiosyncratic fashion. In this way, John gains a point of entry into mathematical activities with the object before he 'knows' that object.

The question now is: how does John move from this (objectively) incoherent usage to a usage which is both personally satisfying and mathematically acceptable?

I suggest that the answer lies in John's imitation of the improper integral sign. That is, John is finally able to appropriate the socially-sanctioned usage of the improper integral sign through interaction with the mathematics textbook (a resource comprising socially sanctified mathematics).

Specifically, it is only after John has seen exemplars in the textbook of improper integrals and their evaluation, that he starts to use the improper integral in a way that is consonant with its definition. Indeed, after seeing textbook exemplars, he is able to answer Question 4 in a coherent fashion. That is, he writes

$$\int_1^\infty \frac{dx}{x^3} = \lim_{b\to\infty}\int_1^b \frac{dx}{x^3} = \lim_{b\to\infty}\left[\frac{-2}{x^2}\right]_1^b = \lim_{b\to\infty}\left[\frac{-2}{b^2}-2\right] = -2.$$

And he states that this integral is convergent.

Although John has integrated $\int \frac{dx}{x^3}$ incorrectly, ($\int \frac{dx}{x^3} = \frac{-1}{2x^2}$), his response is coherent; also he uses correct procedure and appropriate notation. This is a much improved response compared to his response to Question 1.

Furthermore, John tells me that the examples are useful to him and that he is no longer confused. This contrasts with earlier statements that he is very confused about notions of convergence and divergence and the improper integral.

My contention is: it is John's functional use of the improper integral sign (initially association, template-matching and manipulations and then imitation) that enables him to move from activity dominated by complex thinking to conceptual (possibly pseudoconceptual) activity. Allied to this, he is able to move from a confused notion of the improper integral (by his own admission) to a personally meaningful usage (again, in terms of his own assessment).