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Invited review: The new spin foam models and quantum gravity

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In this article, we give a systematic definition of the recently introduced spin foam models for four-dimensional quantum gravity, reviewing the main results on their semiclassical limit on fixed discretizations.

I. Introduction

The quantization of the gravitational interaction is a major open challenge in theoretical physics. This review presents the status of the *spin foam approach* to the problem. Spin foam models are definitions of the path integral formulation of quantum general relativity and are expected to be the covariant counterpart of the background independent canonical quantization of general relativity known as *loop quantum gravity* [1–3].

This article focuses on the definition of the recently introduced Engle-Pereira-Rovelli-Livine (EPRL) model [4,5] and the closely related Freidel-Krasnov (FK) model [6]. An important original feature of the present paper is the explicit derivation of both the Riemannian and the Lorentzian models, in terms of a notation that exhibits the close relationship between the two, at the algebraic level, that might signal a possible deeper relationship at the level of transition amplitudes.

We will take Plebanski's perspective in which general relativity is formulated as a constrained BF theory (for a review introducing the new models from a bottom-up perspective see Ref. [7]; for an

extended version of the present review including a wide collection of related work see Ref. [8]). For that reason, it will be convenient to start this review by introducing the exact spin foam quantization of BF in the following section. In Section III, we present the EPRL model in both its Riemannian and Lorentzian versions. A unified treatment of the representation theory of the relevant gauge groups is presented in that section. In Section IV, we introduce the FK model and discuss its relationship with the EPRL model. In Section V, we describe the structure of the boundary states of these models and emphasize the relationship with the kinematical Hilbert space of loop quantum gravity. In Section VI, we give a compendium of important issues (and associated references) that have been left out but which are important for future development. Finally, in Section VII, we present the recent encouraging results of the nature of the semiclassical limit of the new models.

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II. Spin foam quantization of BF theory

We will start by briefly reviewing the spin foam quantization of BF theory. This section will be the basic building block for the construction of the models of quantum gravity that are dealt with in this article. The key idea is that the quantum transition amplitudes (computed in the path integral representation) of gravity can be obtained by suitably restricting the histories that are summed over in the spin foam representation of exactly solvable BF theory. We describe the nature of these constraints at the end of this section.

Here, one follows the perspective of Ref. [9]. Let G be a compact group whose Lie algebra \mathfrak{g} has an invariant inner product, here denoted $\langle \cdot, \cdot \rangle$, and \mathcal{M} a d -dimensional manifold. Classical BF theory is defined by the action

$$S[B, \omega] = \int_{\mathcal{M}} \langle B \wedge F(\omega) \rangle, \quad (1)$$

where B is a \mathfrak{g} valued $(d-2)$ -form, ω is a connection on a G principal bundle over \mathcal{M} . The theory has no local excitations: All the solutions of the equations of motion are locally related by gauge transformations. More precisely, the gauge symmetries of the action are the local G gauge transformations

$$\delta B = [B, \alpha], \quad \delta \omega = d_\omega \alpha, \quad (2)$$

where α is a \mathfrak{g} -valued 0-form, and the ‘topological’ gauge transformation

$$\delta B = d_\omega \eta, \quad \delta \omega = 0, \quad (3)$$

where d_ω denotes the covariant exterior derivative and η is a \mathfrak{g} -valued 0-form. The first invariance is manifest in the form of the action, while the second one is a consequence of the Bianchi identity, $d_\omega F(\omega) = 0$. The gauge symmetries are so vast that all the solutions to the equations of motion are locally pure gauge. The theory has only global or topological degrees of freedom.

For the time being, we assume \mathcal{M} to be a compact and orientable manifold. The partition function, \mathcal{Z} , is formally given by

$$\mathcal{Z} = \int \mathcal{D}[B] \mathcal{D}[\omega] \exp(i \int_{\mathcal{M}} \langle B \wedge F(\omega) \rangle). \quad (4)$$

Formally integrating over the B field in (4), we obtain

$$\mathcal{Z} = \int \mathcal{D}[\omega] \delta(F(\omega)). \quad (5)$$

The partition function \mathcal{Z} corresponds to the ‘volume’ of the space of flat connections on \mathcal{M} .

In order to give a meaning to the formal expressions above, we replace the d -dimensional manifold \mathcal{M} with an arbitrary cellular decomposition Δ . We also need the notion of the associated dual 2-complex of Δ denoted by Δ^* . The dual 2-complex Δ^* is a combinatorial object defined by a set of vertices $v \in \Delta^*$ (dual to d -cells in Δ) edges $e \in \Delta^*$ (dual to $(d-1)$ -cells in Δ) and faces $f \in \Delta^*$ (dual to $(d-2)$ -cells in Δ). In the case where Δ is a simplicial decomposition of \mathcal{M} , the structure of both Δ and Δ^* is illustrated in Figs. 1, 2 and 3 in two, three, and four dimensions, respectively.

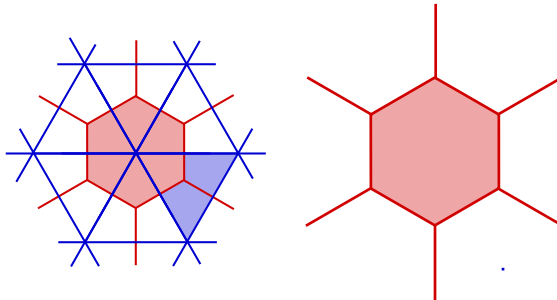


Figure 1: On the left: A triangulation and its dual in two dimensions. On the right: The dual two complex; faces (shaded polygon) are dual to 0-simplices in 2d.

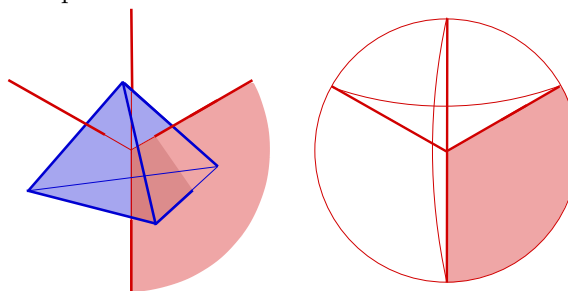


Figure 2: On the left: A triangulation and its dual in three dimensions. On the right: The dual two complex; faces (shaded wedge) are dual to 1-simplices in 3d.

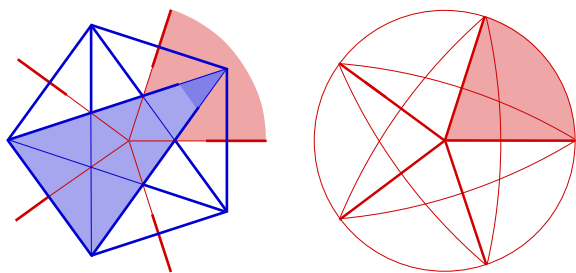


Figure 3: On the left: a triangulation and its dual in four dimensions. On the right: the dual two complex; faces (shaded wedge) are dual to triangles in 4d. The shaded triangle dual to the shaded face is exhibited.

For simplicity, we concentrate on the case when Δ is a triangulation. The field B is associated with Lie algebra elements B_f assigned to faces $f \in \Delta^*$. We can think of it as the integral of the $(d-2)$ -form B on the $(d-2)$ -cell dual to the face $f \in \Delta^*$, namely

$$B_f = \int_{(d-2)\text{-cell}} B. \quad (6)$$

In other words, B_f can be interpreted as the ‘smearing’ of the continuous $(d-2)$ -form B on the $(d-2)$ -cells in Δ . We use the one-to-one correspondence between faces $f \in \Delta^*$ and $(d-2)$ -cells in Δ to label the discretization of the B field B_f . The connection ω is discretized by the assignment of group elements $g_e \in G$ to edges $e \in \Delta^*$. One can think of the group elements g_e as the holonomy of ω along $e \in \Delta^*$, namely

$$g_e = \text{P exp} \left(- \int_e \omega \right), \quad (7)$$

where the symbol “P exp” denotes the path-order-exponential that reminds us of the relationship of the holonomy with the connection along the path $e \in \Delta^*$.

With this, the discretized version of the path integral (4) is

$$\begin{aligned} Z(\Delta) &= \int \prod_{e \in \Delta^*} dg_e \prod_{f \in \Delta^*} dB_f e^{iB_f U_f} \\ &= \int \prod_{e \in \Delta^*} dg_e \prod_{f \in \Delta^*} \delta(g_{e_1} \cdots g_{e_n}), \end{aligned} \quad (8)$$

where $U_f = g_{e_1} \cdots g_{e_n}$ denotes the holonomy around faces, and the second equation is the result of the B integration: It can be, thus, regarded as the analog of (5). The integration measure dB_f is the standard Lebesgue measure, while the integration in the group variables is done in terms of the invariant measure in G (which is the unique Haar measure when G is compact). For given $h \in G$ and test function $F(g)$, the invariance property reads as follows

$$\begin{aligned} \int dg F(g) &= \int dg F(g^{-1}) = \int dg F(gh) \\ &= \int dg F(hg) \end{aligned} \quad (9)$$

The Peter-Weyl’s theorem provides a useful formula or the Dirac delta distribution appearing in (8), namely

$$\delta(g) = \sum_{\rho} d_{\rho} \text{Tr}[\rho(g)], \quad (10)$$

where ρ are irreducible unitary representations of G . From the previous expression, one obtains

$$\begin{aligned} Z(\Delta) &= \sum_{\mathcal{C}: \{\rho\} \rightarrow \{f\}} \int \prod_{e \in \Delta^*} dg_e \prod_{f \in \Delta^*} d_{\rho_f} \text{Tr}[\rho_f(g_e^1 \cdots g_e^N)]. \end{aligned} \quad (11)$$

Integration over the connection can be performed as follows. In a triangulation Δ , the edges $e \in \Delta^*$ bound precisely d different faces. Therefore, the g_e ’s in (11) appear in d different traces. The relevant formula is

$$\begin{aligned} P_{inv}^e(\rho_1, \cdots, \rho_d) \\ := \int dg_e \rho_1(g_e) \otimes \rho_2(g_e) \otimes \cdots \otimes \rho_d(g_e). \end{aligned} \quad (12)$$

For compact G , using the invariance (and normalization) of the the integration measure (9), it is easy to prove that $P_{inv}^e = (P_{inv}^e)^2$ is the projector onto $\text{Inv}[\rho_1 \otimes \rho_2 \otimes \cdots \otimes \rho_d]$. In this way, the spin foam amplitudes of $SO(4)$ BF theory reduce to

$$Z_{BF}(\Delta) = \sum_{\mathcal{C}_f: \{f\} \rightarrow \rho_f} \prod_{f \in \Delta^*} d_{\rho_f} \prod_{e \in \Delta^*} P_{inv}^e(\rho_1, \dots, \rho_d). \quad (13)$$

In other words, the BF amplitude associated to a two-complex Δ^* is simply given by the sum over of all possible assignments of irreducible representations of G to faces of the number obtained by the natural contraction of the network of projectors P_{inv}^e , according to the pattern provided defined by the two-complex Δ^* .

There is a nice graphical representation of the partition function of BF theory that will be very useful for some calculations. On the one hand, using this graphical notation one can easily prove the discretization independence of the BF amplitudes. On the other hand, this graphical notation will simplify the presentation of the new spin foam models of quantum gravity that will be considered in the following sections. This useful notation was introduced by Oeckl [10,11] and used in Ref. [12] to give a general proof of the discretization independence of the BF partition function and the Turaev-Viro invariants for their definition on general cellular decompositions.

We will present this notation in detail: The idea is to represent each representation matrix appearing in (11) by a line (called a wire) labeled by an irreducible representation, and integrations on the group by a box (called a cable). The traces in Eq. (11) imply that there is a wire, labeled by the representation ρ_f , winding around each face $f \in \Delta^*$. In addition, there is a cable (integration on the group) associated with each edge $e \in \Delta^*$. As in (13), there is a projector P_{inv}^e , which is the projector in $\text{Inv}[\rho_1 \otimes \rho_2 \otimes \dots \otimes \rho_d]$ associated to each edge. This will be represented by a cable with d wires, as shown in (14). Such graphical representation allows for a simple diagrammatic expression of the BF quantum amplitudes.

$$P_{inv}^e(\rho_1, \rho_2, \rho_3, \dots, \rho_d) \equiv \begin{array}{c} \rho_1 \rho_2 \rho_3 \dots \rho_d \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \quad (14)$$

The case of physical interest is $d = 4$. In such case, edges are shared by four faces; each cable has

now four wires. The cable wire diagram giving the BF amplitude is dictated by the combinatorics of the dual two complex Δ^* . From Fig. 3, one gets

$$Z_{BF}(\Delta) = \sum_{\mathcal{C}_f: \{f\} \rightarrow \rho_f} \prod_{f \in \Delta^*} d_{\rho} \cdot \text{Diagram} \quad (15)$$

The 10 wires corresponding to the 10 faces $f \in \Delta^*$, sharing a vertex $v \in \Delta^*$, are connected to the neighboring vertices through the 5 cables (representing the projectors in (13) and Fig. 14) associated to the 5 edges $e \in \Delta^*$, sharing the vertex $v \in \Delta^*$.

a. $SU(2) \times SU(2)$ BF theory: a starting point for 4d Riemannian gravity.

We now present the BF quantum amplitudes in the case $G = SU(2) \times SU(2)$. This special case is of fundamental importance in the construction of the gravity models presented in the following sections. The product form of the structure group implies the simple relationship $Z_{BF}(SU(2) \times SU(2)) = Z_{BF}(SU(2))^2$. Nevertheless, it is important for us to present this example in an explicit way as it will provide the graphical notation that is needed to introduce the gravity models in a simple manner. The spin foam representation of the BF partition function follows from expressing the projectors in (15) in the orthonormal basis of intertwiners, i.e., invariant vectors in $\text{Inv}[\rho_1 \otimes \dots \otimes \rho_4]$. From the product form of the structure group, one has

$$\begin{array}{c} \rho_1 \rho_2 \rho_3 \rho_4 \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} = \begin{array}{c} j_1^- j_2^- j_3^- j_4^- \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \begin{array}{c} j_1^+ j_2^+ j_3^+ j_4^+ \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \\ = \sum_{\iota^- \iota^+} \begin{array}{c} j_1^- j_2^- j_3^- j_4^- \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \begin{array}{c} j_1^+ j_2^+ j_3^+ j_4^+ \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \quad (16)$$

where $\rho_f = j_f^- \otimes j_f^+, j_f^\pm$ and ι^\pm are half integers labeling left and right representations of $SU(2)$ that defined the irreducible unitary representations of $G = SU(2) \times SU(2)$. We have used the expression of the right and left $SU(2)$ projectors in a basis of intertwiners, namely

$$\begin{array}{c} j_1 \quad j_2 \quad j_3 \quad j_4 \\ | \quad | \quad | \quad | \\ \text{---} \\ | \quad | \quad | \quad | \\ j_1 \quad j_2 \quad j_3 \quad j_4 \end{array} = \sum_{\iota} \begin{array}{c} j_1 \quad j_2 \quad j_3 \quad j_4 \\ \diagdown \quad \diagup \quad \diagdown \quad \diagup \\ \iota \\ \diagup \quad \diagdown \quad \diagup \quad \diagdown \\ j_1 \quad j_2 \quad j_3 \quad j_4 \end{array}, \quad (17)$$

where the four-leg objects on the right hand side denote the invariant vectors spanning a basis of $\text{Inv}[j_1 \otimes \dots \otimes j_4]$, and ι is a half integer, labeling those elements. Accordingly, when replacing the previous expression in (15), one gets

$$Z_{BF}(\Delta) = \sum_{C_f: \{f\} \rightarrow \rho_f} \prod_{f \in \Delta^*} d_{j_f^-} d_{j_f^+} \begin{array}{c} \text{---} \\ | \quad | \quad | \quad | \\ \text{---} \\ | \quad | \quad | \quad | \\ \text{---} \\ | \quad | \quad | \quad | \\ \text{---} \end{array}, \quad (18)$$

and equivalently,

$$Z_{BF}(\Delta) = \sum_{C_e: \{e\} \rightarrow \iota_e} \prod_{f \in \Delta^*} d_{j_f^-} d_{j_f^+} \begin{array}{c} \text{---} \\ | \quad | \quad | \quad | \\ \text{---} \\ | \quad | \quad | \quad | \\ \text{---} \\ | \quad | \quad | \quad | \\ \text{---} \end{array}, \quad (19)$$

from which we finally obtain the spin foam representation of the $SU(2) \times SU(2)$ partition function

as a product of two $SU(2)$ amplitudes, namely

$$Z_{BF}(\Delta) = \sum_{C_f: \{f\} \rightarrow \rho_f} \prod_{f \in \Delta^*} d_{j_f^-} d_{j_f^+} \sum_{C_e: \{e\} \rightarrow \iota_e} \prod_{v \in \Delta^*} \begin{array}{c} \text{---} \\ | \quad | \quad | \quad | \\ \text{---} \\ | \quad | \quad | \quad | \\ \text{---} \\ | \quad | \quad | \quad | \\ \text{---} \end{array} \quad (20)$$

Extra remarks on four-dimensional BF theory

The state sum (11) is generically divergent (due to the gauge freedom analogous to (3)). A regularized version defined in terms of $SU_q(2) \times SU_q(2)$ was introduced by Crane and Yetter [13,14]. As in three dimensions, if an appropriate regularization of bubble divergences is provided, (11) is topologically invariant and the spin foam path integral is discretization independent.

As in the three-dimensional case, BF theory can be coupled to topological defects [15] in any dimension. In the four-dimensional case, defects are string-like [16] and can carry extra degrees of freedom, such as topological Yang-Mills fields [17]. The possibility that quantum gravity could be defined directly from these simple kinds of topological theories has also been considered outside spin foams [18] (for which the UV problem described in the introduction is absent). This is attractive and should, in my view, be considered further.

It is also possible to introduce one-dimensional particles in four-dimensional BF theory and gravity, as shown in Ref. [19].

Two-dimensional BF theory has been used as the basic theory in an attempt to define a manifold independent model of QFT in Ref. [20]. It is also related to gravity in two dimensions in two ways: On the one hand, it is equivalent to the so-called Jackiw-Teitelboim model [21,22], on the other hand it is related to usual 2d gravity via constraints in a way similar to the one exploited in four dimensions (see next section). The first relationship has been used in the canonical quantization of the Jackiw-Teitelboim model in Ref. [23]. The second relationship has been explored in Ref. [24].

Three-dimensional BF theory and the spin foam quantization presented above are intimately related to classical and quantum gravity in three dimensions (for a classic reference see Ref. [25]). The state sum, as presented above, matches the quantum amplitudes first proposed by Ponzano and Regge in the 60's, based on their discovery of the asymptotic expressions of the 6j symbols [26], often referred to as the Ponzano-Regge model. Divergences in the above formal expression require regularization. Natural regularizations are available so that the model is well-defined [27–29]. For a detailed study of the divergence structure of the model, see Refs. [30–32]. The quantum deformed version of the above amplitudes lead to the so-called Turaev-Viro model [33], which is expected to correspond to the quantization of three-dimensional Riemannian gravity in the presence of a non-vanishing positive cosmological constant. For the definition of observables in the latter context, as well as in the analogue four-dimensional analog, see Ref. [34].

The topological character of BF theory can be preserved by the coupling of the theory with topological defects that play the role of point particles. In the spin foam literature, this has been considered from the canonical perspective in Refs. [35,36] and from the covariant perspective extensively by Freidel and Louapre [37]. These theories have been proved by Freidel and Livine to be dual, in a suitable sense, to certain non-commutative fields theories in three dimensions [38,39].

Concerning coupling BF theory with non-topological matter, see Refs. [40,41] for the case of fermionic matter, and Ref. [42] for gauge fields. A more radical perspective for the definition of matter in 3d gravity is taken in Ref. [43]. For three-dimensional supersymmetric BF theory models, see Refs. [44,45].

Recursion relations for the 6j vertex amplitudes have been investigated in Refs. [46,47]. They provide a tool for studying dynamics in spin foams of 3d gravity and might be useful in higher dimensions [48].

i. The coherent states representation

In this section, we introduce the coherent state representation of the $SU(2)$ and $Spin(4)$ path integral of BF theory. This will be particularly important

for the definition of the models defined by Freidel and Krasnov in Ref. [6] that we will address in Section IV as well as in the semiclassical analysis of the new models reported in Section VII. The relevance of such representation for spin foams was first emphasized by Livine and Speziale in Ref. [49].

a. Coherent states

Coherent states associated with the representation theory of a compact group have been studied by Thiemann and collaborators [50,51,51–59], see also Ref. [60]. Their importance for the new spin foam models was put forward by Livine and Speziale in Ref. [49], where the emphasis was put on coherent states of intertwiners or the so-called quantum tetrahedron (see also [61]). Here we follow the presentation of [6].

In order to build coherent states for $Spin(4)$, we start by introducing them in the case of $SU(2)$. Starting from the representation space \mathcal{H}_j of dimension $d_j \equiv 2j + 1$, one can write the resolution of the identity in terms of the canonical orthonormal basis $|j, m\rangle$ as

$$1_j = \sum_m |j, m\rangle \langle j, m|, \quad (21)$$

where $-j \leq m \leq j$. There exists an over complete basis $|j, g\rangle \in \mathcal{H}_j$, labeled by $g \in SU(2)$, such that

$$1_j = d_j \int_{SU(2)} dg |j, g\rangle \langle j, g|, \quad (22)$$

The states $|j, g\rangle \in \mathcal{H}_j$ are $SU(2)$ coherent states defined by the action of the group on maximum weight states $|j, j\rangle$ (themselves coherent), namely

$$|j, g\rangle \equiv g|j, j\rangle = \sum_m |j, m\rangle D_{mj}^j(g), \quad (23)$$

where $D_{mj}^j(g)$ are the matrix elements of the unitary representations in the $|j, m\rangle$ (Wigner matrices). Equation (22) follows from the orthonormality of unitary representation matrix elements, namely

$$\begin{aligned}
 & d_j \int_{SU(2)} dg |j, g\rangle \langle j, g|, \\
 &= d_j \sum_{mm'} |j, m\rangle \langle j, m'| \int_{SU(2)} dg D_{mj}^j(g) \overline{D_{m'j}^j(g)} \\
 &= \sum_m |j, m\rangle \langle j, m|,
 \end{aligned} \tag{24}$$

where in the last equality we have used the orthonormality of the matrix elements. The decomposition of the identity (22) can be expressed as an integral on the two-sphere of directions $S^2 = SU(2)/U(1)$ by noticing that $D_{mj}^j(g)$ and $D_{m'j}^j(gh)$ differ only by a phase for any group element h from a suitable $U(1) \subset SU(2)$. Thus, one has

$$1_j = d_j \int_{S^2} dn |j, n\rangle \langle j, n|, \tag{25}$$

where $n \in S^2$ is integrated with the invariant measure of the sphere. The states $|j, n\rangle$ form (an over-complete) basis in \mathcal{H}_j . $SU(2)$ coherent states have the usual semiclassical properties. Indeed, if one considers the generators J^i of $su(2)$, one has

$$\langle j, n | \hat{J}^i | j, n \rangle = j n^i, \tag{26}$$

where n^i is the corresponding three-dimensional unit vector for $n \in S^2$. The fluctuations of \hat{J}^2 are also minimal with $\Delta J^2 = \hbar^2 j$, where we have restored \hbar for clarity. The fluctuations go to zero in the limit $\hbar \rightarrow 0$ and $j \rightarrow \infty$, while $\hbar j$ is kept constant. This kind of limit will be used often as a notion of semiclassical limit in spin foams. The state $|j, n\rangle$ is a semiclassical state describing a vector in \mathbb{R}^3 of length j and of direction n . It will be convenient to introduce the following graphical notation for Eq. (25)

$$\left. \begin{array}{c} j \\ | \\ \circ \\ | \end{array} \right| = d_j \int_{S^2} dn \left. \begin{array}{c} j \\ | \\ \circ \\ n \\ | \\ \circ \\ | \end{array} \right| \tag{27}$$

Finally, an important property of $SU(2)$ coherent states stemming from the fact that

$$|j, j\rangle = |\frac{1}{2}, \frac{1}{2}\rangle |\frac{1}{2}, \frac{1}{2}\rangle \cdots |\frac{1}{2}, \frac{1}{2}\rangle \equiv |\frac{1}{2}, \frac{1}{2}\rangle^{\otimes 2j}$$

is that

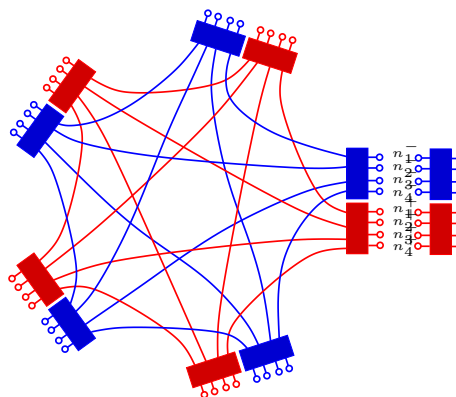
$$|j, n\rangle = |\frac{1}{2}, n\rangle^{\otimes 2j}. \tag{28}$$

The above property will be of key importance in constructing effective discrete actions for spin foam models. In particular, it will play a central role in the study of the semiclassical limit of the EPRL and FK models studied in Sections III, and IV. In the following subsection, we provide an example for $Spin(4)$ BF theory.

b. Spin(4) BF theory: Amplitudes in the coherent state basis

Here we study the coherent states representation of the path integral for $Spin(4)$ BF theory. The construction presented here can be extended to more general cases. The present case is, however, of particular importance for the study of gravity models presented in Sections III, and IV. With the introduction of coherent states, one achieves the most difficult part of the work. In order to express the $Spin(4)$ BF amplitude in the coherent state representation, one simply inserts a resolution of the identity in the form (25) on each and every wire connecting neighboring vertices in the expression (18) for the BF amplitudes. The result is

$$\begin{aligned}
 Z_{BF}(\Delta) &= \sum_{\mathcal{C}_f: \{f\} \rightarrow \rho_f} \prod_{f \in \Delta^*} d_{j_f^-} d_{j_f^+} \\
 &\int \prod_{e \in \Delta^*} d_{j_{ef}^-} d_{j_{ef}^+} dn_{ef}^- dn_{ef}^+
 \end{aligned} \tag{29}$$



where we have explicitly written the $n_{\pm} \in S^2$ integration variables only on a single cable. One observes that there is one $n_{\pm} \in S^2$ per each wire coming out at an edge $e \in \Delta^*$. As wires are in one-to-one correspondence with faces $f \in \Delta^*$, the integration variables $n_{ef}^{\pm} \in S^2$ are labeled by an

edge and face subindex. In order to get an expression of the BF path integral in terms of an affective action, we restore, at this stage, the explicit group integrations represented by the boxes in the previous equation. One gets

$$\begin{aligned}
 Z_{BF}(\Delta) = & \sum_{\mathcal{C}_f: \{f\} \rightarrow \rho_f} \prod_{f \in \Delta^*} d_{j_f^-} d_{j_f^+} \int \\
 & \prod_{e \in \Delta^*} d_{j_{ef}^-} d_{j_{ef}^+} dn_{ef}^- dn_{ef}^+ \prod_{v \in \Delta^*} \prod_{e, e' \in v} \\
 & dg_{ef}^- dg_{ef}^+ (\langle n_{ef}^- | (g^-)_{ef}^{-1} g_{e'f}^- | n_{e'f}^- \rangle)^{2j_f^-} \\
 & (\langle n_{ef}^+ | (g^+)_{ef}^{-1} g_{e'f}^+ | n_{e'f}^+ \rangle)^{2j_f^+}, \quad (30)
 \end{aligned}$$

where we have used the coherent states property (28), and $|n^\pm\rangle$ is a simplified notation for $|\frac{1}{2}, n^\pm\rangle$. The previous equation can be finally written as

$$\begin{aligned}
 Z_{BF}(\Delta) = & \sum_{\mathcal{C}_f: \{f\} \rightarrow \rho_f} \prod_{f \in \Delta^*} d_{j_f^-} d_{j_f^+} \int \\
 & \prod_{e \in \Delta^*} d_{j_{ef}^-} d_{j_{ef}^+} dn_{ef}^- dn_{ef}^+ dg_{ef}^- dg_{ef}^+ \\
 & \exp(S_{j^\pm, \mathbf{n}^\pm}[g^\pm]), \quad (31)
 \end{aligned}$$

where the discrete action

$$S_{j^\pm, \mathbf{n}^\pm}[g^\pm] = \sum_{v \in \Delta^*} S_{j_v, \mathbf{n}_v}^v[g^\pm] \quad (32)$$

with

$$S_{j, \mathbf{n}}^v[g] = \sum_{a < b=1}^5 2j_{ab} \ln \langle n_{ab} | g_a^{-1} g_b | n_{ba} \rangle, \quad (33)$$

and the indices a, b label the five edges of a given vertex. The previous expression is equal to the form (11) of the BF amplitude. In the case of the gravity models studied in what follows, the coherent state path integral representation will be the basic tool for the study of the semiclassical limit of the models and the relationship with Regge discrete formulation of general relativity.

ii. The relationship between gravity and BF theory

The field theory described in the present section has no local degrees of freedom. It represents the simplest example of a topological field theory in four dimensions. The interest of this theory for gravity

models stems from the fact that an action for the gravitational degrees of freedom (basically equivalent to general relativity in the first order formulation) can be obtained by supplementing a 4d BF theory action with internal gauge group $SL(2, \mathbb{C})$ (Lorentzian) or $Spin(4)$ (Riemannian) with the following set of quadratic constraints on the B -field

$$\epsilon_{IJKL} B_{\mu\nu}^{IJ} B_{\rho\sigma}^{KL} - e \epsilon_{\mu\nu\rho\sigma} \approx 0, \quad (34)$$

where $e \equiv \sigma^2 (1/4!) \epsilon_{IJKL} B_{\mu\nu}^{IJ} B_{\rho\sigma}^{KL} \epsilon^{\mu\nu\rho\sigma}$ where $\sigma^2 = \pm 1$ according to whether one is in the Riemannian or Lorentzian case. More generally, a one-parameter family of gravity actions can be obtained from the imposition of the previous constraints on the following modified BF action

$$S_\gamma(B, \omega) = \int_{\mathcal{M}} \langle (*B + \frac{1}{\gamma} B) \wedge F(\omega) \rangle, \quad (35)$$

where γ is the Immirzi parameter. The strategy behind the definition of the new spin foam models for quantum gravity consists of imposing these constraints on the path integral of BF theory on the momenta $J = *B + \frac{1}{\gamma} B$ conjugated to ω . In order to impose the Plebanski constraints above, it will be convenient to express the B field in terms of the momenta J , namely

$$B = \frac{\gamma}{1 - \sigma^2 \gamma^2} (J - \gamma^* J). \quad (36)$$

The imposition of the constraints (34) on the BF path integral on a fixed discretization can be done in two different ways: by directly restricting the spin foam configurations (this is the EPRL approach described in the following section), or by restricting the semiclassical values of the B field in the coherent state representation of the BF path integral (this is the FK strategy described in Section IV).

III. The Engle-Pereira-Rovelli-Livine (EPRL) model

In this section, we introduce the Engle-Pereira-Rovelli-Livine (EPRL) model [4, 5]. The section is organized as follows: The relevant representation theory is introduced in Subsection i. In Subsection ii, we present and discuss the linear simplicity constraints—classically equivalent to the Plebanski constraints—and discuss their implementation in

the quantum theory. In Subsection iii, we introduce the EPRL model of Riemannian gravity. In Subsection iv, we prove the validity of the quadratic Plebanski constraints—reducing BF theory to general relativity—directly in the spin foam representation. In Subsection v, we present the coherent state representation of the Riemannian EPRL model. In Subsection vi, we describe the Lorentzian model. The material of this section will also allow us to describe the construction of the closely related (although derived from a different logic) Riemannian FK constructed in Ref. [6]. The idea that linear simplicity constraints are more convenient for dealing with the constraints that reduce BF theory to gravity was pointed out by Freidel and Krasnov in this last reference.

i. Representation theory of $Spin(4)$ and $SL(2, \mathbb{C})$ and the canonical basis

In this section, we present the representation theory of the groups $Spin(4)$ and $SL(2, \mathbb{C})$ that is necessary for the definition of the new spin foam models for Riemannian and Lorentzian gravity, respectively. To emphasize the highly symmetric structure of the two, we present them in a unified notation where a parameter $\sigma = 1$ for the Riemannian sector and $\sigma = i$ for the Lorentzian one. The simple relationship between the two might be a hint to a possible relationship between model amplitudes in a spirit similar to the interesting link between Euclidean and Lorentzian QFT provided by Wick rotations¹. Unitary irreducible representations $\mathcal{H}_{p,k}$ of $Spin(4)$ and $SL(2, \mathbb{C})$ are labeled by two parameters, p and k . In the case of $Spin(4) = SU(2) \times SU(2)$, the unitary irreducible representations are finite-dimensional and the labels p and k can be expressed in terms of the half integers labeling the right and left $SU(2)$ unitary representations j^\pm , as follows

$$p = j^+ + j^- + 1 \quad k = |j^+ - j^-|. \quad (37)$$

In the $SL(2, \mathbb{C})$ case, the unitary irreducible representations are infinite-dimensional and one has

$$p \in \mathbb{R}^+ \quad k \in \mathbb{N}/2. \quad (38)$$

The two Casimirs are $C_1 = \frac{1}{2} J_{IJ} J^{IJ} = L^2 + \sigma^2 K^2$ and $C_2 = \frac{1}{2} \star J_{IJ} J^{IJ} = K \cdot L$ where L^i are the generators of an arbitrary rotation subgroup and K^i are the generators of the corresponding boosts. The Casimirs act on $|p, k\rangle \in \mathcal{H}_{p,k}$, as follows

$$\begin{aligned} C_1 |p, k\rangle &= \frac{1}{2} (k^2 + \sigma^2 p^2 - 1) |p, k\rangle \\ C_2 |p, k\rangle &= pk |p, k\rangle. \end{aligned} \quad (39)$$

For details on the representation theory of $SL(2, \mathbb{C})$, see Refs. [63–65]. The definition of the EPRL model requires the introduction of an (arbitrary) subgroup $SU(2) \subset Spin(4)$ or $SU(2) \subset SL(2, \mathbb{C})$, according to whether one is working in the Riemannian or in the Lorentzian sector. This subgroup corresponds to the internal gauge group of the gravitational phase space in connection variables in the time gauge (see Ref. [8] for details). Hence, in the quantum theory, the representation theory of this $SU(2)$ subgroup will be important. This importance will soon emerge as apparent from the imposition of the constraints that define the EPRL model. The link between the unitary representations of $SL(2, \mathbb{C})$ and those of $SU(2)$ is given by the decomposition

$$\mathcal{H}_{p,k} = \bigoplus_{j=k}^{p-1} \mathcal{H}_j = \bigoplus_{j=|j^+ - j^-|}^{j^+ + j^-} \mathcal{H}_j, \quad (40)$$

for the Riemannian sector, and

$$\mathcal{H}_{p,k} = \bigoplus_{j=k}^{\infty} \mathcal{H}_j, \quad (41)$$

for the Lorentzian sector. As the unitary irreducible representations of the subgroup $SU(2) \in Spin(4)$ and $SU(2) \in SL(2, \mathbb{C})$ are essential for understanding the link of the EPRL model and the operator canonical formulation of LQG, it will be convenient to express the action of the generators of the Lie algebra of the corresponding group in a basis adapted to the above equation. In order to do this, we first notice that the Lie algebra $spin(4)$ and $sl(2, \mathbb{C})$ can be characterized in terms of the generators of a rotation subgroup L^i and the remaining

¹Such explicit relationship between gravity amplitudes in the Euclidean and Lorentzian sectors can be established by analytic continuation in 3d [62].

boost generators K^i , as follows

$$\begin{aligned}
 [L_3, L_\pm] &= \pm L_\pm \quad [L_+, L_-] = 2 L_3 \\
 [L_+, K_+] &= [L_-, K_-] = [L_3, K_3] = 0 \\
 [K_3, L_\pm] &= \pm K_\pm \quad [L_\pm, K_\mp] = \pm 2 K_3 \\
 [L_3, K_\pm] &= \pm K_\pm \\
 [K_3, K_\pm] &= \pm \sigma^2 L_\pm \\
 [K_+, K_-] &= 2\sigma^2 L_3,
 \end{aligned} \tag{42}$$

where $K_\pm = K^1 \pm iK^2$ and $L_\pm = L^1 \pm iL^2$, respectively. The action of the previous generators in the basis $|p, k; j, m\rangle$ can be shown to be

$$\begin{aligned}
 L^3 |p, k; j, m\rangle &= m |p, k; j, m\rangle, \\
 L^+ |p, k; j, m\rangle &= \sqrt{(j+m+1)(j-m)} \\
 &\quad |p, k; j, m+1\rangle, \\
 L^- |p, k; j, m\rangle &= \sqrt{(j+m)(j-m-1)} \\
 &\quad |p, k; j, m-1\rangle, \\
 K^3 |p, k; j, m\rangle &= \alpha_j \sqrt{j^2 - m^2} |p, k; j-1, m\rangle \\
 &\quad + \gamma_j m |p, k; j, m\rangle \\
 &\quad - \alpha_{j+1} \sqrt{(j+1)^2 - m^2} \\
 &\quad |p, k; j+1, m\rangle, \\
 K^+ |p, k; j, m\rangle &= \alpha_j \sqrt{(j-m)(j-m-1)} \\
 &\quad |p, k; j-1, m+1\rangle \\
 &\quad + \gamma_j \sqrt{(j-m)(j+m+1)} \\
 &\quad |p, k; j, m+1\rangle \\
 &\quad + \alpha_{j+1} \sqrt{(j+m+1)(j+m+2)} \\
 &\quad |p, k; j+1, m+1\rangle, \\
 K^- |p, k; j, m\rangle &= -\alpha_j \sqrt{(j+m)(j+m-1)} \\
 &\quad |p, k; j-1, m-1\rangle \\
 &\quad + \gamma_j \sqrt{(j+m)(j-m+1)} \\
 &\quad |p, k; j, m-1\rangle \\
 &\quad - \alpha_{j+1} \sqrt{(j-m+1)(j-m+2)} \\
 &\quad |p, k; j+1, m-1\rangle,
 \end{aligned} \tag{43}$$

where

$$\gamma_j = \frac{kp}{j(j+1)}, \quad \alpha_j = \sigma \sqrt{\frac{(j^2 - k^2)(j^2 + p^2)}{j^2(4j^2 - 1)}} \tag{44}$$

The previous equations will be important in what follows: they will allow for the characterization of

the solutions of the quantum simplicity constraints, in both the Riemannian and Lorentzian models, in a direct manner. This concludes the review of the representation theory that is necessary for the definition of the EPRL model.

ii. The linear simplicity constraints

As first shown in Ref. [6], the quadratic Plebanski simplicity constraints—and more precisely in their dual version presented below (34)—are equivalent in the discrete setting to the linear constraint on each face of a given tetrahedron

$$D_f^i = L_f^i - \frac{1}{\gamma} K_f^i \approx 0, \tag{45}$$

where the label f makes reference to a face $f \in \Delta^*$, and where (very importantly) the subgroup $SU(2) \subset Spin(4)$ (or $SL(2, \mathbb{C})$) that is necessary for the definition of the above constraints is chosen arbitrarily at each tetrahedron, equivalent on each edge $e \in \Delta^*$. Such choice of the rotation subgroup is the precise analog of the time gauge in the canonical analysis of general relativity. The EPRL model is defined by imposing the previous constraints as operator equations on the Hilbert spaces defined by the unitary irreducible representations of the internal gauge group that take part in the state-sum of BF theory. We will show in Subsection iv that the models constructed on the requirement of a suitable imposition of the linear constraints (45) satisfy the usual quadratic Plebanski constraints—that reduce BF theory to general relativity—in the path integral formulation (up to quantum corrections which are neglected in the usual semiclassical limit).

From the commutation relations (42), from previous section, we can easily compute the commutator of the previous tetrahedron constraints and conclude that, in fact, it does not close, namely

$$\begin{aligned}
 [D_f^i, D_{f'}^j] &= \delta_{ff'} \epsilon_k^{ij} \left[\left(1 + \frac{\sigma^2}{\gamma^2}\right) L_f^k - \frac{2}{\gamma} K_e^k \right] \\
 &= 2\delta_{ee'} \epsilon_k^{ij} D^k + \delta_{ee'} \frac{\sigma^2 - \gamma^2}{\gamma^2} \epsilon_k^{ij} L_f^k.
 \end{aligned} \tag{46}$$

The previous commutation relations imply that the constraint algebra is not closed and cannot therefore be imposed as operator equations on the states summed over in the BF partition function in general. There are two interesting exceptions to the previous statement:

1. The first one is to take $\gamma = \pm\sigma$. This corresponds to the description of the model in terms of self-dual or anti-self-dual variables. Unfortunately, the construction of the new models is not well defined in this case for the Lorentzian theory and leads to a trivial result in the Riemannian sector: $SU(2)$ BF theory.
2. The second possibility is to work in the sector where $L_f^i = 0$. This choice leads to the Barret-Crane model [66], where the degrees of freedom of BF theory seem over constrained: Boundary states satisfying the BC constraints are a very small subset of the allowed boundary states in LQG. This is believed to be problematic if gravity is to be recovered at low energies.

The EPRL model is obtained by restricting the representations appearing in the expression of the BF partition function so that at each tetrahedron the linear constraint (45) is the strongest possible way that is compatible with the uncertainties relations stemming from (46). In addition, one would add the requirement that the state-space of the tetrahedra is compatible with the state-space of the analogous excitation in the canonical context of LQG, so that arbitrary states in the kinematical state of LQG have non-trivial amplitudes in the model.

Due to the fact that the constraints D_f^i do not form a closed (first class) algebra in the generic case, one needs to devise a weaker sense in which they are to be imposed. One possibility is to consider the Gupta-Bleuler criterion consisting of selecting a suitable class of states for which the matrix elements on D_f^i vanish. One notices from (43) that if we chose the subspace $\mathcal{H}_j \subset \mathcal{H}_{p,k}$, we would have

$$\begin{aligned} \langle p, k, j, q | D_f^3 | p, k, j, m \rangle &= \delta_{q,m} m \left(1 - \frac{\gamma_j}{\gamma}\right) \\ \langle p, k, j, q | D_f^\pm | p, k, j, m \rangle &= \delta_{q \pm 1, m} \\ &\times \sqrt{(j \pm m + 1)(j \mp m)} \left(1 - \frac{\gamma_j}{\gamma}\right). \end{aligned}$$

The matrix elements of the linear constraints vanish in this subclass if one chooses

$$\gamma_j = \frac{pk}{j(j+1)} = \gamma \quad (47)$$

There are two cases:

1. **Case $\gamma < 1$:** Following Ref. [67], in this case one restricts the representations to

$$\begin{aligned} \text{Riemannian: } & p = j + 1, k = \gamma j. \\ \text{Lorentzian: } & p = \gamma(j + 1), k = j. \end{aligned} \quad (48)$$

which amounts to choosing the *maximum weight* component $j = p - 1$ in the expansion (41). In the Riemannian case, the above choice translates into $j^\pm = (1 \pm \gamma)j/2$ for the $SU(2)$ right and left representations. Notice that the solutions to the simplicity constraints in the Riemannian and Lorentzian sectors look very different for $\gamma < 1$. Simple algebra shows that condition (47) is met. There are indeed other solutions [68] to the Gupta-Bleuler criterion in this case.

2. **Case $\gamma > 1$:** In this case, according to Ref. [69], one restricts the representations to

$$\begin{aligned} \text{Riemannian: } & p = \gamma(j + 1), k = j. \\ \text{Lorentzian: } & p = \gamma(j + 1), k = j. \end{aligned} \quad (49)$$

which amounts to choosing the *minimum weight* component $j = k$ in the expansion (41). For the Riemannian case, we can write the solutions in terms of $j^\pm = (\gamma \pm 1)\frac{j}{2} + \frac{\gamma-1}{2}$. Notice that for $\gamma > 1$ there is complete symmetry between the solutions of the Riemannian and Lorentzian sectors. In my opinion, this symmetry deserves further investigation as it might be an indication of a deeper connection between the Riemannian and Lorentzian models (again, such relationship is a fact in 3d gravity [62]).

Another criterion for weak imposition can be developed by studying the spectrum of the Master constraint $M_f = D_f \cdot D_f$. Strong imposition of the constraints D_f^i would amount to looking for the kernel of the master constraint M_f . However, generically, the positive operator associated with the master constraint does not contain the zero eigenvalue in the spectrum due to the open nature of the constraint algebra (46).

It is convenient, as in Ref. [70], to express the master constraint in a manifestly invariant way. In order to get a gauge invariant constraint one starts

from the master constraint and uses the $D_f^i = 0$ classically to write it in terms of Casimirs, namely

$$M_f = (1 + \sigma^2 \gamma^2) C_2 - 2C_1 \gamma,$$

where C_1 and C_2 are the Casimirs given in Eq. (39). The minimum eigenvalue condition is

$$\begin{aligned} \text{Riemannian: } & p = j, k = \gamma j. \\ \text{Lorentzian: } & p = \gamma j, k = j. \end{aligned} \quad (50)$$

The minimum eigenvalue is $m_{min} = \hbar^2 \gamma j (\gamma^2 - 1)$ for the Riemannian case and $m_{min} = \gamma$ for the Lorentzian case. The master constraint criterion works better in the Lorentzian case, as pointed out in Ref. [70]. More recently, it has been shown that the constraint solutions $p = \gamma j$ and $k = j$ also follow naturally from a spinor formulation of the simplicity constraints [71–73]. The above criterion is used in the definition of the EPRL model.

It is important to point out that the Riemannian case imposes strong restrictions on the allowed values of the Immirzi parameter if one wants the spin $j \in \mathbb{N}/2$ to be arbitrary (in order to have all possible boundary states allowed in LQG). In this case, the only possibilities are $\gamma = \mathbb{N}$ or $\gamma = 1$. This restriction is not natural from the viewpoint of LQG. Its relevance, if any, remains mysterious at this stage.

Summarizing, in the Lorentzian (Riemannian) EPRL model one restricts the $SL(2, \mathbb{C})$ ($Spin(4)$) representations of BF theory to those satisfying

$$p = \gamma j \quad k = j \quad (51)$$

for $j \in \mathbb{N}/2$. From now on, we denote the subset of admissible representation

$$\mathcal{H}_\gamma \subset \text{Irrep}(SL(2, \mathbb{C}))(\text{Irrep}(Spin(4))) \quad (52)$$

The admissible quantum states Ψ are elements of the subspace $\mathcal{H}_j \subset \mathcal{H}_{\gamma j, j}$ (i.e., minimum weight states) which satisfy the constraints (45) in the following semiclassical sense:

$$(K_f^i - \gamma L_f^i) \Psi = \mathcal{O}_{sc}, \quad (53)$$

where the symbol \mathcal{O}_{sc} (order semiclassical) denotes a quantity that vanishes in limit $\hbar \rightarrow 0, j \rightarrow \infty$

with $\hbar j = \text{constant}$. In the Riemannian case, the previous equation can be written as

$$[(1 - \gamma)J_+^i - (1 + \gamma)J_-^i] \Psi = \mathcal{O}_{sc}, \quad (54)$$

which in turn has a simple graphical representation in terms of spin-network grasping operators, namely

The previous equation will be of great importance in the graphical calculus that will allow us to show that the linear constraint imposed here, at the level of states, implies the vanishing of the quadratic Plebanski constraints (34) and their fluctuations, computed in the path integral sense, in the appropriate large spin semiclassical limit.

iii. Presentation of the Riemannian EPRL amplitude

Here we complete the definition of the EPRL models by imposing the linear constraints on the BF amplitudes constructed in Section II. We will also show that the path-integral expectation value of the Plebanski constraints (34), as well as their fluctuations, vanish in a suitable semiclassical sense. This shows that the EPRL model can be considered as a lattice definition of the a quantum gravity theory.

We start with the Riemannian model for which a straightforward graphical notation is available. The first step is the translation of Eq. (40)—for p and k satisfying the simplicity constraints—in terms of the graphical notation introduced in Section II. Concretely, for $\gamma < 1$, one has $j^\pm = (1 \pm \gamma)j/2 \in \mathcal{H}_\gamma$ and (40) becomes

For $\gamma > 1$ we have

$$= \sum_l \text{[Diagrammatic representation of the simplified vertex]} \quad (62)$$

which follows, basically, from the invariance of the Haar measure (9) (in the last line, we have used (17)). More precisely, the integration of the subgroup $SU(2) \in Spin(4)$, represented by the green box on the right, can be absorbed by suitable redefinition of the integration on the right and left copies of $SU(2)$, represented by the red and blue boxes, respectively. With this, we can already write the spin foam representation of the EPRL model, namely

$$Z_{eprl}^E(\Delta) = \sum_{j_f} \sum_{\iota_e} \prod_{f \in \Delta^*} d_{|1-\gamma|\frac{j}{2}} d_{(1+\gamma)\frac{j}{2}} \times \prod_{v \in \Delta^*} \text{[Diagrammatic representation of the vertex]} \quad (63)$$

where the vertex amplitude (graphically represented) depends on the 10 spins j associated to the face-wires and the 5 intertwiners associated to the five edges (tetrahedra). As in previous equations, we have left the spin labels of wires implicit for notational simplicity. We can write the previous spin foam amplitude in another form by integrating out all the projectors (boxes) explicitly. Using (17), we get

$$\text{[Diagrammatic representation of the vertex]} = \sum_{l+l-l} \text{[Diagrammatic representation of the simplified vertex]} \quad (64)$$

thus replacing this in (61), we get

$$Z_{eprl}^E(\Delta) = \sum_{j_f} \prod_{f \in \Delta^*} d_{|\gamma-1| \frac{j}{2}} d_{(\gamma+1) \frac{j}{2}} \sum_{\iota_e} \prod_{v \in \Delta^*} \sum_{\iota_1^- \dots \iota_5^-} \sum_{\iota_1^+ \dots \iota_5^+} \prod_{a=1}^5 f_{\iota_a^-, \iota_a^+}^{\iota_a} \quad (65)$$

where the coefficients $f_{\iota^+ \iota^-}^{\iota}$ are the so-called fusion coefficients which already appear in their graphical form in (64), more explicitly

$$f_{\iota^+ \iota^-}^{\iota}(j_1, \dots, j_4) =$$
(66)

The previous Eq. (66) is the form of the EPRL model as derived in Ref. [5].

iv. Proof of validity of the Plebanski constraints

In this section, we prove that the quadratic constraints are satisfied in the sense that their path integral expectation value and fluctuation vanish in the appropriate semiclassical limit.

a. The quadratic Plebanski constraints

The quadratic Plebanski constraints are

$$\epsilon_{IJKL} B_{\mu\nu}^{IJ} B_{\rho\sigma}^{KL} - e \epsilon_{\mu\nu\rho\sigma} \approx 0. \quad (67)$$

The constraints in this form are more suitable for the translation into the discrete formulation. More

precisely, according to (6), the smooth fields $B_{\mu\nu}^{IJ}$ are now associated with the discrete quantities $B_{\text{triangles}}^{IJ}$, or equivalently B_f^{IJ} as faces $f \in \Delta^*$ are in one-to-one correspondence to triangles in four dimensions. The constraints (67) are local constraints valid at every spacetime point. In the discrete setting, spacetime points are represented by four-simplexes or (more adapted to our discussion) vertices $v \in \Delta^*$. With this, the constraints (67) are discretized as follows:

Triangle (or diagonal) constraints:

$$\epsilon_{IJKL} B_f^{IJ} B_f^{KL} = 0, \quad (68)$$

for all $f \in v$, i.e., for each and every face of the 10 possible faces touching the vertex v .

Tetrahedron constraints:

$$\epsilon_{IJKL} B_f^{IJ} B_{f'}^{KL} = 0, \quad (69)$$

for all $f, f' \in v$, so that they are dual to triangles sharing a one-simplex, i.e., belonging to the same tetrahedron out of the five possible ones.

4-simplex constraints:

$$\epsilon_{IJKL} B_f^{IJ} B_{\bar{f}}^{KL} = e_v, \quad (70)$$

for any pair of faces $f, \bar{f} \in v$ that are dual to triangles sharing a single point. The last constraint will require a more detailed discussion. At this point, let us point out that the constraint (70) is interpreted as a definition of the four volume e_v of the four-simplex. The constraint requires such definition to be consistent, i.e., the true condition is

$$\begin{aligned} \epsilon_{IJKL} B_f^{IJ} B_{\bar{f}}^{KL} &= \epsilon_{IJKL} B_{f'}^{IJ} B_{\bar{f}'}^{KL} \\ &= \epsilon_{IJKL} B_{f''}^{IJ} B_{\bar{f}''}^{KL} = \dots = e_v \end{aligned} \quad (71)$$

for all five different possible pairs of f and \bar{f} in a four simplex, and where we assume the pairs $f-\bar{f}$ are ordered in agreement with the orientation of the complex Δ^* .

b. The path integral expectation value of the Plebanski constraints

Here we prove that the Plebanski constraints are satisfied by the EPRL amplitudes in the path integral expectation value sense.

The triangle constraints:

We start from the simplest case: The triangle (or diagonal) constraints (68). We choose a face $f \in v$ (dual to a triangle) in the cable-wire-diagram of Eq. (61). This amounts to choosing a pair of wires (right and left representations) connecting two nodes in the vertex cable wire diagram. The two nodes are dual to the two tetrahedra—in the four simplex dual to the vertex—sharing the chosen triangle. Equation (36) shows that

$$\epsilon_{IJKL} B_f^{IJ} B_f^{KL} \propto (1 + \gamma)^2 J_f^- \cdot J_f^- - (1 - \gamma)^2 J_f^+ \cdot J_f^+, \quad (72)$$

where J_f^\pm denotes the self-dual and anti-self-dual parts of Π_f^{IJ} . The path integral expectation value of the triangle constraint is then

$$\langle (1 + \gamma)^2 J_f^- \cdot J_f^- - (1 - \gamma)^2 J_f^+ \cdot J_f^+ \rangle \propto \quad (73)$$

where the double grasplings on the anti-self-dual (blue) wire and the self-dual (red) wire represent the action of the Casimirs $J_f^- \cdot J_f^-$ and $J_f^+ \cdot J_f^+$, on the cable-wire diagram of the corresponding vertex. Direct evaluation shows that the previous diagram is proportional to $\hbar^2 j_f$ which vanishes in the semi-classical limit $\hbar \rightarrow 0, j \rightarrow \infty$ with $\hbar j = \text{constant}$. We use the notation already adopted in (54) and call such quantity \mathcal{O}_{sc} . This proves that the triangle Plebanski constraints are satisfied in the semi-classical sense.

The tetrahedra constraints:

The proof of the validity of the tetrahedra constraints (69). In this case we also have

where we have chosen an arbitrary pair of faces. In order to prove this, let us develop the term on the right. The result follows from

where in the first line we have used the fact that the double grasping can be shifted through the group integration (due to gauge invariance (9)).

$$\begin{aligned}
 e_v &= (1 + \gamma)^2 \left[\text{Diagram 1} \right] - (1 - \gamma)^2 \left[\text{Diagram 2} \right] \\
 &= -(1 + \gamma)^2 \left[\text{Diagram 3} + \text{Diagram 4} + \text{Diagram 5} \right] \\
 &\quad + (1 - \gamma)^2 \left[\text{Diagram 6} + \text{Diagram 7} + \text{Diagram 8} \right] \\
 &= -(1 + \gamma)^2 \left[\text{Diagram 9} \right] + (1 - \gamma)^2 \left[\text{Diagram 10} \right] + \mathcal{O}_{sc}, \tag{77}
 \end{aligned}$$

where going from the first line to the second and third lines we have simply used (76) on the bottom grasplings on the right and left wires. The last line results from the validity of (69). Notice that the second terms in the second and third lines add up to \mathcal{O}_{sc} , as well as the third terms in the second and third line. There is an overall minus sign which amounts for an orientation factor. It should be clear that we can apply the same procedure to arrive at any admissible pair.

c. P_{eprl} is not a projector

We will study in detail the object $P_{eprl}^e(j_1, \dots, j_4)$. We see that it is made of two ingredients. The first one is the projection to the maximum weight subspace \mathcal{H}_j for $\gamma > 1$ in the decomposition of \mathcal{H}_{j^+, j^-} for $j^\pm = (1 \pm \gamma)j/2$ ($j^\pm = (\gamma \pm 1)j/2$ for

$\gamma > 1$) in terms of irreducible representations of an arbitrarily chosen $SU(2)$ subgroup of $Spin(4)$. The second ingredient is to eliminate the dependence on the choice of subgroup by group averaging with respect to the full gauge group $Spin(4)$. This is diagrammatically represented in (60). However, $P_{eprl}^e(j_1, \dots, j_4)$ is not a projector, namely

$$P_{eprl}^e(j_1, \dots, j_4)^2 \neq P_{eprl}^e(j_1, \dots, j_4). \tag{78}$$

Technically, this follows from (59) and the fact that

$$[P_{inv}^e(\rho_1 \dots \rho_4), (\mathcal{Y}_{j_1} \otimes \dots \otimes \mathcal{Y}_{j_4})] \neq 0 \tag{79}$$

i.e., the projection imposing the linear constraints (defined on the frame of a tetrahedron or edge)

and the $Spin(4)$ (or Lorentz) group averaging—rendering the result gauge invariant—do not commute. The fact that the $P_{eprl}^e(j_1, \dots, j_4)$ is not a projection operator has important consequences in the mathematical structure of the model:

1. From (61) one can immediately obtain the following expression for the EPRL amplitude

$$Z_{eprl}(\Delta) = \sum_{\rho_f \in \mathcal{K}} \prod_{f \in \Delta^*} d_{|1-\gamma| \frac{i}{2}} d_{(1+\gamma) \frac{i}{2}} \times \prod_e P_{eprl}^e(j_1, \dots, j_4). \quad (80)$$

This expression has the formal structure of expression (13) for BF theory. The formal similarity, however, is broken by the fact that $P_{eprl}^e(j_1, \dots, j_4)$ is not a projection operator. From the formal perspective, there is the possibility for the amplitudes to be defined in terms of a network of projectors (as in BF theory). This might provide an interesting structure that might be of relevance in the definition of a discretization independent model. On the contrary, the failure of $P_{eprl}^e(j_1, \dots, j_4)$ to be a projector may lead, in my opinion, to difficulties in the limit where the complex Δ is refined: The increasing of the number of edges might produce either trivial or divergent amplitudes ².

2. Another difficulty associated with $P_{eprl}^e(j_1, \dots, j_4)^2 \neq P_{eprl}^e(j_1, \dots, j_4)$ is the failure of the amplitudes of the EPRL model, as defined here, to be consistent with the abstract notion of spin foams as defined in [74]. This is a point of crucial importance under current discussion in the community. The point is that the cellular decomposition Δ has no physical meaning and is to be interpreted as a subsidiary regulating structure to be removed when computing physical quantities. Spin foams configurations can fit in different ways on a given Δ , yet any of these different embeddings represent the same physical process (like the same gravitational field in different coordinates). Consistency requires the spin

foam amplitudes to be independent of the embedding, i.e., well-defined on the equivalence classes of spin foams as defined by Baez in Ref. [74] (the importance of these consistency requirements was emphasized in Ref. [75]). The amplitude (80) fails this requirement due to $P_{eprl}^e(j_1, \dots, j_4)^2 \neq P_{eprl}^e(j_1, \dots, j_4)$.

d. The Warsaw proposal

If one sees the above as difficulties, then there is a simple solution, at least in the Riemannian case. As proposed in Ref. [76, 77], one can obtain a consistent modification of the EPRL model by replacing P_{eprl}^e in (80) by a genuine projector P_w^e , graphically

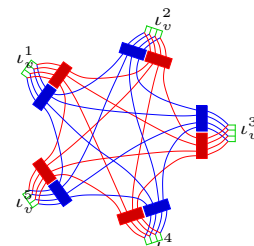
$$P_w^e(j_1 \dots j_4) = \sum_{\alpha\beta} \text{Inv} \left(\begin{array}{c} \alpha \\ \text{[Diagram: A network of red and blue lines connecting two green boxes labeled } \alpha \text{ and } \beta \text{]} \\ \beta \end{array} \right), \quad (81)$$

It is easy to check that by construction

$$(P_w^e(j_1 \dots j_4))^2 = P_w^e(j_1 \dots j_4). \quad (82)$$

The variant of the EPRL model proposed in Refs. [76, 77] takes then the form

$$Z_{eprl}(\Delta) = \sum_{j_f} \prod_{f \in \Delta^*} d_{|1-\gamma| \frac{i}{2}} d_{(1+\gamma) \frac{i}{2}} \times \prod_e P_w^e(j_1, \dots, j_4) \quad (83)$$

$$= \sum_{j_f} \sum_{\ell_{ev}} \prod_{f \in \Delta^*} d_{|1-\gamma| \frac{i}{2}} d_{(1+\gamma) \frac{i}{2}} \times \prod_{e \in \Delta^*} g_{\ell_{vs} \ell_{vt}}^e \prod_{v \in \Delta^*}$$


²This is obviously not clear from the form of (80). We are extrapolating the properties of $(P_{eprl}^e)^N$ for large N to those of the amplitude (80) in the large number of edges limit implied by the continuum limit.

Thus, in the modified EPRL model, edges $e \in \Delta^*$ are assigned pairs of intertwiner quantum numbers $\iota_{v_s}^e$ and $\iota_{v_t}^e$ and an edge amplitude given by the matrix elements $g_{\iota_{v_s}^e, \iota_{v_t}^e}^e$ (where v_s and v_t stand for the source and target vertices of the given oriented edge). The fact that edges are not assigned a single quantum number is not really significant; one could go to a basis of normalized eigenstates of P_w^e and rewrite the modified model above as a spin foam model where edges are assigned a single (basis element) quantum number. As the nature of such basis and the quantum geometric interpretation of its elements are not clear at this stage, it seems simpler to represent the amplitudes of the modified model in the above form.

The advantages of the modified model are important. However, a generalization of the above modification of the EPRL model in the Lorentzian case is still lacking. Notice that this modification does not interfere with the results on the semiclassical limit (to leading order), as reviewed in Section VII. The reason for this is that the matrix elements $g_{\alpha\beta}^e \rightarrow \delta_{\alpha\beta}$ in that limit [78].

v. The coherent states representation

We have written the amplitude defining the EPRL model by constraining the state sum of BF theory. For semiclassical studies that we will review in Section VII, it is convenient to express the EPRL amplitude in terms of the coherent states basis. The importance of coherent states in spin foam models was put forward in Ref. [49] and explicitly used to re-derive the EPRL model in Ref. [79]. The coherent state technology was used by Freidel and Krasnov in [6] to introduce a new kind of spin foam models for gravity: The FK models. In some cases, the FK model is equivalent to the EPRL model. We will review this in detail in Section IV.

The coherent state representation of the EPRL model is obtained by replacing (27) in each of the intermediate $SU(2)$ (green) wires in the expression (61) of the EPRL amplitudes, namely

(84)

The case $\gamma < 1$

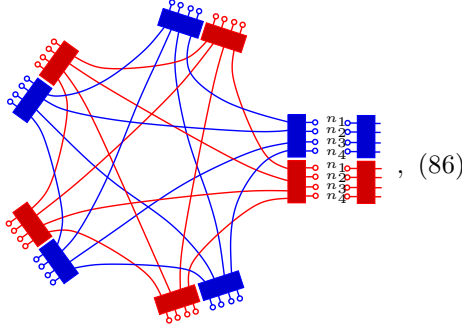
In this case, the coherent state property (28) implies

(85)

where we have used, in the last line, the fact that for $\gamma < 1$ the representations j of the subgroup $SU(2) \in Spin(4)$ are maximum weight, i.e., $j = j^+ + j^-$. Doing this at each edge, we get

where the discrete action

$$Z_{eprl}^E(\Delta) = \sum_{j_f} \prod_{f \in \Delta^*} d_{j_f^-} d_{j_f^+} \int \prod_{e \in \Delta^*} d_{j_{ef}} dn_{ef}$$



$$S_{j^\pm, \mathbf{n}}^{\gamma < 1}[g^\pm] = \sum_{v \in \Delta^*} (S_{(1-\gamma)\frac{j_f}{2}, \mathbf{n}}^v[g^-] + S_{(1+\gamma)\frac{j_f}{2}, \mathbf{n}}^v[g^+]) \quad (89)$$

with

$$S_{j, \mathbf{n}}^v[g] = \sum_{a < b=1}^5 2j_{ab} \ln \langle n_{ab} | g_a^{-1} g_b | n_{ba} \rangle, \quad (90)$$

where we have explicitly written the $n \in S^2$ integration variables on a single cable. The expression above is very similar to the coherent states representation of $Spin(4)$ BF theory given in Eq. (29). In fact, one gets the above expression if one starts from the expression (29) and sets $n_{ef}^+ = n_{ef}^- = n_{ef}$ while dropping, for example, all the sphere integrations corresponding to the n_{ef}^+ (or equivalently n_{ef}^-). Moreover, by construction, the coherent states participating in the previous amplitude satisfy the linear constraints (45) in expectation values, namely

$$\begin{aligned} & \langle j, n_{ef} | D_f^i | j, n_{ef} \rangle \\ &= \langle j, n_{ef} | (1 - \gamma) J_f^{+i} + (1 + \gamma) J_f^{-i} | j, n_{ef} \rangle \\ &= 0. \end{aligned} \quad (87)$$

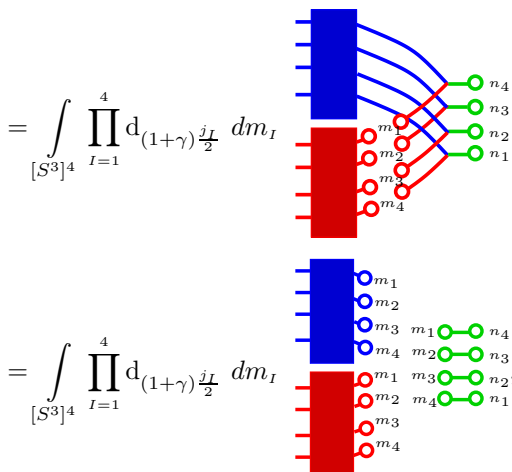
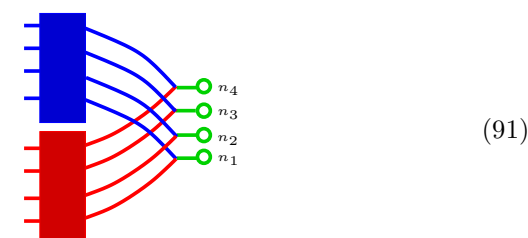
Thus, the coherent states participating in the above representation of the EPRL amplitudes solve the linear simplicity constraints in the usual semiclassical sense. The same manipulations leading to (89) in Section II lead to a discrete effective action for the EPRL model, namely

$$Z_{eprl}^{\gamma < 1} = \sum_{j_f} \prod_{f \in \Delta^*} d_{(1-\gamma)\frac{j_f}{2}} d_{(1+\gamma)\frac{j_f}{2}} \times \int \prod_{e \in \Delta^*} d_{j_{ef}} dn_{ef} dg_{ef}^- dg_{ef}^+ \exp(S_{j^\pm, \mathbf{n}}^{\gamma < 1}[g^\pm]), \quad (88)$$

and the indices a, b label the five edges of a given vertex. The previous expression is exactly equal to the form (11) of the BF amplitude. In the case of the gravity models presented here, the coherent state path integral representation (analogous to (31)) will be the basic tool for the study of the semiclassical limit of the models and the relationship with Regge discrete formulation of general relativity.

The case $\gamma > 1$

The case $\gamma > 1$ is more complicated [80]. The reason for this is that the step (85), directly leading to the discrete action in the previous case, is no longer valid, as the representations of the subgroup $SU(2) \in Spin(4)$ are now minimum instead of maximum weight. However, the representations $j^+ = j^- + j$ are maximum weight. We can, therefore, insert coherent states resolution of the identity on the right representations and get:



where we are representing the relevant part of the diagram appearing in Eq. (85). In the last line, we have used $j^+ = j + j^-$ (i.e., maximum weight), and the graphical notation $m \circ n \equiv \langle m|n \rangle$ as it follows from our previous conventions. With this, one gets

$$\begin{aligned}
 Z_{\text{eprl}}^{\gamma > 1} &= \\
 &\sum_{j_f} \prod_{f \in \Delta^*} d_{(1-\gamma)\frac{j_f}{2}} d_{(1+\gamma)\frac{j_f}{2}} \\
 &\times \int \prod_{e \in \Delta^*} d_{j_{ef}} d_{(1+\gamma)\frac{j_{ef}}{2}} dn_{ef} dm_{ef} dg_{ef}^- dg_{ef}^+ \\
 &\times \exp(S_{j^\pm, \mathbf{n}, \mathbf{m}}^{\gamma > 1}[g^\pm]),
 \end{aligned}
 \tag{92}$$

where the discrete action

$$S_{j^\pm, \mathbf{n}, \mathbf{m}}^{\gamma > 1}[g^\pm] = \sum_{v \in \Delta^*} S_{j^\pm, \mathbf{n}, \mathbf{m}}^v[g^\pm]
 \tag{93}$$

with

$$\begin{aligned}
 S_{j^\pm, \mathbf{n}, \mathbf{m}}^v[g^\pm] &= \\
 &= \sum_{1 \leq a < b \leq 5} [j_{ab}(1 + \gamma) \log(\langle m_{ab} | g_{ab}^+ | m_{ba} \rangle) \\
 &+ j_{ab}(\gamma - 1) \log(\langle m_{ab} | g_{ab}^- | m_{ba} \rangle) \\
 &+ 2j_{ab} (\log(\langle n_{ab} | m_{ab} \rangle) + \log(\langle m_{ba} | n_{ba} \rangle))].
 \end{aligned}
 \tag{94}$$

a. Some additional remarks

It is important to point out that the commutation relations of basic fields—reflecting the simple algebraic structure of $spin(4)$ —used here are the ones induced by the canonical analysis of BF theory presented previously. The presence of constraints generally modifies canonical commutation relations, in particular in the presence of second class constraints. For some investigation of the issue in the context of the EPRL and FK models, see Ref. [69]. In Ref. [81], it is pointed out that the presence of secondary constraints in the canonical analysis of Plebanski action should be translated into additional constraints in the holonomies of the spin foam models here considered (see also Ref. [82]). A possible view is that the simplicity constraints are here imposed *for all times* and thus secondary constraints should be imposed automatically.

There are alternative derivations of the models presented in the previous sections. In particular, one can derive them from a strict Lagrangean approach of Plebanski's action. Such viewpoint is taken in Refs. [83–85]. The path integral formulation of Plebanski theory using commuting B -fields was studied in Ref. [86], where it is shown that only in the appropriate semiclassical limit the amplitudes coincide with the ones presented in the previous sections (this is just another indication that the construction of the models has a certain semiclassical input; see below). The spin foam quantization of the Holst formulation of gravity via cubulations was investigated in Ref. [87]. The simplicity constraints can also be studied from the perspective of the $U(N)$ formulation of quantum geometry [88]. Such $U(N)$ treatment is related to previous work [89, 90] which has been extended to a completely new perspective on quantum geometry with possible advantageous features [91, 92]. For additional discussion on the simplicity constraints, see Ref.

[93].

vi. Presentation of the EPRL Lorentzian model

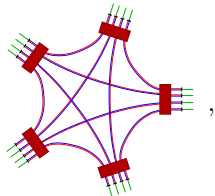
As it was briefly discussed in Section III, unitary irreducible representations of $SL(2, \mathbb{C})$ are infinite dimensional and labeled by a positive real number $p \in \mathbb{R}^+$ and a half-integer $k \in \mathbb{N}/2$. These representations are the ones that intervene in the harmonic analysis of square integrable functions of $SL(2, \mathbb{C})$ [64]. Consequently, one has an explicit expression of the delta function distribution (defined on such test function), namely

$$\delta(g) = \sum_k \int_{\mathbb{R}^+} dp (p^2 + k^2) \sum_{j,m} D_{jmjm}^{p,k}(g) \quad (95)$$

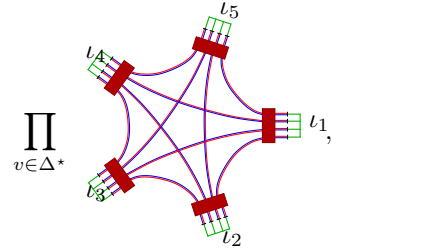
where $D_{jmj'm'}^{p,k}(g)$ with $j \geq k$ and $j \geq m \geq -j$ (similarly for the primed indices) are the matrix elements of the unitary representations $p - k$ in the so-called canonical basis [63]. One can use the previous expression, the Lorentzian version of Eq. (11), in order to introduce a formal definition of the BF amplitudes, which now would involve integration of the continuous labels p_f , in addition to sums over discrete quantum numbers such as k, j and m . The Lorentzian version of the EPRL model can be obtained from the imposition of the linear simplicity constraints to this formal expression. As the continuum labels p_f are restricted to $p_f = \gamma j_f$, the Lorentzian EPRL model becomes a state-sum model as its Riemannian relative. Using the following graphical notation

$$D_{jmj'm'}^{p,k}(g) = j,m \text{ --- } \frac{p}{k} \text{ --- } j',m' \quad (96)$$

the amplitude is

$$Z_{eprl}^L(\Delta) = \sum_{j_f} \prod_{f \in \Delta^*} (1 + \gamma^2) j_f^2$$


where the boxes now represent $SL(2, \mathbb{C})$ integrations with the invariant measure. The previous amplitude is equivalent to its spin foam representation

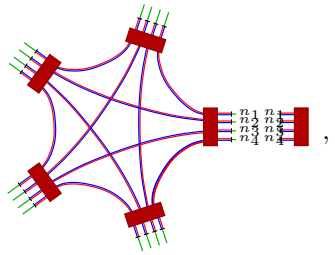
$$Z_{eprl}^L(\Delta) = \sum_{j_f} \sum_{\iota_e} \prod_{f \in \Delta^*} (1 + \gamma^2) j_f^2$$


The vertex amplitude is divergent due to the presence of a redundant integration over $SL(2, \mathbb{C})$. It becomes finite by dropping an arbitrary integration, i.e., removing any of the 5 boxes in the vertex expression [94].

a. The coherent state representation

It is immediate to obtain the coherent states representation of the Lorentzian models. As in the Riemannian case, one simply inserts resolution of the identities (22) on the intermediate $SU(2)$ (green) wires in (97), from where it results

$$Z_{eprl}^L(\Delta) = \sum_{j_f} \prod_{f \in \Delta^*} (1 + \gamma^2) j_f^2 \quad (97)$$

$$\int \prod_{e \in \Delta^*} d_{j_{e_f}} dn_{e_f}$$


IV. The Freidel-Krasnov (FK) model

Shortly after the appearance of the paper in Ref. [4], Freidel and Krasnov [6] introduced a set of new spin foam models for four-dimensional gravity using the coherent state basis of the quantum tetrahedron of Livine and Speziale [49]. The idea is to

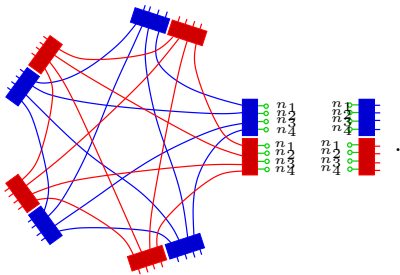
impose the linearized simplicity constraints (45) directly as a semiclassical condition on the coherent state basis. As we have seen above, coherent states are quantum states of the right and left tetrahedra in BF theory which have a clear-cut semiclassical interpretation through their property (26). We have also seen that the imposition of the linear constraints (45) *a la* EPRL is in essence semiclassical as they are strictly valid only in the large spin limit. In the FK approach one simply accepts from the starting point that, due to their property of a non-defining set that is closed under commutation relations, the Plebansky constraints are to be imposed semiclassically. One defines new models by restricting the set of coherent states entering in the coherent state representation of $Spin(4)$ BF theory (29) to those that satisfy condition (45) in expectation values. They also emphasize how the model [4] corresponds, indeed, to the sector $\gamma = \infty$ which has been shown to be topological [95].

The case $\gamma < 1$

For $\gamma < 1$, the vertex amplitude is identical to the EPRL model. This is apparent in the coherent state expression of the EPRL model (88). Thus, we have

$$Z_{fk}^{\gamma < 1}(\Delta) = \sum_{j_f} \prod_{f \in \Delta^*} d_{|1-\gamma|\frac{j_f}{2}} d_{(1+\gamma)\frac{j_f}{2}} \quad (98)$$

$$\prod_{e \in \Delta^*} \int d_{(1+\gamma)\frac{j_{ef}}{2}} d_{(\gamma-1)\frac{j_{ef}}{2}} dn_{ef}$$



From the previous expression, we conclude that the vertex amplitudes of the FK and EPRL model coincide for $\gamma < 1$

$$A_{v\,fk}^{\gamma < 1} = A_{v\,eprl}^{\gamma < 1}. \quad (99)$$

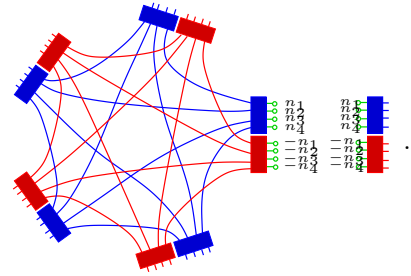
Notice, however, that different weights are assigned to edges in the FK model. This is due to the

fact that one is restricting the $Spin(4)$ resolution of identity in the coherent basis in the previous expression, while in the EPRL model the coherent state resolution of the identity is used for $SU(2)$ representations. This difference is important and has to do with the still unsettled discussion concerning the measure in the path integral representation.

The case $\gamma > 1$

For the case $\gamma > 1$, the FK amplitude is given by

$$Z_{fk}^{\gamma > 1}(\Delta) = \sum_{j_f} \prod_{f \in \Delta^*} d_{|1-\gamma|\frac{j_f}{2}} d_{(1+\gamma)\frac{j_f}{2}} \prod_{e \in \Delta^*} \int d_{(1+\gamma)\frac{j_{ef}}{2}} d_{(\gamma-1)\frac{j_{ef}}{2}} dn_{ef} \quad (100)$$



The study of the coherent state representation of the FK model for $\gamma > 1$, in comparison with Eq. (92) for the EPRL model, clearly shows the difference between the two models in this regime.

$$Z_{fk}^{\gamma} = \sum_{j_f} \prod_{f \in \Delta^*} d_{(1-\gamma)\frac{j_f}{2}} d_{(1+\gamma)\frac{j_f}{2}} \int \prod_{e \in \Delta^*} d_{|1-\gamma|\frac{j_{ef}}{2}} d_{(1+\gamma)\frac{j_{ef}}{2}} dn_{ef} dg_{ef}^- dg_{ef}^+ \exp(S_{j_{\pm}, \mathbf{n}}^{fk, \gamma}[g^{\pm}]), \quad (101)$$

where the discrete action

$$S_{j_{\pm}, \mathbf{n}}^{fk, \gamma}[g^{\pm}] = \sum_{v \in \Delta^*} \left[S_{(1-\gamma)\frac{j_f}{2}, \mathbf{n}}^v[g^-] + S_{(1+\gamma)\frac{j_f}{2}, s(\gamma)\mathbf{n}}^v[g^+] \right], \quad (102)$$

where $s(\gamma) = \text{sign}(1 - \gamma)$ and

$$S_{j,n}^v[g] = \sum_{a < b=1}^5 2j_{ab} \ln \langle n_{ab} | g_a^{-1} g_b | n_{ba} \rangle, \quad (103)$$

with the indices a, b labeling the five edges of a given vertex.

V. Boundary data for the new models and relationship with the canonical theory

So far, we have considered cellular complexes with no boundaries. Transition amplitudes are expected to be related to the definition of the physical scalar product. In order to define them, one needs to consider complexes with boundaries. Boundary states are defined on the boundary of the dual two-complex Δ^* that we denote $\partial\Delta^*$. The object $\partial\Delta^*$ is a one-complex (a graph). According to the construction of the model (Section III), boundary states are in one-to-one correspondence with $SU(2)$ spin networks. This comes simply from the fact that links (one-cells) $\ell \in \partial\Delta^*$ inherit the spins labels (unitary irreducible representations of the subgroup $SU(2)$) from the boundary faces while nodes (zero-cells) $n \in \partial\Delta^*$ inherit the intertwiner levels from boundary edges.

At this stage, one can associate the boundary data with elements of a Hilbert space. Being in one-to-one correspondence with $SU(2)$ spin networks, a natural possibility is to associate to them an element of the kinematical Hilbert space of LQG. More precisely, with a given colored boundary graph γ with links labeled by spins j_ℓ and nodes labeled by intertwiners ι_n , we associate a cylindrical function $\Psi_{\gamma, \{j_\ell\}, \{\iota_n\}} \in \mathcal{L}^2(SU(2)^{N_\ell})$, where here N_ℓ denotes number of links in the graph γ . In this way, the boundary Hilbert space associated with $\partial\Delta^*$ is isomorphic (if one used the natural AL measure) with the Hilbert space of LQG truncated to that fixed graph. Moreover, geometric operators, such as volume and area defined in the covariant context, are shown to coincide with the corresponding operators defined in the canonical formulation [67, 96]. Now, if cellular complexes are dual to triangulations, then the boundary spin networks can have at most four-valent nodes. This limitation can be easily overcome: As in BF theory, the

EPRL amplitudes can be generalized to arbitrary complexes with boundaries given by graphs with nodes of arbitrary valence. The extension of the model to arbitrary complexes has been first studied in Refs. [97, 98]. It has also been revisited in Refs. [68].

Alternatively, one can associate the boundary states with elements of $\mathcal{L}^2(Spin(4)^{N_\ell})$ (in the Riemannian models)—or carefully define the analog of spin network states as distributions in the Lorentzian case (see Refs. [99] for some insights on the problem of defining a gauge invariant Hilbert space of graphs for non-compact gauge groups). In this case, one gets special kinds of spin network states. These are a subclass of the so-called projected spin networks introduced in Refs. [100, 101] in order to define a heuristic quantization of the (non-commutative and very complicated) Dirac algebra of a Lorentz connection formulation of the phase space of gravity [100, 102–107]. The fact that this special subclass of projected spin networks appears naturally as a boundary state of the new spin foams is shown in Ref. [108].

Due to their similarity to $\gamma < 1$, the same relationship between boundary data and elements of the kinematical Hilbert space holds for the FK model. However, such simple relationship does not hold for the model in the case $\gamma > 1$.

It is important to mention that the knotting properties of boundary spin networks do not seem to play a role in present definitions of transition amplitudes [109].

VI. Further developments and related models

The spin foam amplitudes discussed in the previous sections have been introduced by constraining the BF histories through the simplicity constraints. However, in the path integral formulation, the presence of constraints has the additional effect of modifying the weights with which those histories are to be summed: Second class constraints modify the path integral measure (in the spin foam context this issue was raised in Ref. [75]). As pointed out before, this question has not been completely settled in the spin foam community yet. The explicit modification of the formal measure in terms of continuous variables for the Plebansky action was pre-

sented in Ref. [110]. A systematic investigation of the measure in the spin foam context was attempted in Ref. [111] and [112]. As pointed out in Ref. [75], there are restrictions in the manifold of possibilities coming from the requirement of background independence. The simple BF measure chosen in the presentation of the amplitudes in the previous sections satisfies these requirements. There are other consistent possibilities; see, for instance, Ref. [113] for a modified measure which remains extremely simple and is suggested from the structure of LQG.

An important question is the relationship between the spin foam amplitudes and the canonical operator formulation. The question of whether one can reconstruct the Hamiltonian constraints out of spin foam amplitudes has been analysed in detail in three dimensions. For the study of quantum three-dimensional gravity from the BF perspective, see Ref. [114]. We will, in fact, present this perspective in detail in the three dimensional part of this article. For the relationship with the canonical theory using variables that are natural from the Regge gravity perspective, see [115, 116]. There are generalizations of Regge variables more adapted to the interpretation of spin foams [117]. In four dimensions, the question has been investigated in Ref. [118] in the context of the new spin foam models. In the context of group field theories, this issue is explored in Ref. [119]. Finally, spin foams can, in principle, be obtained directly from the implementation of the Dirac program using path integral methods. This has been explored in Refs. [120, 121], from which a discrete path integral formulation followed [122]. The question of the relationship between covariant and canonical formulations in the discrete setting has been analyzed also in Ref. [123].

By construction, all tetrahedra in the FK and EPRL models are embedded in a spacelike hypersurface and hence have only spacelike triangles. It seems natural to ask the question of whether a more general construction allowing for timelike faces is possible. The models described in previous sections have been generalized in order to include timelike faces in the work of F. Conrady [124–126]. An earlier attempt to define such models in the context of the Barrett-Crane model can be found in Refs. [127].

The issue of the coupling of the new spin foam

models to matter remains to a large extent unexplored territory. Nevertheless, some results can be found in the literature. The coupling of the Barrett-Crane model (the $\gamma \rightarrow \infty$ limit of the EPRL model) to Yang-Mills fields was studied in Ref. [128]. More recently, the coupling of the EPRL model to fermions has been investigated in Refs. [129, 130]. A novel possibility of unification of the gravitational and gauge fields was recently proposed in Ref. [131].

The introduction of a cosmological constant in the construction of four-dimensional spin foam models has a long history. Barrett and Crane introduced a vertex amplitude [132], in terms of the Crane and Yetter model [13], for BF theory with cosmological constant. The Lorentzian quantum deformed version of the previous model was studied in Ref. [133]. For the new models, the coupling with a cosmological constant is explored in terms of the quantum deformation of the internal gauge symmetry in Refs. [134, 135], as well as (independently) in Ref. [136]. The asymptotics of the vertex amplitude are shown to be consistent with a cosmological constant term in the semiclassical limit in Ref. [137].

The spin foam approach applied to quantum cosmology has been explored in Refs. [138–143]. The spin foam formulation can also be obtained from the canonical picture provided by loop quantum cosmology (see Ref. [144] and references therein). This has been explored systematically in Refs. [145–148].

As we have discussed in the introduction of the new models, Heisenberg uncertainty principle precludes the strong imposition of the Plebanski constraints that reduce BF theory to general relativity. The results of the semiclassical limit of these models seem to indicate that metric gravity should be recovered in the low energy limit. However, it seems likely that the semiclassical limit could be related to certain modifications of Plebanski's formulation of gravity [149–153]. A simple interpretation of the new models in the context of the bi-gravity paradigm proposed in Ref. [154] could be of interest.

As it was already pointed out in Ref. [74], spin foams can be interpreted in close analogy to Feynman diagrams. Standard Feynman graphs are generalized to 2-complexes and the labeling of propagators by momenta to the assignment of spins to

faces. Finally, momentum conservation at vertices in standard feynmanology is now represented by spin-conservation at edges, ensured by the assignment of the corresponding intertwiners. In spin foam models, the non-trivial content of amplitudes is contained in the vertex amplitude which, in the language of Feynman diagrams, can be interpreted as an interaction. This analogy is indeed realized in the formulation of spin foam models in terms of a group field theory (GFT) [155, 156].

The GFT formulation resolves, by definition, the two fundamental conceptual problems of the spin foam approach: Diffeomorphism gauge symmetry and discretization dependence. The difficulties are shifted to the question of the physical role of λ and the convergence of the corresponding perturbative series.

This idea has been studied in more detail in three dimensions. In Ref. [157], the scaling properties of the modification of the Boulatov group field theory introduced in Ref. [158] were studied in detail. In a further modification of the previous model (known as colored tensor models [159], new techniques based on a suitable $1/N$ expansion imply that amplitudes are dominated by spherical topology [160]. Moreover, it seems possible that the continuum limit might be critical as in certain matrix models [161–165]. However, it is not yet clear if there is a sense in which these models correspond to a physical theory. The naive interpretation of the models is that they correspond to a formulation of 3d quantum gravity including a dynamical topology.

VII. Results on the semiclassical limit of EPRL-FK models

Having introduced the relevant spin foam models in the previous sections, we now present the results of the large spin asymptotics of the spin foam amplitudes suggesting that on a fixed discretization the semiclassical limit of the EPRL-FK models is given by Regge’s discrete formulation of general relativity [80, 166].

The semiclassical limit of spin foams is based on the study of the the large spin limit asymptotic behavior of coherent state spin foam amplitudes. The notion of large spin can be defined by the rescaling of quantum numbers and Planck constant accord-

ing to $j \rightarrow \lambda j$ and $\hbar \rightarrow \hbar/\lambda$ and taking $\lambda \gg 1$. In this limit, the quantum geometry approximates the classical one when tested with suitable states (e.g., coherent states). However, the geometry remains discrete during this limiting process as the limit is taken on a fixed regulating cellular structure. That is why one usually makes a clear distinction between semiclassical limit and the continuum limit. In the semiclassical analysis presented here, one can only hope to make contact with discrete formulations of classical gravity. Hence, the importance of Regge calculus in the discussion of this section.

The key technical ingredient in this analysis is the representation of spin foam amplitudes in terms of the coherent state basis introduced in Subsection i. Here we follow Refs. [80, 166–169]. The idea of using coherent states and discrete effective actions for the study of the large spin asymptotics of spin foam amplitudes was put forward in Refs. [170, 171]. The study of the large spin asymptotics has a long tradition in the context of quantum gravity, dating back to the study of Ponzano-Regge [26]. More directly related to our discussion, here are the early works [172, 173]. The key idea is to use asymptotic stationary phase methods for the amplitudes written in terms of the discrete actions presented in the previous section.

In this section, we review the results of the analysis of the large spin asymptotics of the EPRL vertex amplitude for both the Riemannian and Lorentzian models. We follow the notation and terminology of Ref. [80] and related papers.

b. $SU(2)$ $15j$ -symbol asymptotics

As $SU(2)$ BF theory is quite relevant for the construction of the EPRL-FK models, the study of the large spin asymptotics of the $SU(2)$ vertex amplitude is a key ingredient in the analysis of [80]. The coherent state vertex amplitude is

$$15j(j, \mathbf{n}) \quad (104)$$

$$= \int \prod_{a=1}^5 dg_a \prod_{1 \leq a < b \leq 5} \langle n_{ab} | g_a^{-1} g_b | n_{ba} \rangle^{2j_{ab}},$$

which depends on 10 spins j_{ab} and 20 normals $n_{ab} \neq n_{ba}$. The previous amplitude can be expressed as

$$15j(j, \mathbf{n}) = \int \prod_{a=1}^5 dg_a \prod_{1 \leq a < b \leq 5} \exp S_{j, \mathbf{n}}[g], \quad (105)$$

$$S_{j, \mathbf{n}}[g] = \sum_{a < b=1}^5 2j_{ab} \ln \langle n_{ab} | g_a^{-1} g_b | n_{ba} \rangle, \quad (106)$$

and the indices a, b label the five edges of a given vertex. The previous expression is equal to the form (11) of the BF amplitude. In the case of the EPRL model studied in Section III, the coherent state representation—see Eqs. 88, 92, and 97—is the basic tool for the study of the semiclassical limit of the models and the relationship with Regge discrete formulation of general relativity.

In order to study the asymptotics of (105), one needs to use extended stationary phase methods due to the fact that the action (106) is complex (see Refs. [170, 171]). The basic idea is that, in addition to stationarity, one requires real part of the action to be maximal. Points satisfying these two conditions are called *critical points*. As the real part of the action is negative definite, the action at critical points is purely imaginary.

Notice that the action (106) depends parametrically on 10 spins j and 20 normals \mathbf{n} . These parameters define the so-called boundary data for the four simplex $v \in \Delta^*$. Thus, there is an action principle for every given boundary data. The number of critical points and their properties depend on these boundary data, hence the asymptotics of the vertex amplitude is a function of the boundary data. Different cases are studied in detail in Ref. [80]. Here we present their results in the special case where the boundary data describe a non-degenerate Regge geometry for the boundary of a four simplex. These data are referred to as Regge-like, and satisfy the gluing constraints. For such boundary data, the action (106) has exactly two critical points, leading to the asymptotic formula

$$15j(\lambda j, \mathbf{n}) \sim \frac{1}{\lambda^{12}} \left[N_+ \exp\left(i \sum_{a < b} \lambda j_{ab} \Theta_{ab}^E\right) + N_- \exp\left(-i \sum_{a < b} \lambda j_{ab} \Theta_{ab}^E\right) \right], \quad (107)$$

where Θ_{ab} are the appropriate dihedral angles defined by the four-simplex geometry. Finally, the N_{\pm} are constants that do not scale with λ .

c. The Riemannian EPRL vertex asymptotics

The previous result, together with the fact that the EPRL amplitude for $\gamma < 1$ is a product of $SU(2)$ amplitudes with the same \mathbf{n} in the coherent state representation (88), implies the asymptotic formula for the vertex amplitude to be given by the unbalanced square of the above formula [167], namely

$$A_v^{eprl} \sim \frac{1}{\lambda^{12}} \left[N_+ e^{i \frac{(1-\gamma)}{2} \sum_{a < b} \lambda j_{ab} \Theta_{ab}^E} + N_- e^{-i \frac{(1-\gamma)}{2} \sum_{a < b} \lambda j_{ab} \Theta_{ab}^E} \right] \times \left[N_+ e^{i \frac{(1+\gamma)}{2} \sum_{a < b} \lambda j_{ab} \Theta_{ab}^E} + N_- e^{-i \frac{(1+\gamma)}{2} \sum_{a < b} \lambda j_{ab} \Theta_{ab}^E} \right].$$

One can write the previous expression as

$$A_v^{eprl} \sim \frac{1}{\lambda^{12}} \left[2N_+ N_- \cos(S_{Regge}^E) + N_+^2 e^{i \frac{1}{\gamma} S_{Regge}^E} + N_-^2 e^{-i \frac{1}{\gamma} S_{Regge}^E} \right]. \quad (108)$$

where

$$S_{Regge}^E = \sum_{a < b} \lambda \gamma j_{ab} \Theta_{ab}^E \quad (109)$$

is the Regge-like action for $\lambda \gamma j_{ab} = A_{ab}$, the ten triangle areas (according to the LQG area spectrum [1, 2]). Remarkably, the above asymptotic formula is also valid for the case $\gamma > 1$ [80]. The first term in the vertex asymptotics is in essence the expected one: It is the analog of the $6j$ symbol asymptotics in three-dimensional spin foams. Due to their explicit dependence on the Immirzi parameter, the last two terms are somewhat strange from the theoretical point of view of the continuum field. However, this seems to be a peculiarity of the Riemannian theory alone, as shown by the results of Ref. [166] for the Lorentzian models. Non-geometric configurations are exponentially suppressed

d. Lorentzian EPRL model

To each solution, one can associate a second solution corresponding to a parity related 4-simplex and, consequently, the asymptotic formula has two terms. It is given, up to a global sign, by the expression

$$A_v^{eprl} \sim \frac{1}{\lambda^{12}} \left[N_+ \exp \left(i\lambda\gamma \sum_{a<b} j_{ab} \Theta_{ab}^L \right) + N_- \exp \left(-i\lambda\gamma \sum_{a<b} j_{ab} \Theta_{ab}^L \right) \right], \quad (110)$$

where N_{\pm} are constants that do not scale. Non-geometric configurations are exponentially suppressed

In Ref. [171], Freidel and Conrady gave a detailed description of the coherent state representation of the various spin foam models described so far. In particular, they provided the definition of the effective discrete actions associated to each case which we presented in (101). This provides the basic elements for setting up the asymptotic analysis presented in Ref. [170] (the first results of the semiclassical limit of the new spin foam models). This is similar to the studies of the asymptotic of the vertex amplitude reviewed above, but more general in the sense that the semiclassical limit of a full spin foam configuration (involving many vertices) is studied. The result is technically more complex as one studies now critical points of the action associated to a colored complex which, in addition of depending on group variables g , depends on the coherent state parameters \mathbf{n} . The authors of Ref. [170] write Eq. (101) in the following way:

$$Z_{fk}^{\gamma} = \sum_{j_f} \prod_{f \in \Delta^*} d_{(1-\gamma)\frac{j_f}{2}} d_{(1+\gamma)\frac{j_f}{2}} W_{\Delta^*}^{\gamma}(j_f), \quad (111)$$

where

$$W_{\Delta^*}^{\gamma}(j_f) = \int \prod_{e \in \Delta^*} d_{|1-\gamma|\frac{j_{ef}}{2}} d_{(1+\gamma)\frac{j_{ef}}{2}} dn_{ef} dg_{ef}^- dg_{ef}^+ \times \exp(S_{j_{\pm}, \mathbf{n}}^{\gamma}[g^{\pm}]). \quad (112)$$

They show that those solutions of the equations of motion of the effective discrete action that are non-geometric (i.e., the contrary of Regge-like) are not critical and hence exponentially suppressed in the scaling $j_f \rightarrow \lambda j_f$ with $\lambda \gg 1$. If configurations are geometric (i.e., Regge-like), one has two kinds of contributions to the amplitude asymptotics: Those coming from degenerate and non-degenerate configurations. If one (by hand) restricts to the non-degenerate configurations, then one has

$$W_{\Delta^*}^{\gamma}(j_f) \sim \frac{c}{\lambda^{(33n_e - 6n_v - 4n_f)}} \times \exp(i\lambda S_{Regge}^E(\Delta^*, j_f)), \quad (113)$$

where n_e , n_v , and n_f denote the number of edges, vertices, and faces in the two complex Δ^* , respectively. There are recent works by M. Han in which asymptotics of general simplicial geometry amplitudes are studied in the context of the EPRL model [174, 175].

The problem of computing the two-point function and higher correlation functions in the context of spin foam has received a lot of attention recently. The framework for the definition of the correlation functions in the background independent setting has been generally discussed by Rovelli in Ref. [176], and corresponds to a special application of a more general proposal investigated by Oeckl [177–184]. It was then applied to the Barrett-Crane model in Refs. [185–187], where it was discovered that certain components of the two-point function could not yield the expected result compatible with Regge gravity in the semiclassical limit. This was used as the main motivation for the weakening of the imposition of the Plebanski constraints, leading to the new models. Soon thereafter, it was argued that the difficulties of the Barrett-Crane model where indeed absent in the EPRL model [188]. The two-point function for the EPRL model was calculated in Ref. [189] and it was shown to produce a result in agreement with that of Regge calculus [190, 191], in the limit $\gamma \rightarrow 0$.

The fact that, for the new model, the double scaling limit $\gamma \rightarrow 0$ and $j \rightarrow \infty$ with $\gamma j = \text{constant}$ defines the appropriate regime where the fluctuation behave as in Regge gravity (in the leading order) has been further clarified in Ref. [192]. This indicates that the quantum fluctuations in the new models are more general than simply metric fluc-

tuations. The fact that the new models are not metric at all scales should not be surprising as we know that the Plebanski constraints that produce metric general relativity out of BF theory have been implemented only semiclassically (in the large spin limit). At the deep Planckian regime, fluctuations are more general than metric. However, it is not clear at this stage why this is controlled by the Immirzi parameter.

All the previous calculations involve a complex with a single four-simplex. The first computation involving more than one simplex was performed in Refs. [187,193], for the case of the Barrett-Crane model. Certain peculiar properties were found and it is not clear at this stage whether these issues remain in the EPRL model. Higher order correlation functions have been computed in Ref. [194], the results are in agreement with Regge gravity in the $\gamma \rightarrow 0$ limit.

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