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The Parallel Simulation Method for d-dimensional Abelian Sandpile Automata

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Abstract

In this paper, the star-packing problem introduced in [1] for a square lattice is generalized for d-dimensional lattice $\mathcal{L}_d$, $d \in \mathbb{N}$. The problem is to pack the lattice $\mathcal{L}_d$ with star graphs S_{2d}. Using the solution of this problem, a parallel algorithm for the simulation of d-dimensional cellular automata is developed. As an example of cellular automata, the relaxation process of unstable states of Abelian sandpile model is considered. Appropriate software packages have been developed using OpenMP and CUDA technologies. The parallel simulation results, carried out for 3-dimensional lattices of different sizes, are presented.

Keywords: Abelian sandpile model, Dense packing problem, Parallel algorithm, Cellular automata.

1. Introduction

Cellular automata (CA) are discrete models studied in computability theory, mathematics, physics, complexity science, theoretical biology and microstructure modeling. Modern multicore computers with shared memory and multiprocessor clusters with distributed memory make it possible to efficiently parallelize simulation algorithms for CA. In the work [2], a cluster-based parallel algorithm and a package to simulate two-dimensional CA is introduced.

The concept of self-organized criticality was first introduced by Bak, Tang and Wiesenfeld in 1987 [4], and gave rise to growing interest in the study of self-organizing systems. Bak et al. argued that in many natural phenomena, the dissipative dynamics of the system is such that it drives the system to a critical state, thereby leading to ubiquitous power law behaviors. This mechanism has been invoked to understand the power law distributions observed in turbulent fluids, earthquakes, distribution of visible matter in the universe, solar flares and surface roughening of growing interfaces.

The Sandpile models, being a class of cellular automata, are among the simplest theoretical models which show self-organized criticality. A special subclass of interest consists of so called Abelian sandpile models (ASM). The Abelian property means that the final stable state of the CA is independent of the order in which the updates of cells are carried out. This property plays a key role during the numerical, as well as analytical studies of the ASM [5, 6, 7, 8, 9, 10, 11, 12, 13, 14]. In numerous works, in the absence of analytical relations for

various physical characterizers, different methods of statistical analysis have been applied for calculation or verification of hypothetical and analytical expressions for sandpile models. The luck of such systems, also the demand for accurate calculations of physical observables would dramatically increase the acceptable sizes of lattice as well as the time constraints for calculations. An approach of efficient parallel simulation of sandpile model on multicore computers with shared memory, which is based on the idea that there are few active cells (unstable vertices) at each moment of time, is given in [3]. In general, when the initial unstable configuration contains many unstable vertices, this approach does not give good results for big lattices. In what follows we present a more efficient parallel algorithm for the simulation of d-dimensional CA implemented on systems with shared memory, which is based on the star-packing of the d-dimensional lattices. A solution to the star-packing problem is given in [1] for the 2-dimensional square lattice. In this paper we generalize the solution to the case of general d dimensions. Below a cellular automaton for implementing the Abelian Sandpile model on a d-dimensional lattice is considered aiming at the development of parallel simulation.

2. The Star-Packing Problem

Consider a d-dimensional lattice $\mathcal{L}_d$ of linear size n with periodic boundary conditions, which is a graph with the vertex set

$$V = V(\mathcal{L}_d) = \{v = (x_1, x_2, \ldots, x_d) : \ x_i = 0, 1, 2, \ldots, n-1; \ i = 1, 2, \ldots, d\} \tag{1}$$

and the set of undirected edges $E = E(\mathcal{L}_d)$ defined by the following rule. Each vertex $v = (x_1, x_2, \ldots, x_d) \in V$ is connected with $2d$ vertices given by adjacency list

$$\mathrm{Adj(v)} = \{(\mathrm{x_1}, \mathrm{x_2}, \ldots, (\mathrm{x_k} \pm 1) \bmod \mathrm{n}, \ldots, \mathrm{x_d}) : \ \mathrm{k} = 1, 2, \ldots, \mathrm{d}\}. \tag{2}$$

The number of vertices is $|V| = n^d$, and the number of edges is $|E| = d\, n^d$. Two different vertices v_1 and v_2 are called independent, if they are not adjacent and they do not have any common adjacent vertex:

$$v_1 \notin \mathrm{Adj(v_2)}, \ \ \mathrm{v_2} \notin \mathrm{Adj(v_1)} \ \text{ and } \ \mathrm{Adj(v_1)} \bigcap \mathrm{Adj(v_2)} = \emptyset. \tag{3}$$

The star-packing problem is to pack the lattice $\mathcal{L}_d$ with non-overlapping star graphs S_{2d}. Consider a star packing with a set of central vertices $V' \subset V$ of stars. The following two rules define the set V':

1. Full coverage condition:
$$\left(\bigcup_{v \in V'} \mathrm{Adj(v)} \right) \bigcup V' = V.$$
2. Non-overlapping condition (independence of stars): Any two different vertices $v_1, v_2 \in V'$ are independent.

It is obvious, that a star packing exists if $|V|$ is dividable by $2d + 1$.

Given a vertex $v = (x_1, x_2, \ldots, x_d) \in V$ and a vector $\vec{r} = (p_1, p_2, \ldots, p_d) \in \mathbb{Z}$. Define the sum $v + \vec{r}$ to be a vertex in V with coordinates

$$v + \vec{r} \equiv ((x_1 + p_1) \bmod n, (x_2 + p_2) \bmod n, \ldots, (x_d + p_d) \bmod n) \in V. \tag{4}$$

Equivalently, the sum of a vector $\vec{r}$ and an arbitrary subset $V' \subseteq V$ to be the following subset of V

$$V' + \vec{r} \equiv \{v + \vec{r} : \ v \in V'\} \subseteq V. \tag{5}$$

Note that, given a star packing defined by a set V', the set $V' + \vec{r}$ also defines a star packing, where $\vec{r} \in \mathbb{Z}$.

Theorem: *Consider a d-dimensional lattice $\mathcal{L}_d$ of linear size n with periodic boundary conditions. Assume that n is dividable by $2d+1$. Then a star packing of $\mathcal{L}_d$ exists, and the set of central vertices of stars is*

$$V' = \{v = (x_1, x_2, \ldots, x_d) : \ (x_1 + 2x_2 + 3x_3 + \ldots + d\,x_d) \bmod (2d+1) = 0, \ v \in V\}. \tag{6}$$

Proof: Given a vector $\vec{r} = (p_1, p_2, \ldots, p_d) \in \mathbb{Z}$, define a function

$$Q(\vec{r}) = p_1 + 2p_2 + 3p_3 + \ldots + d\,p_d. \tag{7}$$

Let us introduce the d-dimensional basis vectors

$$\vec{e}_k = (\underbrace{0, 0, \ldots, 0}_{\text{k-1}}, 1, \underbrace{0, 0, \ldots, 0}_{\text{d-k}}), \quad k = 1, 2, 3, \ldots, d, \tag{8}$$

and the null vector $\vec{e}_0 = (0, 0 \ldots, 0)$. First, we prove the full coverage condition. Given an arbitrary vertex $v \in V \setminus V'$, it is necessary to prove that there exists a vertex $u \in \mathrm{Adj(v)}$, which is a central vertex of some star, i.e., $u \in V'$. In other words,

$$\forall \vec{r} = (p_1, p_2, \ldots, p_d) \in \mathbb{Z}, 0 \le p_i \le n-1, \tag{9}$$

one of the following 3 conditions holds

1. $Q(\vec{r}) \bmod (2d+1) = 0$.
2. $\exists k = 1, 2, \ldots, d$; $Q(\vec{r} - \vec{e}_k) \bmod (2d+1) = 0$.
3. $\exists k = 1, 2, \ldots, d$; $Q(\vec{r} + \vec{e}_k) \bmod (2d+1) = 0$.

Assume that $Q(\vec{r})$ has the following form:

$$Q(\vec{r}) = (2d+1)s + t, \ 0 \le t < 2d+1, \ s \ge 0. \tag{10}$$

Let the value of k be

$$k = \begin{cases} 0, & \text{if } t = 0; \Rightarrow Q(\vec{r}) \bmod (2d+1) = 0 \\ t, & \text{if } t < d; \Rightarrow Q(\vec{r} - \vec{e}_k) \bmod (2d+1) = 0 \\ 2d+1-t, & \text{if } t > d; \Rightarrow Q(\vec{r} + \vec{e}_k) \bmod (2d+1) = 0 \end{cases}, \tag{11}$$

which proves the coverage condition.

Now we prove the non-overlapping condition. It states, $\forall\, \vec{r} = (p_1, p_2, \ldots, p_d) \in \mathbb{Z}$, $0 \le p_i \le n-1$, $i = 1, 2, \ldots, d$, which satisfies the condition

$$Q(\vec{r}) \bmod (2d+1) = 0, \tag{12}$$

and $\forall \vec{\Delta r} = (\Delta p_1, \Delta p_2, \ldots, \Delta p_d)$ with

$$0 < |\Delta p_1| + |\Delta p_2| + \ldots + |\Delta p_d| \le 2, \tag{13}$$

we have

$$Q(\vec{r} + \vec{\Delta r}) \bmod (2d+1) \neq 0. \tag{14}$$

Since the Q function is linear, we have to show that $Q(\vec{\Delta r}) \bmod (2d+1) \neq 0$. Then, it is sufficient to prove the inequality $0 < |Q(\vec{\Delta r})| < 2d+1$. Note that there are 3 possible cases, which satisfy the condition (13):

1. $\exists k_0 = 1, 2, \ldots, d$, for which $\Delta p_{k_0} = \pm 2$ and $\Delta p_s = 0$, $\forall s \neq k_0$. Then

$$0 < |Q(\vec{\Delta r})| = 2k_0 < 2d+1. \tag{15}$$

2. $\exists k_0 = 1, 2, \ldots, d$, for which $\Delta p_{k_0} = \pm 1$ and $\Delta p_s = 0$, $\forall s \neq k_0$. Then

$$0 < |Q(\vec{\Delta r})| = k_0 < 2d+1. \tag{16}$$

3. $\exists k_1, k_2 = 1, 2, \ldots, d$; $k_1 \neq k_2$, for which $\Delta p_{k_1} = \pm 1$, $\Delta p_{k_2} = \pm 1$ and $\Delta p_s = 0$, $\forall s \neq k_1, k_2$. Then

$$0 < |Q(\vec{\Delta r})| = |k_1 \Delta p_{k_1} + k_2 \Delta p_{k_2}| \leq k_1 + k_2 < 2d+1. \tag{17}$$

□

Fig.1 and Fig.2 illustrate the star packing of the 2 and 3-dimensional lattices, respectively.

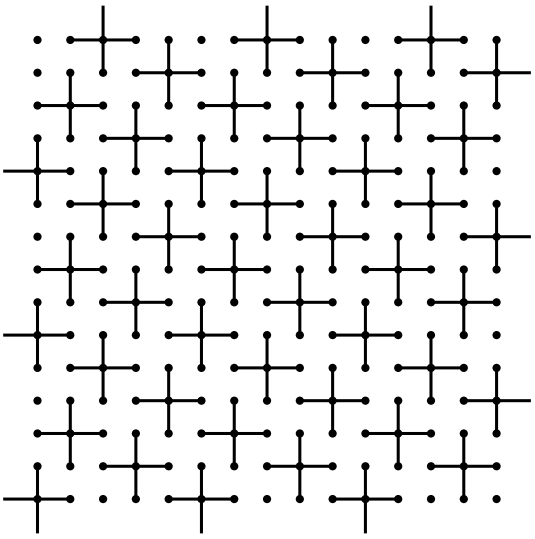

Fig. 1. A star-packing of a 2-dimensional lattice.

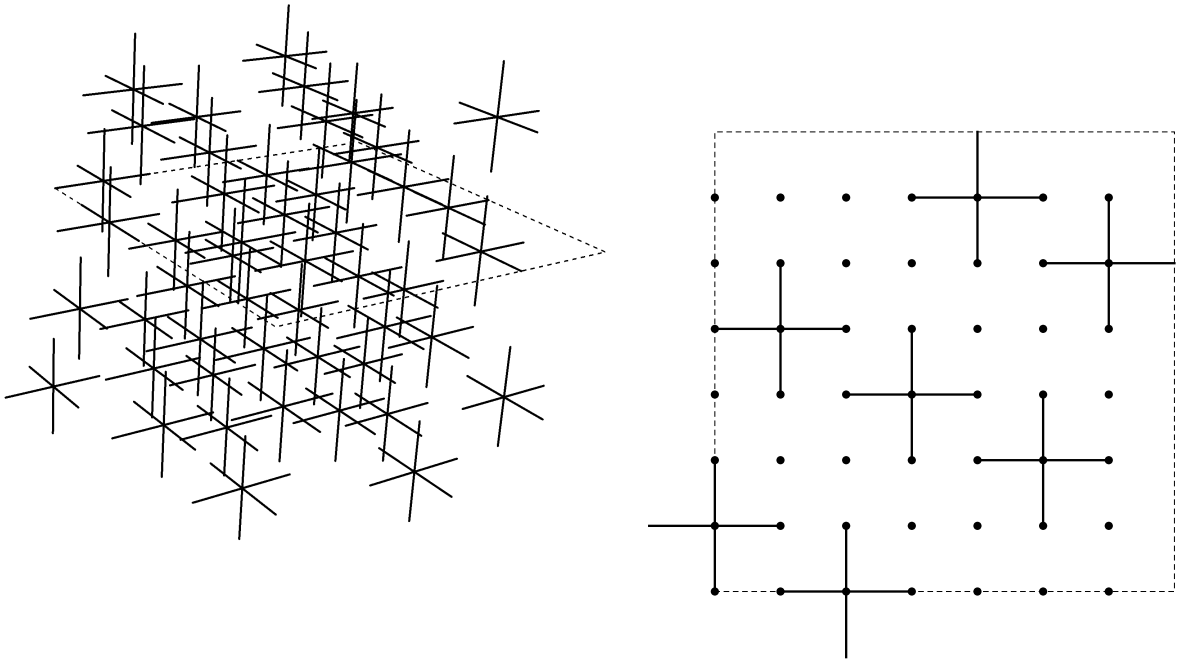

Fig. 2. A star-packing of a 3-dimensional lattice. The layer z=4 is depicted separately.

3. Abelian Sandpile Model

Consider an undirected graph $G = (V, E)$ with vertex set $V = \{v_1, v_2, \ldots, v_N\}$ and edge set E. A random variable h_i, which takes integer values, is attached to each vertex $v_i \in V$, representing the height of the sand at that vertex. h_i^{max} denotes the maximum allowed height of vertex v_i of the graph G. For d-dimensional lattice we take $h_i^{max} = 2d-1$. C_T denotes the collection of heights h_i, which defines a configuration of the system at a given discrete time T. A configuration is called stable, if all heights satisfy $h_i \leq h_i^{max}$. Vertex v_i is called closed, if $h_i^{max} = \deg(v_i) - 1$, where $\deg(v_i)$ indicates the number of adjacent vertices of v_i. Vertex v_i is called open, if $h_i^{max} \geq \deg(v_i)$. The dynamics of the system is defined by the following rules. Consider a stable configuration C_T at a given time T. We add a grain of sand at a random vertex $v_i \in V$ by setting h_i to $h_i + 1$ (we assume that the vertex is chosen randomly with a uniform distribution on the set V). This new configuration, if stable, defines C_{T+1}. If $h_i > h_i^{max}$, v_i becomes unstable and topples losing $h_i^{max}+1$ grains of sand, while all neighbors of v_i receive one grain. Note that if the vertex is open, the system loses grains. During the toppling of the closed vertices, the number of grains is conserved. Note also that toppling of a vertex may cause some of its neighboring vertices to become unstable. In this case those vertices also topple according to the same toppling rule. Once all unstable vertices have been toppled, a new stable configuration C_{T+1} is obtained. If the finite connected graph G has at least one open vertex, then all vertices become stable after finite number of topplings. Moreover, the new stable configuration is independent of the toppling order. Therefore, the dynamics is well defined. Let $\hat{a}_i$ be an operator, which acts on sandpile configurations and adds a grain at vertex i. It can easily be shown that $\hat{a}_i\hat{a}_j = \hat{a}_j\hat{a}_i$. This is the reason why the sandpile model is called Abelian.

4. Simulation Results

In this section we present the simulation results of the implemented software with built-in parallelization algorithms. The Abelian sandpile model on the finite d-dimensional lattice of linear size n is considered. The number of nodes is $N = n^d$. The value of maximal height at all vertices $v_i \in V$ is $h_i^{max} = 2d - 1$, $i = 1, 2, \ldots, N$. To perform parallel simulation, the concurrently toppled unstable vertices should meet the requirement of being independent. Partitioning the set of vertices into the sets of independent vertices is carried out by star-packing discussed above. Let T_{wait} be the time needed for a cell to finish its job, and H denotes the initial number of grains at each node.

In order to obtain a parallel software environment, the OpenMP and CUDA technologies have been used, which was implemented on the CPU i7 2670QM by Intel and Geforce GT. Simulation results include 2 types of Cellular Automata simulating Sandpile model. Difference is that for the second system an additional job is introduced during each toppling, which takes T_{wait} constant time to finish.

Four types of implementations have been analyzed and compared:

1. Standard − topplings are implemented sequentially on CPU. There is a loop around all nodes, and once an unstable node is found, it becomes stable.

2. No OMP − topplings are implemented sequentially on CPU. By a parallelization algorithm, we partition the set of nodes into 7 sections for a 3-dimensional lattice. There are two types of loops, one type over all groups, and the second type over all vertices for each group.

3. OMP − topplings are implemented parallel for each group on CPU, also there is one loop over all groups.

4. GPU − topplings are implemented in parallel for each group on GPU, also there is one loop over all groups.

The simulations have been done on the same CPU and GPU architecture. In Figs. 3-7 the simulation results are presented.

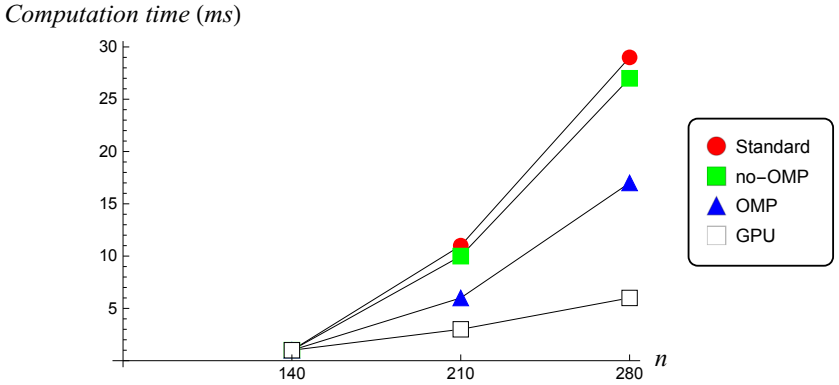

Fig. 3. Sandpile on a 3-dimensional lattice. Grains are added randomly with uniform distribution. Total amount of added grains is $10 \times n^3$.

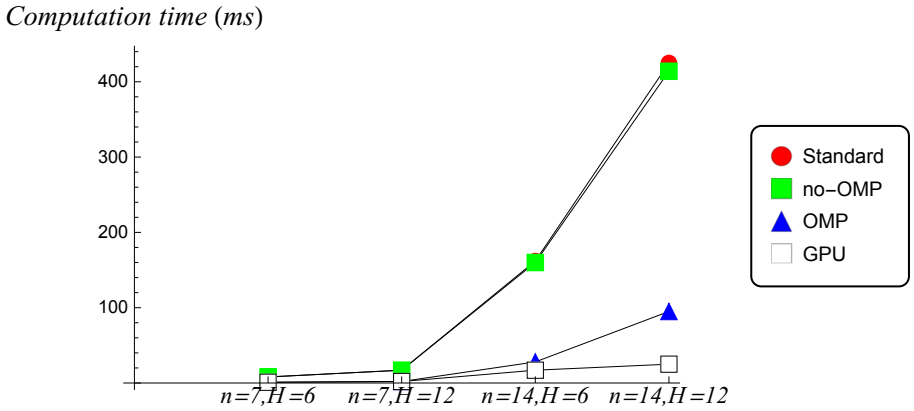

Fig. 4. Sandpile on a 3-dimensional lattice with initial height H=6, H=12. $T_{\text{wait}} = 5ms$.

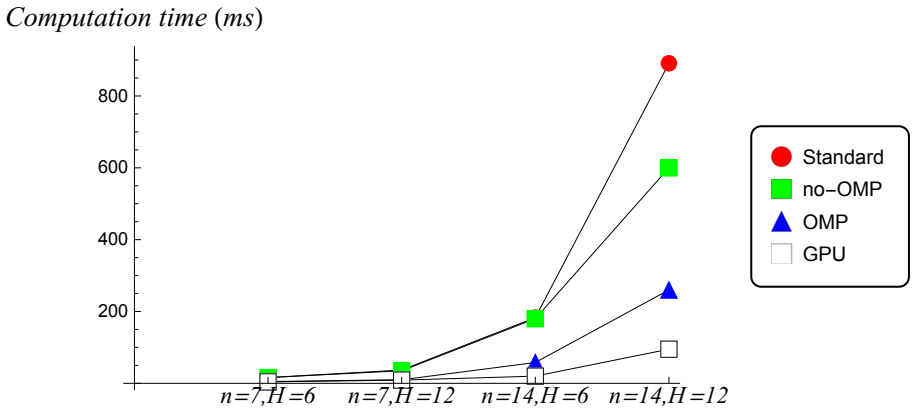

Fig. 5. Sandpile on a 3-dimensional lattice with initial height H=6, H=12. $T_{\text{wait}} = 10ms$.

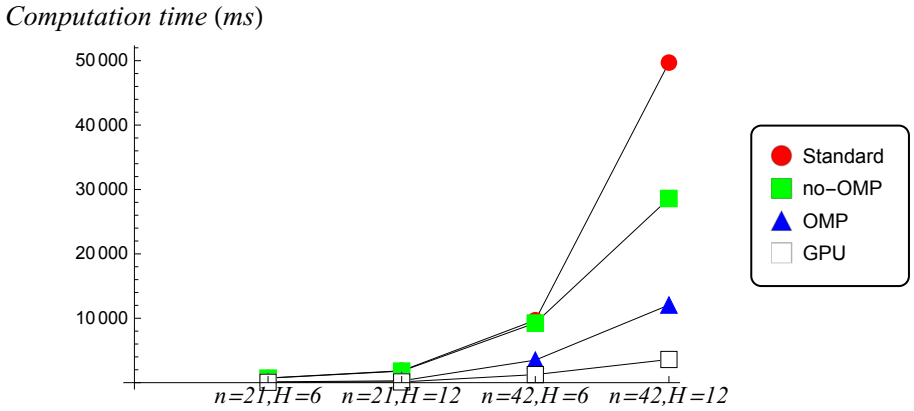

Fig. 6. Sandpile on a 3-dimensional lattice with start height H=6, H=12. $T_{\text{wait}} = 3ms$.

5. Conclusion

In this paper, a parallel algorithm for simulating d-dimensional CA has been described. The algorithm is based on the idea of star-packing of d-dimensional lattices. Delmas and Perennes in their paper [15] found a star packing for 3-dimensional lattice of linear size $n = 7^i$ with integer i. We found an explicit star packing of d-dimensional lattice, which meets weaker restriction on n, namely, $n \bmod (2d + 1) = 0$. As the results show, the algorithm developed gains about 8× speedup with openMP, and more than 50× on CUDA (depending on GPU architecture and cellular automata, it can be 100× and more faster than non-paralyzed

performance). Our next aim is to use algorithms for computing different observables of the sandpile model on d-dimensional lattices, meanwhile analytical calculations have been obtained for 2-dimension.

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Ավազակույտի d-չափանի Աբելյան ավտոմատի զուգահեռ մոդելավորման մեթոդ

Հ. Նահապետյան, Ս. Պողոսյան, Վ. Պողոսյան և Յու. Շուքուրյան

Ամփոփում

Այս հոդվածում բերված է քառակուսի ցանցերի համար [1]-ում ներմուծված աստղային ծածկույթի խնդրի ընդհանրացումը $\mathcal{L}_d$, $d \in \mathbb{N}$ d -չափանի ցանցի համար: Խնդիրը կայանում է S_{2d} գրաֆով $\mathcal{L}_d$ ցանցի ծածկման մեջ: Տվյալ խնդրի լուծման հիման վրա մշակվել է d-չափանի բջջային ավտոմատի մոդելավորման զուգահեռացված ալգորիթմ: Որպես բջջային ավտոմատի օրինակ է դիտարկվել Աբելյան ավազակույտի անկայուն վիճակների ռելաքսացիոն պրոցեսը: Համապատասխան ծրագրային փաթեթները նախագծվել են OpenMP և CUDA տեխնոլոգիաների կիրառմամբ: Բերված են տարբեր չափերի եռաչափ ցանցերի զուգահեռացված մոդելավորման արդյունքները:

Метод параллельной симуляции d-мерных автоматов Абелевой песочной горки

А. Нагапетян, С. Погосян, В. Погосян и Ю. Шукурян

Аннотация

В этой статье приведено обобщение проблемы звездного покрытия, выдвинутой в [1] для квадратных решеток. Обобщение достигнуто для d-мерной решетки $\mathcal{L}_d$, $d \in \mathbb{N}$. Проблема состоит в покрытии $\mathcal{L}_d$ решетки звездным графом S_{2d}. На основании решения данной проблемы, разработана параллелизованная программа симуляции d-мерного клеточного автомата. В качестве примера клеточного автомата рассмотрен процесс релаксации нестабильных состояний Абелевой модели песочной горки. Соответствующие программные пакеты разработаны с использованием технологий OpenMP и CUDA. Представлены результаты параллельной симуляции, проведенной для 3-мерной решетки различных размеров.