The Confidence Interval for the Periodic Intensity Function in the Presence of Power Function Trend on the Nonhomogeneous Poisson Process

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ABSTRACT

The nonhomogeneous Poisson process is one of the most widely applied stochastic processes. In this article, we provide a confidence interval of the intensity estimator in the presence of a periodic multiplied by trend power function. This estimator's confidence interval is an application of the formulation of the estimator asymptotic distribution that has been given in previous studies. By using the asymptotic theorem, the distribution was derived in the form of a confidence interval for the intensity function. In addition, constructive proof of the convergent in probability has been provided for all power functions. The results of this study contribute to the study of statistical analysis of the estimators that have been formulated previously.

Keywords: asymptotic distribution; interval confidence; intensity function; Poisson process.

INTRODUCTION

There are many events in nature can be modeled by stochastic modeling processes. The stochastic process is a set of random variables that map the sample space to a state space. One of the stochastic processes is a counting process which states the number of events at a time interval. The counting process assuming the number of events has a Poisson distribution is called the Poisson process. Some basic theories related Poisson process can be seen in [1]–[3].

Due to the intensity function, the Poisson process is divided into two categories, namely the homogeneous Poisson process and the nonhomogeneous Poisson process. A homogeneous Poisson process has a constant intensity function (independent of time), while a nonhomogeneous Poisson process has a time-dependent intensity function. This nonhomogeneous Poisson process is widely applied to real phenomena, such as the phenomenon of earthquakes [4], traffic accidents [5], and radio burst rates [6].

On the other hand, the study of the nonhomogeneous Poisson process in the form a periodic intensity function also has been conducted in recent years. [7] studied the

estimation of the intensity function in a nonhomogeneous Poisson process by including a trend component in the periodic intensity function. The trend component began with a trend in the form of an additive linear function. Then the study was continued with a trend in the form of a multiplicative linear function [8]. The estimation of the intensity function was carried out using the general kernel function approach, and [9] examined this Poisson process with a uniform kernel approach. Other related studies can be seen in [10]–[13].

[14] studied the estimation of the periodic Poisson process intensity function with the power function trend using a general kernel. Furthermore, the statistical properties of these estimators have also been proven. [15] has given strong consistency of these estimators. In addition, the asymptotic normality of the estimator has also been formulated and given a numerical simulation of the consistency of the estimator [16]. The results obtained in that study are the estimator of the periodic component which converges to the normal distribution by providing certain conditions.

As an application of the asymptotic normality, it can be determined the confidence interval of the estimator for the periodic component. This study provides the theorems for the confidence interval for the intensity function parameters and their proofs. The contribution of this study is to provide the characteristics of the estimator, especially in terms of accuracy. With a certain number of samples (interval length), it can be determined how accurately the estimator predicts the value of the parameter in the form of a confidence interval.

METHODS

The Estimator for Periodic Component of the Intensity Function

Suppose that $\{N(t), t \geq 0\}$ is a nonhomogeneous Poisson process with intensity function λ which locally integrable and unknown. Suppose also that λ is a periodic function with the trend of the power function, then λ which depends on the time variable s can be expressed as

$$\lambda(s) = \lambda_c^*(s). as^b. \tag{1}$$

The values of the a and b constants are assumed to be known, so that what is not known is the function of the periodic component of the intensity function, namely λ_c^*. Equation (1) can also be stated as follows

$$\lambda(s) = \lambda_c(s). s^b, \tag{2}$$

with $\lambda_c(s) = a\lambda_c^*(s)$. [14] has been given the kernel type estimator for $\lambda_c(s)$ by using general kernel functions. The estimator for periodic component of the intensity is

$$\hat{\lambda}_{c,n,k}(s) = \frac{\tau}{n} \sum_{k=0}^{\infty} \frac{1}{h_n(s+k\tau)^b} \int_0^n K\left(\frac{x-(s+k\tau)}{h_n}\right) N(dx). \tag{3}$$

On Equation (3), the constant τ is a period of the intensity function which satisfies

$$\lambda_c(s+k\tau) = \lambda_c(s), \text{ for } k \in Z.$$

With n is the length of the time interval used. In this case, since the Poisson process is a discrete stochastic process, it is clear that n is a natural number. The function K called a kernel function if it satisfies the following properties: (K1) K is a probability density function, (K2) K is bounded, and (K3) K is defined in [-1,1] [17].

The Asymptotic Normality of the Estimator

Theorem 1 (The Asymptotic Normal Distribution for $\hat{\lambda}_{c,n,k}(s), \; 0 < b < 1$)

Suppose that the intensity λ satisfies (1) and locally integrable. The kernel function K satisfies (K1), (K2), (K3), λ_c has a bounded second derivative around of s, $0 < b < 1$, $n^{1-b}h_n \to 0, n^{b+1}h_n \to \infty, h_n \downarrow 0$ as $n \to \infty$,

a) If $(n^{1+b}h_n^5)^{\frac{1}{2}} \to 0$, then

$$\left(n^{1+b}h_n\right)^{\frac{1}{2}}\left(\hat{\lambda}_{c,n,K}(s) - \lambda_c(s)\right) \xrightarrow{d} \text{Normal}(0, \sigma^2) \tag{4}$$

as $n \to \infty$, with $\sigma^2 = \frac{\tau\lambda_c(s)}{(1-b)}\int_{-1}^{1} K^2(z)dz$.

b) If $(n^{1+b}h_n^5)^{\frac{1}{2}} \to 1$, then

$$\left(n^{1+b}h_n\right)^{\frac{1}{2}}\left(\hat{\lambda}_{c,n,K}(s) - \lambda_c(s)\right) \xrightarrow{d} \text{Normal}(\mu, \sigma^2) \tag{5}$$

as $n \to \infty$, with $\mu = \frac{\lambda_c''(s)}{2}\int_{-1}^{1} z^2 K(z)dz$ and $\sigma^2 = \frac{\tau\lambda_c(s)}{(1-b)}\int_{-1}^{1} K^2(z)dz$.

Theorem 2 (The Asymptotic Normal Distribution for $\hat{\lambda}_{c,n,k}(s), \; b = 1$)

Suppose that the intensity λ satisfies (1) and locally integrable. The kernel function K satisfies (K1), (K2), (K3), λ_c has a bounded second derivative around of s, $b = 1$, $\ln(n)h_n \to 0, \frac{n^2 h_n}{\ln(n)} \to \infty, h_n \downarrow 0$ as $n \to \infty$,

a) If $\left(\frac{n^2 h_n^5}{\ln(n)}\right)^{\frac{1}{2}} \to 0$, then

$$\left(\frac{n^2 h_n}{\ln(n)}\right)^{\frac{1}{2}}\left(\hat{\lambda}_{c,n,K}(s) - \lambda_c(s)\right) \xrightarrow{d} \text{Normal}(0, \sigma^2) \tag{6}$$

as $n \to \infty$, with $\sigma^2 = \tau\lambda_c(s)\int_{-1}^{1} K^2(z)dz$.

b) If $\left(\frac{n^2 h_n^5}{\ln(n)}\right)^{\frac{1}{2}} \to 1$, then

$$\left(\frac{n^2 h_n}{\ln(n)}\right)^{\frac{1}{2}}\left(\hat{\lambda}_{c,n,K}(s) - \lambda_c(s)\right) \xrightarrow{d} \text{Normal}(\mu, \sigma^2) \tag{7}$$

as $n \to \infty$, with $\mu = \frac{\lambda_c''(s)}{2}\int_{-1}^{1} z^2 K(z)dz$ and $\sigma^2 = \tau\lambda_c(s)\int_{-1}^{1} K^2(z)dz$.

Theorem 3 (The Asymptotic Normal Distribution for $\hat{\lambda}_{c,n,k}(s), \; b > 1$)

Suppose that the intensity λ satisfies (1) and locally integrable. The kernel function K satisfies (K1), (K2), (K3), and λ_c has a bounded second derivative around of s, $b > 1$, $n^2 h_n \to \infty, \; h_n \downarrow 0$ as $n \to \infty$,

a) If $(n^2 h_n^5)^{\frac{1}{2}} \to 0$, then

$$(n^2 h_n)^{\frac{1}{2}}\left(\hat{\lambda}_{c,n,K}(s) - \lambda_c(s)\right) \xrightarrow{d} \text{Normal}(0, \sigma^2) \tag{8}$$

as $n \to \infty$, with $\sigma^2 = \tau^{2-b}\lambda_c(s)\zeta(b)\int_{-1}^{1} K^2(z)dz$,

and $\zeta(b) = \lim_{n\to\infty}\left(\sum_{k=1}^{\infty}\frac{1}{k^b}I(y + s + k\tau \in [0,n])\right)$.

b) If $(n^2 h_n^5)^{\frac{1}{2}} \to 1$, then

$$(n^2 h_n)^{\frac{1}{2}}\big(\hat{\lambda}_{c,n,K}(s) - \lambda_c(s)\big) \xrightarrow{d} \text{Normal}(\mu, \sigma^2) \tag{9}$$

as $n \to \infty$, with $\mu = \frac{\lambda_c''(s)}{2}\int_{-1}^{1} z^2 K(z)dz$, $\sigma^2 = \tau^{2-b}\lambda_c(s)\zeta(b)\int_{-1}^{1} K^2(z)dz$, and $\zeta(b) = \lim_{n\to\infty}\left(\sum_{k=1}^{\infty}\frac{1}{k^b}I(y+s+k\tau \in [0,n])\right)$.

The proofs of Theorem 1, Theorem 2, and Theorem 3 above can be proved through a rough analysis, [18]. It is recommended to study the basic theory to proof these theorems in [19]–[21].

RESULTS AND DISCUSSION

Suppose that ϕ denotes the standard normal distribution with ϕ^{-1} is the inverse. Based on Theorem 1, Theorem 2, and Theorem 3 above, it can be given some confidence interval for λ_c with significant level $1-\alpha$ as follows:

Corollary 1 (The confidence interval for λ_c for $0 < b < 1$)

Suppose that all conditions on Theorem 1 are satisfied, the for a significant level α where $0 < \alpha < 1$, the confidence interval for λ_c for $0 < b < 1$ has been given in the following conditions:

a) If $(n^{1+b} h_n^5)^{\frac{1}{2}} \to 0$ then

$$I_{\lambda_c} = \left(\hat{\lambda}_{c,n,k}(s) - \frac{\sigma\phi^{-1}\left(1-\frac{\alpha}{2}\right)}{\sqrt{n^{1+b}h_n}}, \hat{\lambda}_{c,n,k}(s) + \frac{\sigma\phi^{-1}\left(1-\frac{\alpha}{2}\right)}{\sqrt{n^{1+b}h_n}}\right),$$

where $\sigma^2 = \frac{\tau\lambda_c(s)}{(1-b)}\int_{-1}^{1} K^2(z)dz$.

b) If $(n^{1+b} h_n^5)^{\frac{1}{2}} \to 1$ then

$$I_{\lambda_c} = \left(\hat{\lambda}_{c,n,k}(s) - \frac{\sigma\phi^{-1}\left(1-\frac{\alpha}{2}\right)+\mu}{\sqrt{n^{1+b}h_n}}, \hat{\lambda}_{c,n,k}(s) + \frac{\sigma\phi^{-1}\left(1-\frac{\alpha}{2}\right)-\mu}{\sqrt{n^{1+b}h_n}}\right),$$

where $\mu = \frac{\lambda_c''(s)}{2}\int_{-1}^{1} z^2 K(z)dz$ and $\sigma^2 = \frac{\tau\lambda_c(s)}{(1-b)}\int_{-1}^{1} K^2(z)dz$.

Corollary 2 (The confidence interval for λ_c for $b = 1$)

Suppose that all conditions on Theorem 2 are satisfied, the for a significant level α where $0 < \alpha < 1$, the confidence interval for λ_c for $b = 1$ has been given in the following conditions:

a) If $\left(\frac{n^2 h_n^5}{\ln(n)}\right)^{\frac{1}{2}} \to 0$ then

$$I_{\lambda_c} = \left(\hat{\lambda}_{c,n,k}(s) - \sigma\sqrt{\frac{\ln(n)}{n^2 h_n}}\phi^{-1}\left(1-\frac{\alpha}{2}\right), \hat{\lambda}_{c,n,k}(s) + \sigma\sqrt{\frac{\ln(n)}{n^2 h_n}}\phi^{-1}\left(1-\frac{\alpha}{2}\right)\right),$$

where $\sigma^2 = \tau\lambda_c(s)\int_{-1}^{1} K^2(z)dz$.

b) If $\left(\frac{n^2 h_n^5}{\ln(n)}\right)^{\frac{1}{2}} \to 1$ then

$$I_{\lambda_c} = \left(\hat{\lambda}_{c,n,k}(s) - (\sigma\phi^{-1}\left(1 - \frac{\alpha}{2}\right) + \mu)\sqrt{\frac{\ln(n)}{n^2 h_n}}, \hat{\lambda}_{c,n,k}(s) + (\sigma\phi^{-1}\left(1 - \frac{\alpha}{2}\right) - \mu)\sqrt{\frac{\ln(n)}{n^2 h_n}}\right),$$

where $\mu = \frac{\lambda_c''(s)}{2}\int_{-1}^{1} z^2 K(z)dz$ and $\sigma^2 = \tau\lambda_c(s)\int_{-1}^{1} K^2(z)dz$.

Corollary 3 (The confidence interval for λ_c for $b > 1$)

Suppose that all conditions on Theorem 3 are satisfied, the for a significant level α where $0 < \alpha < 1$, the confidence interval for λ_c for $b > 1$ has been given in the following conditions:

a) If $(n^2 h_n^5)^{\frac{1}{2}} \to 0$ then

$$I_{\lambda_c} = \left(\hat{\lambda}_{c,n,k}(s) - \frac{\sigma\phi^{-1}\left(1 - \frac{\alpha}{2}\right)}{\sqrt{n^2 h_n}}, \hat{\lambda}_{c,n,k}(s) + \frac{\sigma\phi^{-1}\left(1 - \frac{\alpha}{2}\right)}{\sqrt{n^2 h_n}}\right),$$

where $\sigma^2 = \tau^{2-b}\lambda_c(s)\zeta(b)\int_{-1}^{1} K^2(z)dz$

b) If $(n^{1+b} h_n^5)^{\frac{1}{2}} \to 1$ then

$$I_{\lambda_c} = \left(\hat{\lambda}_{c,n,k}(s) - \frac{\sigma\phi^{-1}\left(1 - \frac{\alpha}{2}\right) + \mu}{\sqrt{n^2 h_n}}, \hat{\lambda}_{c,n,k}(s) + \frac{\sigma\phi^{-1}\left(1 - \frac{\alpha}{2}\right) - \mu}{\sqrt{n^2 h_n}}\right),$$

where $\mu = \frac{\lambda_c''(s)}{2}\int_{-1}^{1} z^2 K(z)dz$, $\sigma^2 = \tau^{2-b}\lambda_c(s)\zeta(b)\int_{-1}^{1} K^2(z)dz$,

and $\zeta(b) = \lim_{n\to\infty}\left(\sum_{k=1}^{\infty}\frac{1}{k^b}I(y + s + k\tau \in [0, n])\right)$.

To strengthen the reasons for the above confidence intervals, it is given the probability convergence theorems for these confidence interval.

Theorem 4. Convergence in Probability of the Confidence Interval for λ_c and $0 < b < 1$

If $\hat{\lambda}_{c,n,k}$ is the estimator for periodic component of the intensity function that is given in equation (3). Also, $I_{\lambda_{c,n}}$ is a confidence interval that is given in Corollary 1, then for the value $0 < b < 1$ satisfies

$$P\left(\lambda_{c,n}(s)\epsilon I_{\lambda_{c,n}}\right) \to 1 - \alpha + o(1),$$

provided $n \to \infty$.

The proof of Theorem 4:

Case (a) Assumption $\left(n^{1+b} h_n^5\right)^{\frac{1}{2}} \to 0$

$$P\left(\lambda_c(s)\epsilon I_{\lambda_c}\right) = P\left(\hat{\lambda}_{c,n,k} - \frac{\sigma\phi^{-1}\left(1 - \frac{\alpha}{2}\right)}{\sqrt{n^{1+b} h_n}} \leq \lambda_{c,n}(s) \leq \hat{\lambda}_{c,n,k} + \frac{\sigma\phi^{-1}\left(1 - \frac{\alpha}{2}\right)}{\sqrt{n^{1+b} h_n}}\right)$$

$$= P\left(-\frac{\sigma\phi^{-1}\left(1-\frac{\alpha}{2}\right)}{\sqrt{n^{1+b}h_n}} \leq \lambda_{c,n}(s) - \hat{\lambda}_{c,n,k} \leq \frac{\sigma\phi^{-1}\left(1-\frac{\alpha}{2}\right)}{\sqrt{n^{1+b}h_n}}\right)$$

$$= P\left(-\frac{\sigma\phi^{-1}\left(1-\frac{\alpha}{2}\right)}{\sqrt{n^{1+b}h_n}} \leq \hat{\lambda}_{c,n,k} - \lambda_{c,n}(s) \leq \frac{\sigma\phi^{-1}\left(1-\frac{\alpha}{2}\right)}{\sqrt{n^{1+b}h_n}}\right)$$

$$= P\left(-\sigma\phi^{-1}\left(1-\frac{\alpha}{2}\right) \leq \sqrt{n^{1+b}h_n}\left(\hat{\lambda}_{c,n,k} - \lambda_{c,n}(s)\right) \leq \sigma\phi^{-1}\left(1-\frac{\alpha}{2}\right)\right),$$

Let $Y = \sqrt{n^{1+b}h_n}\left(\hat{\lambda}_{c,n,k} - \lambda_c(s)\right)$, then based on Theorem 1 $Y \sim \text{Normal}(0, \sigma^2)$, By using Central Limit Theorem

$$Z = \frac{Y}{\sigma} \sim \text{Normal}(0,1).$$

Therefore

$$P\left(\lambda_c(s)\epsilon I_{\lambda_c}\right) = P\left(-\phi^{-1}\left(1-\frac{\alpha}{2}\right) \leq Z \leq \phi^{-1}\left(1-\frac{\alpha}{2}\right)\right)$$

$$= P\left(Z \leq \phi^{-1}\left(1-\frac{\alpha}{2}\right)\right) - P\left(Z < -\phi^{-1}\left(1-\frac{\alpha}{2}\right)\right).$$

Since the normal distribution has a symmetricity property,

$$P\left(Z < -\phi^{-1}\left(1-\frac{\alpha}{2}\right)\right) = P\left(Z \geq -\phi^{-1}\left(1-\frac{\alpha}{2}\right)\right),$$

then

$$P\left(\lambda_c(s)\epsilon I_{\lambda_c}\right) = P\left(Z \leq \phi^{-1}\left(1-\frac{\alpha}{2}\right)\right) - \left(Z \geq -\phi^{-1}\left(1-\frac{\alpha}{2}\right)\right)$$

$$= P\left(Z \leq \phi^{-1}\left(1-\frac{\alpha}{2}\right)\right) - 1 + P\left(Z \leq \phi^{-1}\left(1-\frac{\alpha}{2}\right)\right)$$

$$= \phi\left(\phi^{-1}\left(1-\frac{\alpha}{2}\right)\right) - 1 + \phi\left(\phi^{-1}\left(1-\frac{\alpha}{2}\right)\right)$$

$$= 1 - \frac{\alpha}{2} - 1 + 1 - \frac{\alpha}{2} = 1 - \alpha, \text{ provided } n \to \infty.$$

Case (b). Assumption $\left(n^{1+b}h_n^5\right)^{\frac{1}{2}} \to 1$

$$P\left(\lambda_c(s)\epsilon I_{\lambda_c}\right)$$

$$= P\left(\hat{\lambda}_{c,n,k} - \frac{\sigma\phi^{-1}\left(1-\frac{\alpha}{2}\right) + \mu}{\sqrt{n^{1+b}h_n}} \leq \lambda_{c,n}(s) \leq \hat{\lambda}_{c,n,k} + \frac{\sigma\phi^{-1}\left(1-\frac{\alpha}{2}\right) - \mu}{\sqrt{n^{1+b}h_n}}\right)$$

$$= P\left(-\frac{\sigma\phi^{-1}\left(1-\frac{\alpha}{2}\right) + \mu}{\sqrt{n^{1+b}h_n}} \leq \lambda_{c,n}(s) - \hat{\lambda}_{c,n,k} \leq \frac{\sigma\phi^{-1}\left(1-\frac{\alpha}{2}\right) - \mu}{\sqrt{n^{1+b}h_n}}\right)$$

$$= P\left(-\frac{\sigma\phi^{-1}\left(1-\frac{\alpha}{2}\right) - \mu}{\sqrt{n^{1+b}h_n}} \leq \hat{\lambda}_{c,n,k} - \lambda_{c,n}(s) \leq \frac{\sigma\phi^{-1}\left(1-\frac{\alpha}{2}\right) + \mu}{\sqrt{n^{1+b}h_n}}\right)$$

$$= P\left(-\left(\sigma\varphi^{-1}\left(1-\frac{\alpha}{2}\right)-\mu\right) \le \sqrt{n^{1+b}h_n}\left(\hat{\lambda}_{c,n,k}-\lambda_{c,n}(s)\right) \le \left(\sigma\varphi^{-1}\left(1-\frac{\alpha}{2}\right)+\mu\right)\right)$$

Suppose that $Y = \sqrt{n^{1+b}h_n}\left(\hat{\lambda}_{c,n,k}-\lambda_c(s)\right)$ then based on Theorem 1b $Y \sim \text{Normal}(\mu, \sigma^2)$, and

$$Z = \frac{Y-\mu}{\sigma} \sim \text{Normal}(0,1).$$

Therefore

$$P\left(\lambda_c(s)\epsilon I_{\lambda_c}\right) = P\left(-\varphi^{-1}\left(1-\frac{\alpha}{2}\right) \le Z \le \varphi^{-1}\left(1-\frac{\alpha}{2}\right)\right)$$

$$= P\left(Z \le \varphi^{-1}\left(1-\frac{\alpha}{2}\right)\right) - P\left(Z < -\varphi^{-1}\left(1-\frac{\alpha}{2}\right)\right).$$

Since the normal distribution has a symmetricity property

$$P\left(Z < -\varphi^{-1}\left(1-\frac{\alpha}{2}\right)\right) = P\left(Z \ge -\varphi^{-1}\left(1-\frac{\alpha}{2}\right)\right),$$

so

$$P\left(\lambda_c(s)\epsilon I_{\lambda_c}\right) = P\left(Z \le \varphi^{-1}\left(1-\frac{\alpha}{2}\right)\right) - \left(Z \ge -\varphi^{-1}\left(1-\frac{\alpha}{2}\right)\right)$$

$$= P\left(Z \le \varphi^{-1}\left(1-\frac{\alpha}{2}\right)\right) - 1 + P\left(Z \le \varphi^{-1}\left(1-\frac{\alpha}{2}\right)\right)$$

$$= \varphi\left(\varphi^{-1}\left(1-\frac{\alpha}{2}\right)\right) - 1 + \varphi\left(\varphi^{-1}\left(1-\frac{\alpha}{2}\right)\right)$$

$$= 1 - \frac{\alpha}{2} - 1 + 1 - \frac{\alpha}{2} = 1 - \alpha, \text{ provided } n \to \infty.$$

Theorem 5. Convergence in Probability of the Confidence Interval for λ_c and $b = 1$

If $\hat{\lambda}_{c,n,k}$ is the estimator for periodic component of the intensity function that is given in equation (3). Also, I_{λ_c} is an confidence interval that is given in Corollary 2, then for the value b = 1 satisfies

$$P\left(\lambda_c(s)\epsilon I_{\lambda_c}\right) \to 1 - \alpha + o(1),$$

provided n $\to \infty$.

The proof of Theorem 5

Case (a) Assumption $\left(\frac{n^2 h_n^5}{\ln(n)}\right)^{\frac{1}{2}} \to 0$

$$P\left(\lambda_c(s)\epsilon I_{\lambda_c}\right)$$

$$= P\left(\hat{\lambda}_{c,n,k} - \sigma\sqrt{\frac{\ln(n)}{n^2 h_n}}\varphi^{-1}\left(1-\frac{\alpha}{2}\right) \le \lambda_{c,n}(s) \le \hat{\lambda}_{c,n,k} + \sigma\sqrt{\frac{\ln(n)}{n^2 h_n}}\varphi^{-1}\left(1-\frac{\alpha}{2}\right)\right)$$

$$= P\left(-\sigma\sqrt{\frac{\ln(n)}{n^2 h_n}}\varphi^{-1}\left(1-\frac{\alpha}{2}\right) \le \lambda_{c,n}(s) - \hat{\lambda}_{c,n,k} \le \sigma\sqrt{\frac{\ln(n)}{n^2 h_n}}\varphi^{-1}\left(1-\frac{\alpha}{2}\right)\right)$$

$$= P\left(-\sigma\sqrt{\frac{\ln(n)}{n^2 h_n}}\,\phi^{-1}\left(1-\frac{\alpha}{2}\right) \leq \hat{\lambda}_{c,n,k} - \lambda_{c,n}(s) \leq \sigma\sqrt{\frac{\ln(n)}{n^2 h_n}}\,\phi^{-1}\left(1-\frac{\alpha}{2}\right)\right)$$

$$= P\left(-\sigma\phi^{-1}\left(1-\frac{\alpha}{2}\right) \leq \sqrt{\frac{n^2 h_n}{\ln(n)}}\,(\hat{\lambda}_{c,n,k} - \lambda_{c,n}(s)) \leq \sigma\phi^{-1}\left(1-\frac{\alpha}{2}\right)\right).$$

Let $= \sqrt{\frac{n^2 h_n}{\ln(n)}}\,(\hat{\lambda}_{c,n,k} - \lambda_c(s))$, then based on Theorem 2 $Y \sim \text{Normal}(0, \sigma^2)$, By using Central Limit Theorem

$$Z = \frac{Y}{\sigma} \sim \text{Normal}(0,1).$$

Therefore

$$P\big(\lambda_c(s)\epsilon I_{\lambda_c}\big) = P\left(-\phi^{-1}\left(1-\frac{\alpha}{2}\right) \leq Z \leq \phi^{-1}\left(1-\frac{\alpha}{2}\right)\right)$$

By using the same arguments before, it is obtained

$$P\big(\lambda_c(s) \in I_{\lambda_c}\big) = 1 - \alpha, \text{ provided } n \to \infty.$$

Case (b). Assumption $\left(\frac{n^2 h_n^5}{\ln(n)}\right)^{\frac{1}{2}} \to 1$

$$P\big(\lambda_c(s)\epsilon I_{\lambda_c}\big)$$

$$= P\left(\hat{\lambda}_{c,n,k} - \left(\sigma\phi^{-1}\left(1-\frac{\alpha}{2}\right) + \mu\right)\sqrt{\frac{\ln(n)}{n^2 h_n}} \leq \lambda_c(s) \leq \hat{\lambda}_{c,n,k} + \left(\sigma\phi^{-1}\left(1-\frac{\alpha}{2}\right) - \mu\right)\sqrt{\frac{\ln(n)}{n^2 h_n}}\right)$$

$$= P\left(-\sigma\phi^{-1}\left(1-\frac{\alpha}{2}\right) + \mu\sqrt{\frac{\ln(n)}{n^2 h_n}} \leq \lambda_c(s) - \hat{\lambda}_{c,n,k} \leq \sigma\phi^{-1}\left(1-\frac{\alpha}{2}\right) - \mu\sqrt{\frac{\ln(n)}{n^2 h_n}}\right)$$

$$= P\left(-\phi^{-1}\left(1-\frac{\alpha}{2}\right) - \mu\sqrt{\frac{\ln(n)}{n^2 h_n}} \leq \hat{\lambda}_{c,n,k} - \lambda_c(s) \leq \sigma\phi^{-1}\left(1-\frac{\alpha}{2}\right) + \mu\sqrt{\frac{\ln(n)}{n^2 h_n}}\right)$$

$$= P\left(-\left(\sigma\phi^{-1}\left(1-\frac{\alpha}{2}\right) - \mu\right) \leq \sqrt{\frac{n^2 h_n}{\ln(n)}}\,(\hat{\lambda}_{c,n,k} - \lambda_c(s)) \leq \left(\sigma\phi^{-1}\left(1-\frac{\alpha}{2}\right) + \mu\right)\right),$$

Suppose that $Y = \sqrt{\frac{n^2 h_n}{\ln(n)}}\,(\hat{\lambda}_{c,n,k} - \lambda_c(s))$, then according to Theorem 1b $Y \sim \text{Normal}(\mu, \sigma^2)$ and

$$Z = \frac{Y - \mu}{\sigma} \sim \text{Normal}(0,1).$$

Therefore

$$P\big(\lambda_{c,n}(s) \in I_{\lambda_{c,n}}\big) = P\left(-\phi^{-1}\left(1-\frac{\alpha}{2}\right) \leq Z \leq \phi^{-1}\left(1-\frac{\alpha}{2}\right)\right).$$

The same arguments gave us

$$P\big(\lambda_c(s) \in I_{\lambda_c}\big) = 1 - \alpha, \text{ provided } n \to \infty.$$

Theorem 6. Convergence in Probability of the Confidence Interval for $\lambda_{c,n}$ and $b > 1$

If $\hat{\lambda}_{c,n,k}$ is the estimator for periodic component of the intensity function that is given in equation (3). Also, $I_{\lambda_{c,n}}$ is a confidence interval that is given in Corollary 3, then for the value b > 1 satisfies

$$P\left(\lambda_c(s)\epsilon I_{\lambda_c}\right) \to 1 - \alpha + o(1),$$

provided n → ∞.

The proof of Theorem 6

Case a. Assumption $(n^2 h_n^5)^{\frac{1}{2}} \to 0$

$P\left(\lambda_c(s)\epsilon I_{\lambda_c}\right)$

$$= P\left(\hat{\lambda}_{c,n,k} - \frac{\sigma\phi^{-1}\left(1-\frac{\alpha}{2}\right)}{\sqrt{n^2 h_n}} \le \lambda_c(s) \le \hat{\lambda}_{c,n,k} + \frac{\sigma\phi^{-1}\left(1-\frac{\alpha}{2}\right)}{\sqrt{n^2 h_n}}\right)$$

$$= P\left(-\frac{\sigma\phi^{-1}\left(1-\frac{\alpha}{2}\right)}{\sqrt{n^2 h_n}} \le \lambda_c(s) - \hat{\lambda}_{c,n,k} \le \frac{\sigma\phi^{-1}\left(1-\frac{\alpha}{2}\right)}{\sqrt{n^2 h_n}}\right)$$

$$= P\left(-\sigma\frac{\sigma\phi^{-1}\left(1-\frac{\alpha}{2}\right)}{\sqrt{n^2 h_n}} \le \hat{\lambda}_{c,n,k} - \lambda_c(s) \le \sigma\frac{\sigma\phi^{-1}\left(1-\frac{\alpha}{2}\right)}{\sqrt{n^2 h_n}}\right)$$

$$= P\left(-\sigma\phi^{-1}\left(1-\frac{\alpha}{2}\right) \le \sqrt{n^2 h_n}\left(\hat{\lambda}_{c,n,k} - \lambda_c(s)\right) \le \sigma\phi^{-1}\left(1-\frac{\alpha}{2}\right)\right).$$

Let $Y = \sqrt{n^2 h_n}\left(\hat{\lambda}_{c,n,k} - \lambda_c(s)\right)$, then according to Theorem 3a $Y \sim \text{Normal}(0, \sigma^2)$ and

$$Z = \frac{Y}{\sigma} \sim \text{Normal}(0,1).$$

Therefore

$$P\left(\lambda_c(s)\epsilon I_{\lambda_c}\right) = P\left(-\phi^{-1}\left(1-\frac{\alpha}{2}\right) \le Z \le \phi^{-1}\left(1-\frac{\alpha}{2}\right)\right).$$

The same arguments gave us

$$P\left(\lambda_c(s) \in I_{\lambda_c}\right) = 1 - \alpha, \text{ provided } n \to \infty.$$

Case b. Assumption $(n^2 h_n^5)^{\frac{1}{2}} \to 1$

$P\left(\lambda_c(s) \in I_{\lambda_c}\right)$

$$= P\left(\hat{\lambda}_{c,n,k} - \frac{\sigma\phi^{-1}\left(1-\frac{\alpha}{2}\right) + \mu}{\sqrt{n^2 h_n}} \le \lambda_c(s) \le \hat{\lambda}_{c,n,k} + \frac{\sigma\phi^{-1}\left(1-\frac{\alpha}{2}\right) - \mu}{\sqrt{n^2 h_n}}\right)$$

$$= P\left(-\frac{\sigma\phi^{-1}\left(1-\frac{\alpha}{2}\right) + \mu}{\sqrt{n^2 h_n}} \le \lambda_c(s) - \hat{\lambda}_{c,n,k} \le \sigma\frac{\sigma\phi^{-1}\left(1-\frac{\alpha}{2}\right) - \mu}{\sqrt{n^2 h_n}}\right)$$

$$= P\left(-\frac{\sigma\phi^{-1}\left(1-\frac{\alpha}{2}\right)-\mu}{\sqrt{n^2 h_n}} \leq \hat{\lambda}_{c,n,k} - \lambda_c(s) \leq \frac{\sigma\phi^{-1}\left(1-\frac{\alpha}{2}\right)+\mu}{\sqrt{n^2 h_n}}\right)$$

$$= P\left(-\left(\sigma\phi^{-1}\left(1-\frac{\alpha}{2}\right)-\mu\right) \leq \sqrt{n^2 h_n}\left(\hat{\lambda}_{c,n,k} - \lambda_c(s)\right) \leq \left(\sigma\phi^{-1}\left(1-\frac{\alpha}{2}\right)+\mu\right)\right).$$

Suppose that $Y = \sqrt{n^2 h_n}\left(\hat{\lambda}_{c,n,k} - \lambda_c(s)\right)$, then based on Theorem 3b $Y \sim \text{Normal}(\mu, \sigma^2)$ and

$$Z = \frac{Y - \mu}{\sigma} \sim \text{Normal}(0,1).$$

Therefore

$$P\left(\lambda_c(s) \in I_{\lambda_c}\right) = P\left(-\phi^{-1}\left(1-\frac{\alpha}{2}\right) \leq Z \leq \phi^{-1}\left(1-\frac{\alpha}{2}\right)\right).$$

By using the same arguments, it is obtained that
$$P\left(\lambda_c(s) \in I_{\lambda_c}\right) = 1 - \alpha, \text{ provided } n \to \infty.$$

CONCLUSIONS

From the results that have been studied, the formula to determine the confidence interval for parameter of the periodic component of the nonhomogeneous Poisson process with the intensity in the form of periodic function has been obtained. These confidence intervals have been given for each case of the values of b, this is because the results of previous studies show that the variance of the estimator is given in a different function for each case of the values of b. These confidence intervals have been proved to converge in probability $1 - \alpha$.

The recommendation for further research that can be done is providing numerical simulations for each confidence interval case, there are 6 cases. The simulation can be started by determining the bandwidth function h_n which satisfies all the conditions in the given case and determining the probability of the estimator being in the confidence interval.

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