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ON THE WEAK SOLUTION OF THE VON-KARMAN MODEL WITH THERMOELASTIC PLATES

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Abstract. In this article we aim to study the dynamic Von-Karman model coupled with thermoelastic equations without rotational terms, subject to a thermal dissipation. We establish the existence as well as the uniqueness of a weak solution related to the dynamic model. At the end, we apply the finite difference method for approximating the solution of our problem.

1. Introduction

Nonlinear oscillation of an elastic model, for dynamic von-Karman model without rotational terms, subject to a thermal dissipation [4] describes the phenomenon of small nonlinear vibration with a vertical displacement to the elastic plates. The case of nonlinear thermoelastic plate interaction coupled with thermal dissipation plays an interesting place in this subject and will be our fundamental target in the present paper. The model with clamped boundary conditions, in the note account of rotational terms, can be formulated as follows ([4]):

Find $(u,\phi,\theta)\in L^2\big([0,T],H_0^2(\omega)\big)\times H_0^2(\omega)\times H_0^1(\omega)$ such that

$$(\mathcal{P}_0)\begin{cases} u_{tt}+\Delta^2u+\mu\Delta\theta-[\phi+F_0,u]=p(x) & in\quad \omega\times[0,T]\,,\\ k\theta_t-\eta\Delta\theta-\mu\Delta u_t=0 & in\quad \omega\times[0,T]\,,\\ u_{|t=0}=u_0,\quad (u_t)_{|t=0}=\overline{u},\quad \theta_{|t=0}=\theta_0 & in\quad \omega,\\ u=\partial_\nu u=0 & on\quad \Gamma\times[0,T]\,,\\ \theta=0 & on\quad \Gamma, \end{cases}$$

and

$$(\mathcal{Q})\begin{cases} \Delta^2\phi+[u,u]=0 & in\quad \omega\times[0,T]\,,\\ \phi=0,\ \partial_\nu\phi=0 & on\quad \Gamma\times[0,T]\,. \end{cases}$$

Here, u is the displacement, ϕ denotes the Airy stress function and θ is the thermal function, ω is the surface plate, u_0, u_1, θ_0 refer to the initial data and $[.,.]$ stands for the Monge-Ampère symbol defined through ([1])

$$[\phi,u]=\partial_{11}\phi\partial_{22}u+\partial_{11}u\partial_{22}\phi-2\partial_{12}\phi\partial_{12}u. \tag{1.1}$$

The parameters $\mu,\eta>0$ are fixed real numbers and $k>0$ measures the capacity of the heat/thermal. The plate is subject to the internal force F_0 and the external force p. In [4], the authors studied the problem of the von-Karman model for the case $0\le k\le 1$.

Our fundamental target in this paper is to explore a condition that should be satisfied by the external/internal loads and the initial data for ensuring the existence and the uniqueness of a weak solution for to the von-Karman evolution, without rotational terms nor clamped boundary conditions, subject to the thermal dissipation when $k>0$ and $0<\mu\le 2\eta$. The present approach turns out of to

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construct an iterative process converging, in an appropriate sense, to the unique solution of the initial problem.

This paper will be organized as follows. In Section 2 we present the mathematical structure of the model that will be studied in the sequel together with some basic tools and results. In Section 3 we use an iterative method that will be a good tool for establishing the existence and the uniqueness of a weak solution of the dynamical plate problem without rotational terms, subject to a thermal dissipation. Section 4 deals with a numerical simulation for approaching the solution of the initial problem.

2. **Preliminaries and main results**

Throughout this paper, we denote by ω a nonempty bounded domain in $\mathbb{R}^2$ with regular boundary $\Gamma = \partial\omega$. We suppose that the parameters k, μ, η in the problem $(\mathcal{P}_0)$ are such that $k > 0$ and $0 < \mu \leq 2\eta$.

Let $p \geq 1$ be a real number and $m \geq 1$ be an integer. The notation $|.|_{p,\omega}$ refers to the standard norm of $L^p(\omega)$ while $\|.\|_{m,\omega}$ stands for the classical norm of $H^m(\omega)$. For $u \in H_0^2(\omega)$, we put

$$\|u\| =: |\Delta u|_{2,\omega} = \Big(\int_\omega (\Delta u)^2 \Big)^{\frac{1}{2}},$$

which is obviously a norm in $H_0^2(\omega)$ ([4, 5]). Otherwise, we set

$$\|u\|_0^2 =: \|u\|^2 + |u_t|^2_{2,\omega} \,. \tag{2.1}$$

We state the following result which will be needed in the sequel, see [7].

Theorem 2.1. *Let $f \in L^2(\omega)$. Then the following problem*

$$(\mathcal{R}) \begin{cases} \Delta^2 v = f & in \quad \omega, \\ v = 0 & on \quad \Gamma, \\ \partial_\nu v = 0 & on \quad \Gamma, \end{cases}$$

has one and only one solution $v \in H_0^2(\omega) \cap H^4(\omega)$ satisfying

$$\|v\| \leq c_0 \, |f|_{2,\omega} \,,$$

where $c_0 > 0$ is a constant depending only on $mes(\omega)$.

Let us mention the following remark.

Remark 2.1. (i) Under the condition that $f \in L^2\Big([0,T] \, , L^2(\omega)\Big)$, the solution of $(\mathcal{R})$ belongs to the set $L^2\Big([0,T] \, , H_0^2(\omega) \cap H^4(\omega)\Big)$. (ii) We mention again that the constant $c_0 > 0$ in Theorem 2.1 depends only on $mes(\omega)$ and does not depend on f.

The following result will be needed later, see [7, 3].

Theorem 2.2. *Let $g \in L^2\Big([0,T] \, , L^2(\omega)\Big)$, $u_0 \in L^2(\omega)$ and $k, \eta, \mu > 0$. Then the following problem :*

$$(\mathcal{D}) \begin{cases} ku_t - \eta\Delta u = \mu g & in \quad \omega \times [0,T] \, , \\ u_{|_{t=0}} = u_0 & in \ \ \omega, \\ u = 0 & on \ \ \Gamma \times [0,T] \, , \end{cases}$$

has one and only one solution $u \in C\big([0,T] \, ; H^2(\omega) \cap H_0^1(\omega)\big) \cap C^1\big([0,T] \, ; L^2(\omega)\big)$.

We have the following result as well.

Proposition 2.3. *Let $f \in H^2(\omega)$, $k > 0$ and $0 < \mu \le 2\eta$. Then the solution u of $(\mathcal{D})$, when $g = -\Delta f$, satisfies the following inequality*

$$\forall t \in [0,T], \quad 0 \le k\,|u|^2_{2,\omega} + (2\eta - \mu)\int_0^t |\nabla u|^2_{2,\omega} \le k\,|u_0|^2_{2,\omega} + \mu\int_0^t |\nabla f|^2_{2,\omega}. \tag{2.2}$$

Proof. Since u is the solution of the problem $(\mathcal{D})$, with $g = -\Delta f$, then $ku_t - \eta\Delta u = -\mu\Delta f$ and so $k\langle u_t, u\rangle - \eta\langle \Delta u, u\rangle = -\mu\langle \Delta f, u\rangle$, where $\langle .,.\rangle$ refers to the standard inner product of $L^2(\omega)$. This latter equation is equivalent to

$$\frac{k}{2}\frac{d}{dt}\,|u|^2_{2,\omega} + \eta\,|\nabla u|^2_{2,\omega} = \mu\langle \nabla f, \nabla u\rangle. \tag{2.3}$$

By Hölder inequality in $L^2(\omega)$ and the standard inequality $ab \le \frac{1}{2}a^2 + \frac{1}{2}b^2$, valid for any $a, b > 0$, we can write

$$\langle \nabla f, \nabla u\rangle \le \Big|\langle \nabla f, \nabla u\rangle\Big| \le |\nabla f|_{2,\omega}\,|\nabla u|_{2,\omega} \le \frac{1}{2}\,|\nabla f|^2_{2,\omega} + \frac{1}{2}\,|\nabla u|^2_{2,\omega}.$$

Substituting this in (2.3) we get

$$\frac{k}{2}\frac{d}{dt}\,|u|^2_{2,\omega} + \eta\,|\nabla u|^2_{2,\omega} \le \frac{\mu}{2}\,|\nabla f|^2_{2,\omega} + \frac{\mu}{2}\,|\nabla u|^2_{2,\omega}.$$

Integrating side by side this latter inequality with respect to $t > 0$, and using the fact that $(u)_{|t=0} = u_0$ in ω, we obtain

$$\frac{k}{2}\,|u|^2_{2,\omega} + \eta\int_0^t |\nabla u|^2_{2,\omega} \le \frac{k}{2}\,|u_0|^2_{2,\omega} + \frac{\mu}{2}\int_0^t |\nabla f|^2_{2,\omega} + \frac{\mu}{2}\int_0^t |\nabla u|^2_{2,\omega}.$$

This implies (2.2) and the proof is finished. □

The following result will be needed as well, see [4].

Theorem 2.4. *Let $f \in L^2([0,T], L^2(\omega))$ and $(u_0, u^1) \in H_0^2(\omega) \times L^2(\omega)$. Then the problem*

$$(\mathcal{S}_1)\begin{cases} u_{tt} + \Delta^2 u = f & \text{in } \ \omega \times [0,T], \\ u = \partial_\nu u = 0 & \text{on } \ \Gamma \times [0,T], \\ u_{|t=0} = u_0, \ \ (u_t)_{|t=0} = \overline{u} & \text{in } \ \omega, \end{cases}$$

has a unique solution such that $(u, u_t) \in C^0([\,0,T\,], H_0^2(\omega) \times L^2(\omega))$.

For the sake of simplicity, we set

$$F_1(u,\phi) = \big[\phi + F_0, u\big]. \tag{2.4}$$

With this, the following result may be stated.

Proposition 2.5. *Let $u, v \in H_0^2(\omega)$ be with small norms and $F_0 \in H^4(\omega)$ be such that $\|F_0\|_{4,\omega} < \frac{1}{4}$. Let $\phi, \varphi \in H_0^2(\omega)$ be the solutions of $\Delta^2\phi = -[u,u]$ and $\Delta^2\varphi = -[v,v]$, respectively. Then there exists $0 < c_1 < 1$ such that*

$$\Big|\,[u,\phi] - [v,\varphi]\,\Big|_{2,\omega} \le c_1\|u - v\|$$

and

$$\Big|F_1(u,\phi) - F_1(v,\varphi)\Big|_{2,\omega} \le c_1\,\|u - v\|.$$

Proof. According to [4], we have

$$\Big|\,[u,\phi] - [v,\varphi]\,\Big|_{2,\omega} \le c_0\Big(\,\|u\|^2 + \|v\|^2\,\Big)\,\|u - v\|,$$

for some $c_0 > 0$ depending only on $mes(\omega)$. If we assume that $\|u\| \leq c$ and $\|v\| \leq c$, for some $c > 0$ enough small, then we get

$$\Big| [u, \phi] - [v, \varphi] \Big|_{2,\omega} \leq 2c_0 c^2 \|u - v\|.$$

Otherwise, by using (1.1) it is not hard to check that

$$\Big| \, [F_0, u - v] \, \Big|_{2,\omega} \leq 4 \|F_0\|_{4,\omega}.$$

It follows that we have

$$\begin{aligned} \Big| F_1(u, \phi) - F_1(v, \varphi) \Big|_{2,\omega} &\leq \Big| \, [\phi + F_0, u] - [\varphi + F_0, v] \, \Big|_{2,\omega}, \\ &\leq \Big| \, [\phi, u] - [\varphi, v] \, \Big|_{2,\omega} + \Big| \, [F_0, u - v] \, \Big|_{2,\omega}, \\ &\leq \Big(2c_0 c^2 + 4 \|F_0\|_{4,\omega} \Big) \|u - v\|. \end{aligned}$$

If $\|F_0\|_{4,\omega} < \frac{1}{4}$ and

$$0 < c < \sqrt{\frac{1 - 4\|F_0\|_{4,\omega}}{2c_0}},$$

then

$$0 < 2c_0 c^2 < c_1 =: 2c_0 c^2 + 4 \|F_0\|_{4,\omega} < 1.$$

In summary, the proposition is completely proved. □

Remark 2.2. According to Remark 2.1,(ii), the constant c_1 in Proposition 2.5 depends only on $mes(\omega)$ and $\|F_0\|_{4,\omega}$.

Now, we are in the position to state and establish the following main result.

Theorem 2.6. *Let $f \in L^2\big([0,T], L^2(\omega)\big)$, $\theta_0 \in H_0^1(\omega)$ and $(u_0, \overline{u}) \in H_0^2(\omega) \times L^2(\omega)$. The following problem:*

$$(\mathcal{S}) \begin{cases} u_{tt} + \Delta^2 u + \mu \Delta \theta = f & in \quad \omega \times [0,T], \\ k\theta_t - \eta \Delta \theta = \mu \Delta u_t & in \quad \omega \times [0,T], \\ u = \partial_\nu u = \theta = 0 & on \quad \Gamma \times [0,T], \\ (u)_{|t=0} = u_0, (u_t)_{|t=0} = \overline{u}, (\theta)_{|t=0} = \theta_0 & in \quad \omega, \end{cases}$$

has one and only one solution $(u, \theta) \in L^2([\,0,T\,], H_0^2(\omega) \times H_0^1(\omega))$ satisfying that $u_t \in L^2([\,0,T\,], L^2(\omega))$ and, for any $t \in [0,T]$,

$$\|u\|_0^2 + k\,|\theta|_{2,\omega}^2 + 2\eta \int_0^t |\nabla\theta|_{2,\omega}^2 \leq e^T \Big(\|u_0\|^2 + |\overline{u}|_{2,\omega}^2 + k\,|\theta_0|_{2,\omega}^2 + \int_0^T |f|_{2,\omega}^2 \Big). \tag{2.5}$$

Further, the so-called energy equality holds true:

$$\|u\|_0^2 + 2\eta \int_0^t |\nabla\theta|_{2,\omega}^2 + k\,|\theta|_{2,\omega}^2 = \|u_0\|^2 + |\overline{u}|_{2,\omega}^2 + k\,|\theta_0|_{2,\omega}^2 + 2\int_0^t \int_\omega f u_t. \tag{2.6}$$

Proof. To prove our result, we will study the problem $(\mathcal{S})$ by considering the nth-order approximate solution and its associate variational problem. We divide the proof into fourth steps.

Step 1: Let $\{\, e_k, e_k^1 \,\}$ be a basis in the space $H_0^2(\omega) \times H_0^1(\omega)$. The n-order Galerkin approximate solution to the problem $(\mathcal{S}_1)$, with clamped boundary conditions on the interval $[\,0,T\,]$, is a function $(u^n(t), \theta^n(t))$ of the form, [1, 6],

$$u^n(t) = \sum_{k=1}^n h_k(t) e_k \text{ and } \theta^n(t) = \sum_{k=1}^n l_k(t) e_k^1, \;\; n = 1, 2, 3, ...,$$

where $(h_k, l_k) \in W^{2,\infty}(0,T;\mathbb{R}) \times W^{1,\infty}(0,T;\mathbb{R})$. Let (u^n, ϕ^n, θ^n) be a solution of $(\mathcal{P}_0)$ and $(\mathcal{Q})$ corresponding to the initial data (u_{n0}, θ_{n0}) and u_{n1} such that the two following requirements are satisfied:

$$(u_{n0}, \theta_{n0}) \text{ converges to } (u_0, \theta_0) \text{ in } L^2([0,T], H_0^2(\omega) \times H_0^1(\omega)) \tag{2.7}$$

$$(u_{n1}) \text{ converges to } \overline{u} \text{ in } L^2([0,T], L^2(\omega)). \tag{2.8}$$

Now, let us consider the iterative problem $(\mathcal{S}_n)$ associated to the problem $(\mathcal{S})$ given by:

$$(\mathcal{S}_n) \begin{cases} u_{tt}^n + \Delta^2 u^n + \mu \Delta \theta^n = f & in \quad \omega \times [0,T], \\ k\theta_t^n - \eta \Delta \theta^n = \mu \Delta u_t^n & in \quad \omega \times [0,T], \\ u^n = \partial_\nu u^n = \theta^n = 0 & on \quad \Gamma \times [0,T], \\ (u^n)_{|t=0} = u_{n0}, (u_t^n)_{|t=0} = u_{n1}, (\theta^n)_{|t=0} = \theta_{n0} & in \quad \omega, \end{cases}$$

We now multiply the first equation of $(\mathcal{S}_n)$ by u_t^n and the second equation by θ^n and we then integrate both them over ω, with the help of some standard integral rules, we get

$$\begin{cases} \int_\omega u_{tt}^n u_t^n + \int_\omega \Delta u^n \Delta u_t^n + \mu \int_\omega \Delta \theta^n u_t^n = \int_\omega f u_t^n \\ k \int_\omega \theta_t^n \theta^n + \eta \int_\omega (\nabla \theta^n)^2 = \mu \int_\omega \Delta u_t^n \theta^n. \end{cases}$$

Since $(u_t^n, \theta^n) \in H_0^1(\omega) \times H_0^1(\omega)$ and $\int_\omega \Delta \theta^n u_t^n = \int_\omega \theta^n \Delta u_t^n$, the two last equations imply that

$$\begin{cases} \frac{1}{2}\frac{d}{dt}\Big(|u_t^n|_{2,\omega}^2 + \|u^n\|^2 \Big) + \mu \int_\omega \theta^n \Delta u_t^n = \int_\omega f u_t^n, \\ \frac{k}{2}\frac{d}{dt} |\theta^n|_{2,\omega}^2 + \eta |\nabla \theta^n|_{2,\omega}^2 = \mu \int_\omega \theta^n \Delta u_t^n. \end{cases}$$

From these two latter equalities we deduce that we have

$$\frac{1}{2}\frac{d}{dt}\Big(|u_t^n|_{2,\omega}^2 + \|u^n\|^2 \Big) + \frac{k}{2}\frac{d}{dt} |\theta^n|_{2,\omega}^2 + \eta |\nabla \theta^n|_{2,\omega}^2 = \int_\omega f u_t^n.$$

Integrating this latter equality over $[0,t]$, and using (2.1) with the fact that

$$u^n_{|t=0} = u_{n0},\ (u_t^n)_{|t=0} = u_{n1},\ \theta^n_{|t=0} = \theta_{n0},$$

we get

$$\frac{1}{2}\Big(\|u^n\|_0^2 + k |\theta^n|_{2,\omega}^2 \Big) + \eta \int_0^t |\nabla \theta^n|_{2,\omega}^2 = \frac{1}{2}\Big(|u_{n1}|_{2,\omega}^2 + \|u_{n0}\|^2 + k |\theta_{n0}|_{2,\omega}^2 \Big) + \int_0^t \int_\omega f u_t^n. \tag{2.9}$$

Let $s \in [0,T]$. By the Hölder inequality in $L^2(\omega)$, with (2.1), we have

$$\begin{aligned} \int_0^t \int_\omega f u_t^n &\le \int_0^t |f|_{2,\omega} |u_t^n|_{2,\omega} \\ &\le \frac{1}{2}\int_0^t |f|_{2,\omega}^2 + \frac{1}{2}\int_0^t |u_t^n|_{2,\omega}^2 \\ &\le \frac{1}{2}\int_0^T |f|_{2,\omega}^2 + \frac{1}{2}\int_0^t \Big(\|u^n\|_0^2 + k |\theta^n|_{2,\omega}^2 + 2\eta \int_0^s |\nabla \theta^n|_{2,\omega}^2 \Big). \end{aligned} \tag{2.10}$$

Combining (2.9) and (2.10), we have shown that

$$\|u^n\|_0^2 + k\,|\theta^n|_{2,\omega}^2 + 2\eta\int_0^t |\nabla\theta^n|_{2,\omega}^2 \le |u_{n1}|_{2,\omega}^2 + \|u_{n0}\|^2 + k\,|\theta_{n0}|_{2,\omega}^2 + \int_0^T |f|_{2,\omega}^2 + \int_0^t \Big(\|u^n\|_0^2 + k\,|\theta^n|_{2,\omega}^2 + 2\eta\int_0^s |\nabla\theta^n|_{2,\omega}^2 \Big). \tag{2.11}$$

Step 2: For $0 \le s \le t$, we put

$$I(s) = \|u^n\|_0^2 + k\,|\theta^n|_{2,\omega}^2 + 2\eta\int_0^s |\nabla\theta^n|_{2,\omega}^2 .$$

The inequality (2.11), yields

$$\begin{aligned} I(s) - \int_0^s I(z)dz &= \|u^n\|_0^2 + k\,|\theta^n|_{2,\omega}^2 + 2\eta\int_0^s |\nabla\theta^n|_{2,\omega}^2 - \int_0^s \Big(\|u^n\|_0^2 + k\,|\theta^n|_{2,\omega}^2 + 2\eta\int_0^z |\nabla\theta^n|_{2,\omega}^2 \Big)dz \\ &\le |u_{n1}|_{2,\omega}^2 + \|u_{n0}\|^2 + k\,|\theta_{n0}|_{2,\omega}^2 + \int_0^T |f|_{2,\omega}^2 . \end{aligned}$$

It follows that

$$\begin{aligned} \frac{d}{ds}\Big(e^{-s}\int_0^s I(z)dz\Big) &= e^{-s}\Big(I(s) - \int_0^s I(z)dz\Big) \\ &\le e^{-s}\Big(|u_{n1}|_{2,\omega}^2 + \|u_{n0}\|^2 + k\,|\theta_{n0}|_{2,\omega}^2 + \int_0^T |f|_{2,\omega}^2 \Big). \end{aligned} \tag{2.12}$$

Now, if we remark that

$$|u_{n1}|_{2,\omega}^2 + \|u_{n0}\|^2 + k\,|\theta_{n0}|_{2,\omega}^2 + \int_0^T |f|_{2,\omega}^2 = I(0) + \int_0^T |f|_{2,\omega}^2$$

does not depend on s, and we integrate (2.12) over $[0,t]$, then we get

$$\int_0^t \frac{d}{ds}\left(e^{-s}\int_0^s I(z)dz\right)ds \le \Big(\int_0^t e^{-s}ds\Big)\left(|u_{n1}|_{2,\omega}^2 + \|u_{n0}\|^2 + k\,|\theta_{n0}|_{2,\omega}^2 + \int_0^T |f|_{2,\omega}^2 \right),$$

from which we deduce

$$e^{-t}\int_0^t I(z)dz \le (1-e^{-t})\Big(|u_{n1}|_{2,\omega}^2 + \|u_{n0}\|^2 + k\,|\theta_{n0}|_{2,\omega}^2 + \int_0^T |f|_{2,\omega}^2 \Big).$$

It follows that

$$\begin{aligned} \int_0^t I(z)dz &\le \tfrac{(1-e^{-t})}{e^{-t}}\Big(|u_{n1}|_{2,\omega}^2 + \|u_{n0}\|^2 + k\,|\theta_{n0}|_{2,\omega}^2 + \int_0^T |f|_{2,\omega}^2 \Big) \\ &= (e^t-1)\Big(|u_{n1}|_{2,\omega}^2 + \|u_{n0}\|^2 + k\,|\theta_{n0}|_{2,\omega}^2 + \int_0^T |f|_{2,\omega}^2 \Big) \\ &\le (e^T-1)\Big(|u_{n1}|_{2,\omega}^2 + \|u_{n0}\|^2 + k\,|\theta_{n0}|_{2,\omega}^2 + \int_0^T |f|_{2,\omega}^2 \Big). \end{aligned}$$

This, with (2.11), yields

$$\begin{aligned} \|u^n\|_0^2 + k\,|\theta^n|_{2,\omega}^2 + 2\eta\int_0^t |\nabla\theta^n|_{2,\omega}^2 &\le \Big(|u_{n1}|_{2,\omega}^2 + \|u_{n0}\|^2 + k\,|\theta_{n0}|_{2,\omega}^2 + \int_0^T |f|_{2,\omega}^2 \Big) \\ &\quad + (e^T-1)\Big(|u_{n1}|_{2,\omega}^2 + \|u_{n0}\|^2 + k\,|\theta_{n0}|_{2,\omega}^2 + \int_0^T |f|_{2,\omega}^2 \Big), \end{aligned}$$

and therefore

$$\|u^n\|_0^2 + k\,|\theta^n|_{2,\omega}^2 + 2\eta\int_0^t |\nabla\theta^n|_{2,\omega}^2 \le e^T\Big(|u_{n1}|_{2,\omega}^2 + \|u_{n0}\|^2 + k\,|\theta_{n0}|_{2,\omega}^2 + \int_0^T |f|_{2,\omega}^2 \Big). \tag{2.13}$$

According to (2.7) and (2.8), the sequences (u_{n0}, θ_{n0}) and (u_{n1}) are, respectively, bounded in the spaces $L^2\big([0,T], H_0^2(\omega) \times H_0^1(\omega) \times L^2(\omega)\big)$ and $L^2\big([0,T], L^2(\omega) \times L^2(\omega)\big)$. This, with (2.13), imply that the sequences (u^n, θ^n) and (u_t^n) are also bounded, respectively, in $L^2\big([0,T], H_0^2(\omega) \times H_0^1(\omega) \times L^2(\omega)\big)$ and $L^2\big([0,T], L^2(\omega) \times L^2(\omega)\big)$. These latter Banach spaces are reflexive and therefore there exists a subsequence (u^{n_l}, θ^{n_l}) such that $(u^{n_l}, \theta^{n_l}) \rightharpoonup (u, \theta)$ weakly in $L^2\big([0,T], H_0^2(\omega) \times L^2(\omega)\big)$ and $\big((u^{n_l})_t, \nabla\theta^{n_l}\big) \rightharpoonup \big((u)_t, \nabla\theta\big)$ weakly in $L^2\big([0,T], L^2(\omega) \times L^2(\omega)\big)$.

Step 3: In this step, we will establish that (u, θ), previously defined, is a weak solution of the problem $(\mathcal{S})$, by following the same way as in [8].

Let $\varphi_j \in C^1(0,T)$ be such that $\varphi_j(T) = 0$ for any $1 \le j \le j_0$, and we set

$$\psi = \sum_{j=1}^{j_0} \psi_j \otimes e_j, \quad \varphi = \sum_{j=1}^{j_0} \varphi_j \otimes e_j^1.$$

According to $(\mathcal{S}_n)$, with further elementary manipulations and operations, we may infer that

$$-\int_0^T \int_\omega u_t^{n_l} \psi_t + \mu \int_0^T \int_\omega \nabla\theta^{n_l} \nabla\psi + \int_0^T \int_\omega \Delta u^{n_l} \Delta\psi = \int_0^T \int_\omega f\psi - \int_\omega u_{n_l 1}\psi(0) \tag{2.14}$$

and

$$\int_0^T \Big(-k \int_\omega \theta^{n_l} \varphi_t + \eta \int_\omega \nabla\theta^{n_l} \nabla\varphi - \mu \int_\omega \nabla u^{n_l} \nabla\varphi_t \Big) = -k \int_\omega \theta_{n_l 0}\varphi(0) - \mu \int_\omega \nabla u_{n_l 1} \nabla\varphi(0). \tag{2.15}$$

Letting $n_l \to +\infty$ in (2.14) and (2.15) we deduce that the two following equalities

$$-\int_0^T \int_\omega u_t \psi_t + \mu \int_0^T \int_\omega \nabla\theta \nabla\psi + \int_0^T \int_\omega \Delta u \Delta\psi = \int_0^T \int_\omega f\psi - \int_\omega \overline{u}\psi(0)$$

and

$$\int_0^T \Big(-k \int_\omega \theta \varphi_t + \eta \int_\omega \nabla\theta \nabla\varphi - \mu \int_\omega \nabla u \nabla\varphi_t \Big) = -k \int_\omega \theta_0 \varphi(0) - \mu \int_\omega \nabla\overline{u} \nabla\varphi(0),$$

hold true for all $\psi \in L^2([0,T], H_0^2(\omega))$, with $\psi_t \in L^2([0,T], H^1(\omega))$ and $\varphi \in L^2([0,T], H_0^1(\omega))$, with $\varphi_t \in L^2([0,T], L^2(\omega))$, such that $\psi(T) = \varphi(T) = 0$. This means that (u, θ) is a weak solution of the problem $(\mathcal{S})$.

Further, by analogous way as for proving (2.13), we may show that for all $t \in [0,T]$, we have the following inequality

$$\|u\|_0^2 + k\,|\theta|_{2,\omega}^2 + 2\eta \int_0^t |\nabla\theta|_{2,\omega}^2 \le e^T \Big(|\overline{u}|_{2,\omega}^2 + \|u_0\|^2 + k\,|\theta_0|_{2,\omega}^2 + \int_0^T |f|_{2,\omega}^2 \Big). \tag{2.16}$$

Step 4: We now show the uniqueness. Let (u_1, θ_1) and (u_2, θ_2) are two solutions of $(\mathcal{S})$. Then $(u_1 - u_2, \theta_1 - \theta_2)$ is a solution of the following problem

$$\begin{cases} (u_1 - u_2)_{tt} + \Delta^2(u_1 - u_2) + \mu\Delta(\theta_1 - \theta_2) = 0 & in \quad \omega \times [0,T], \\ k(\theta_1 - \theta_2)_t - \eta\Delta(\theta_1 - \theta_2) = \mu\Delta(u_1 - u_2)_t & in \quad \omega \times [0,T], \\ \theta_1 - \theta_2 = u_1 - u_2 = \partial_\nu(u_1 - u_2) = 0 & on \quad \Gamma \times [0,T], \\ (u_1 - u_2)_{|t=0} = 0, ((u_1 - u_2)_t)_{|t=0} = 0, (\theta_1 - \theta_2)_{|t=0} = 0 & in \quad \omega. \end{cases}$$

According to (2.16) we have

$$\begin{aligned} &\|u_1 - u_2\|_0 + k\,|\theta_1 - \theta_2|_{2,\omega}^2 + 2\eta \int_0^t |\nabla(\theta_1 - \theta_2)|_{2,\omega}^2 \\ &\qquad \le e^T \Big(|\overline{u_1} - \overline{u_2}|_{2,\omega}^2 + \|(u_1)_0 - (u_2)_0\|^2 + k\,|(\theta_1)_0 - (\theta_2)_0|_{2,\omega}^2 \Big). \end{aligned}$$

Then we deduce $u_1 = u_2$ and $\theta_1 = \theta_2$.

Finally, by (2.9) we have for all $n \ge 0$

$$\frac{1}{2}\Big(\|u^n\|_0 + k\,|\theta^n|_{2,\omega}^2 \Big) + \eta \int_0^t |\nabla\theta^n|_{2,\omega}^2 = \frac{1}{2}\Big(|u_{n1}|_{2,\omega}^2 + \|u_{n0}\|^2 + k\,|\theta_{n0}|_{2,\omega}^2 \Big) + \int_0^t \int_\omega f u_t^n,$$

from which by letting $n \to +\infty$ we get (2.6), so completing the proof. $\square$

3. Iterative approach and convergence result

To establish the existence and the uniqueness of a weak solution for $(\mathcal{P}_0)$, without rotational terms i.e. $\alpha = 0$, we will use the following iterative approach.

Let $n \geq 2$ and let $0 \neq u_1 \in H_0^2(\omega)$ be given. We define $\phi_{n-1} \in H_0^2(\omega)$ as the unique solution of $\Delta^2 \phi_{n-1} = -[u_{n-1}, u_{n-1}]$ and (u_n, θ_n) as the solution of the following problem :

$$(\mathcal{P}_n) \begin{cases} (u_n)_{tt} + \Delta^2 u_n = F(u_{n-1}, \phi_{n-1}, \theta_n) & in \quad \omega \times [0,T], \\ k(\theta_n)_t - \eta \Delta \theta_n = \mu \Delta (u_n)_t & in \quad \omega \times [0,T], \\ u_n = \partial_\nu u_n = \theta_n = 0 & on \quad \Gamma \times [0,T], \\ (u_n)_{|t=0} = u_0, ((u_n)_t)_{|t=0} = \overline{u}, (\theta_n)_{|t=0} = \theta_0 & in \quad \omega, \end{cases}$$

where we set $F(u, \phi, \theta) = F_1(u, \phi) - \mu \Delta \theta + p$ and, F_1 is defined by (2.4).

The main result of this section is recited in the following.

Theorem 3.1. *Let $p \in L^2(\omega)$, $(u_0, \overline{u}) \in H_0^2(\omega) \times L^2(\omega)$ and $\theta_0 \in H_0^1(\omega)$. We suppose that all the following norms*

$$\|F_0\|_{4,\omega}\,,\ |p|_{2,\omega}\,,\ \|u_0\|^2\,,\ |\overline{u}|_{2,\omega}^2 \ \text{and}\ \|\theta_0\|_{1,\omega}^2$$

are small enough, and $0 < \mu \leq 2\eta$. Then the problem $(\mathcal{P}_0)$, without rotational forces, has one and only one weak solution (u, ϕ, θ) in $L^2\big([0,T], H_0^2(\omega) \times H_0^2(\omega) \times H_0^1(\omega)\big)$ such that $u_t \in L^2\big([0,T], L^2(\omega)\big)$.

Proof. We divide it into four steps.

Step 1: Let us consider the problem $(\mathcal{P}_n)$ with $0 \neq u_1$ does not depend on t. For the sake of simplicity we use the notation

$$\|(u, \theta)\|_* = \|u\|_0^2 + k\,|\theta|_{2,\omega}^2 + 2\eta \int_0^t |\nabla \theta|_{2,\omega}^2\,,$$

where, $\|.\|_0$ is defined by (2.1). Let $c_0 > 0$ be the constant defined by Proposition 2.5. For $\|F_0\|_{4,\omega} < \frac{1}{4}$ we choose $c =: c(\|F_0\|_{4,\omega}, c_0, T) > 0$ such that

$$0 < 4c_0 c < 1, \ \text{ and } \ 0 < c < \sqrt{\frac{1 - 4\|F_0\|_{4,\omega}}{2c_0}}.$$

We also choose u_1 (independent on t) such that $0 < \|u_1\|_{2,\omega} < c < 1$.

By using an induction method, we will prove that the two following inequalities

$$\|u\|_0^2 =: \|u_n\|^2 + |(u_n)_t|_{2,\omega}^2 \leq \|u_1\|_{2,\omega}^2 \quad \text{and} \quad \|\phi_n\|_{2,\omega} \leq \|u_1\|_{2,\omega} \tag{3.1}$$

are satisfied for all $n \geq 1$ and $t \in [0,T]$.

Since u_1 does not depend on t, then we have

$$\|u_1\|_0^2 =: \|u_1\|^2 + |(u_1)_t|_{2,\omega}^2 = \|u_1\|_{2,\omega}^2\,.$$

Now, let ϕ_1 be the solution of $\Delta^2 \phi_1 = -[\, u_1, u_1\,]$. Theorem 2.1 tells us that there exists $c_0 > 0$ such that

$$\|\phi_1\|_{2,\omega} \leq c_0\,|[\, u_1, u_1\,]|_{1,\omega}\,,$$

and by using the same way as in the proof of Proposition 2.5, with $\|u_1\|_{2,\omega} < c$ and $0 < 4c_0 c < 1$, we may deduce that

$$\|\phi_1\|_{2,\omega} \leq 4c_0\,\|u_1\|_{2,\omega}^2 \leq 4c_0 c\,\|u_1\|_{2,\omega} \leq \|u_1\|_{2,\omega}\,.$$

Hence, the inequalities (3.1) are satisfied for $n = 1$.

Assume that for $k = 2, ..., n$ and $t \in [0,T]$, we have

$$\|u_k\|_0^2 \leq \|u_1\|_{2,\omega}^2 \quad \text{and} \quad \|\phi_k\|_{2,\omega} \leq \|u_1\|_{2,\omega}\,.$$

According to Theorem 2.1 and Proposition 2.5, with Remark 2.1 and Remark 2.2, we have

$$\|\phi_n\|_{2,\omega} \leq c_0 \, |[\, u_n, u_n \,]|_{1,\omega} \leq 4c_0 \, \|u_n\|^2 \leq 4c_0 c \, \|u_n\| \leq c_1 \, \|u_n\| \, .$$

Since u_{n+1} is a solution of $(\mathcal{P}_{n+1})$, Proposition 2.6, Proposition 2.5 and Theorem 2.1 imply that, there exist $c_1 > 0$, with

$$0 < c_1 =: 2c_0 c^2 + 4 \, \|F_0\|_{4,\omega} < 1, \tag{3.2}$$

such that

$$\begin{aligned}
\|(u_{n+1}, \theta_{n+1})\|_* &\leq e^T \Big(\|u_0\|^2 + k \, |\theta_0|^2_{2,\omega} + |\overline{u}|^2_{2,\omega} + \int_0^T |F_1(u_n, \phi_n) + p|^2_{2,\omega} \Big) \\
&\leq e^T \Big(\|u_0\|^2 + k \, |\theta_0|^2_{2,\omega} + |\overline{u}|^2_{2,\omega} + 2 \int_0^T \big(|F_1(u_n, \phi_n)|^2_{2,\omega} + |p|^2_{2,\omega} \big) \Big) \\
&\leq e^T \Big(\|u_0\|^2 + k \, |\theta_0|^2_{2,\omega} + |\overline{u}|^2_{2,\omega} + 2c_1^2 \int_0^T \|u_n\|^2 + 2T \, |p|^2_{2,\omega} \Big) \\
&\leq e^T \Big(\|u_0\|^2 + |\overline{u}|^2_{2,\omega} + k \, |\theta_0|^2_{2,\omega} + 2T c_1^2 \, \|u_1\|^4_{2,\omega} + 2T \, |p|^2_{2,\omega} \Big).
\end{aligned}$$

This, with the fact that $0 < c_1 < 1$, $\|u_1\| < 1$ and $c_1^2 \, \|u_1\|^4_{2,\omega} \leq c_1 \, \|u_1\|^2_{2,\omega}$, implies that

$$\|(u_{n+1}, \theta_{n+1})\|_* \leq e^T \Big(\|u_0\|^2 + |\overline{u}|^2_{2,\omega} + k \, |\theta_0|^2_{2,\omega} + 2T c_1 \, \|u_1\|^2_{2,\omega} + 2T \, |p|^2_{2,\omega} \Big).$$

If we choose $c > 0$ and $\|F_0\|_{4,\omega}$ small enough then c_1 defined by (3.2) is also small enough and so $0 < c_2 =: 2Te^T c_1 < 1$. We can then write

$$\|(u_{n+1}, \theta_{n+1})\|_* \leq e^T \Big(\|u_0\|^2 + |\overline{u}|^2_{2,\omega} + k \, |\theta_0|^2_{2,\omega} + 2T \, |p|^2_{2,\omega} \Big) + c_2 \, \|u_1\|^2_{2,\omega} \, . \tag{3.3}$$

In another part we can write

$$\|u_0\|^2 + |\overline{u}|^2_{2,\omega} + k \, |\theta_0|^2_{2,\omega} + 2T \, |p|^2_{2,\omega} \leq \frac{(1 - c_2)}{e^T} \, \|u_1\|^2_{2,\omega} \, , \tag{3.4}$$

since the left quantity of this inequality was assumed to be small enough. Otherwise, it is not hard to check that

$$\|u_{n+1}\|_0^2 =: \|u_{n+1}\|^2 + |(u_{n+1})_t|^2_{2,\omega} \leq \|(u_{n+1}, \theta_{n+1})\|_* \tag{3.5}$$

and

$$\|\phi_n\|_{2,\omega} \leq c_1 \, \|u_n\|_{2,\omega} \leq \|u_1\|_{2,\omega} \, .$$

According to (3.3), (3.5) and (3.5) we deduce that we have

$$\begin{aligned}
\|u_{n+1}\|_0^2 &\leq e^T \Big(\|u_0\|^2 + |\overline{u}|^2_{2,\omega} + k \, |\theta_0|^2_{2,\omega} + 2T \, |p|^2_{2,\omega} \Big) + c_2 \, \|u_1\|^2_{2,\omega} \\
&\leq e^T \frac{(1 - c_2)}{e^T} \, \|u_1\|^2_{2,\omega} + c_2 \, \|u_1\|^2_{2,\omega} = \|u_1\|^2_{2,\omega} \, .
\end{aligned}$$

Furthermore, we have

$$\|\phi_{n+1}\|_{2,\omega} \leq c_0 \, |[\, u_{n+1}, u_{n+1} \,]|_{1,\omega} \, ,$$

which with, $\|u_1\|_{2,\omega} < c$ and $0 < 4c_0 c < 1$, immediately yields

$$\|\phi_{n+1}\|_{2,\omega} \leq 4c_0 \, \|u_{n+1}\|^2 \leq 4c_0 \, \|u_1\|^2_{2,\omega} \leq 4c_0 c \, \|u_1\|_{2,\omega} \leq \|u_1\|_{2,\omega} \, .$$

In summary, we have shown that for all $n \geq 1$ and any $t \in [0, T]$ one has

$$\|u_n\|_0^2 \leq \|u_1\|^2_{2,\omega} \quad \text{and} \quad \|\phi_n\|_{2,\omega} \leq \|u_1\|_{2,\omega} \, .$$

Moreover we have

$$k \, |\theta_n|^2_{2,\omega} + 2\eta \int_0^t |\nabla \theta_n|^2_{2,\omega} \leq \|(u_n, \theta_n)\|_* \leq \|u_1\|^2_{2,\omega} \, .$$

Step 2: For $n \geq 2$, let (u_n, θ_n) be a solution of $(\mathcal{P}_n)$. Let $2 \leq m \leq n$. It is not hard to see that $\theta^{nm} =: \theta_n - \theta_m$ and $u^{nm} =: u_n - u_m$ satisfy the following:

$$\begin{cases} u_{tt}^{nm} + \Delta^2 u^{nm} + \mu\Delta\theta^{nm} = F_1(u_{n-1}, \phi_{n-1}) - F_1(u_{m-1}, \phi_{m-1}) & in \quad \omega \times [0,T]\,, \\ k\theta_t^{nm} - \eta\Delta\theta^{nm} = \mu\Delta(u^{nm})_t & in \quad \omega \times [0,T]\,, \\ u^{nm} = \theta^{nm} = \partial_\nu u^{nm} = 0 & on \quad \Gamma \times [0,T]\,, \\ (u^{nm})_{|t=0} = ((u^{nm})_t)_{|t=0} = ((\theta^{nm})_t)_{|t=0} = 0 & in \quad \omega. \end{cases}$$

According to Proposition 2.5 and Theorem 2.1 we deduce that

$$\|\phi_{n-1} - \phi_{m-1}\|_{2,\omega} \leq 4c_0 c \|u_{n-1} - u_{m-1}\|\,.$$

Using Proposition 2.6 and Proposition 2.5 again we have

$$\begin{aligned} \|(u_n - u_m, \theta_n - \theta_m)\|_* &\leq e^T \int_0^T |F_1(u_{n-1}, \phi_{n-1}) - F_1(u_{m-1}, \phi_{m-1})|_{2,\omega}^2 \\ &\leq c_1 e^T \int_0^t \|u_{n-1} - u_{m-1}\|^2\,. \end{aligned}$$

This, with $0 < c_3 = e^T c_1 < \frac{1}{T}$, yields

$$\begin{aligned} \|(u_n - u_m, \theta_n - \theta_m)\|_* \leq c_3 \int_0^t &\|(u_{n-1} - u_{m-1}, \theta_{n-1} - \theta_{m-1})\|_* \\ &\leq (c_3)^{m-2} \int_0^t \dots \int_0^t \|(u_{n-m+2} - u_1, \theta_{n-m-2} - \theta_1\|_* \\ &\leq (c_3)^{m-2} \int_0^t \dots \int_0^t \sum_{k=0}^{n-m+1} (c_3)^k \int_0^t \dots \int_0^t \|(u_2 - u_1 \theta_2 - \theta_1)\|_* \\ &\leq (c_3)^{m-2} \int_0^t \dots \int_0^t \sum_{k=0}^{n-m+1} (c_3)^k \int_0^t \dots \int_0^t \Big(\|(u_2, \theta_2)\|_* + \|(u_1, \theta_1)\|_* \Big) \\ &\leq (c_3 T)^{m-2} \sum_{k=0}^{n-m+1} (c_3 T)^k \big(4 \|u_1\|_{2,\omega}^2\big). \end{aligned}$$

It follows that we have

$$\int_0^T \|(u_n - u_m, \theta_n - \theta_m)\|_* \leq T(c_3 T)^{m-2} \sum_{k=0}^{n-m+1} (c_3 T)^k (4 \|u_1\|_{2,\omega}^2),$$

and so we infer that

$$\|\phi_n - \phi_m\|_{2,\omega} \leq 4c_0 c \|u_n - u_m\|\,.$$

The sequence $(u_n, \phi_{n-1})_{n\geq 2}$ is a Cauchy sequence in the Banach space $L^2\big([0,T]\,, H_0^2(\omega) \times H_0^2(\omega)\big)$. It follows that (u_n, ϕ_{n-1}) converges to (u, ϕ) in $L^2\big([0,T]\,, H_0^2(\omega) \times H_0^2(\omega)\big)$ and $(u_n)_t$ converges to $(u)_t$ in $L^2\big([0,T]\,, L^2(\omega)\big)$.

Step 3: Now, let us rewrite that $\theta^{nm} =: \theta_n - \theta_m$ is a solution of the following problem

$$\begin{cases} k\theta_t^{nm} - \eta\Delta\theta^{nm} = \mu\Delta(u^{nm})_t & in \quad \omega \times [0,T]\,, \\ \theta^{nm} = 0, & on \quad \Gamma \times [0,T]\,, \\ ((\theta^{nm})_t)_{|t=0} = 0 & in \quad \omega. \end{cases}$$

Using Theorem 2.2, Proposition 2.3 and inequality (2.2), we have

$$k\,|\theta_{n-1} - \theta_{m-1}|_{2,\omega}^2 + (2\eta - \mu) \int_0^t |\nabla(\theta_{n-1} - \theta_{m-1})|_{2,\omega}^2 \leq \mu \int_0^t (|\nabla(u_{n-1} - u_{m-1})_t|_{2,\omega})^2.$$

We deduce that (θ_n) is a Cauchy sequence in the Banach space $L^2([0,T], H_0^1(\omega))$ and so (θ_n) converges to θ in $L^2([0,T], H_0^1(\omega))$. Otherwise, by Proposition 2.5 we may deduce that $F_1(u_{n-1}, \phi_{n-1})$ converges to $F_1(u,\phi)$ in $(L^2(\omega))^2$.

Thanks to Theorem 2.4, we have $(u_n, (u_n)_t) \in C^0([0,T], H_0^2(\omega) \times L^2(\omega))$ with $(u_n)_{|t=0} = u_0$ and $((u_n)_t)_{|t=0} = u_1$, and so $(u)_{|t=0} = u_0$, $((u)_t)_{|t=0} = \overline{u}$.

For showing that (u,θ) is a weak solution of the problem $(\mathcal{P}_0)$, we follow the same way as in [8]. Let $\{e_j, e_j^1\}$ be a basis in the space $H_0^2(\omega) \times H_0^1(\omega)$ and let $\varphi_j \in C^1(0,T)$, $1 \le j \le j_0$, be such that $\varphi_j(T) = 0$. We set

$$\psi = \sum_{j=1}^{j_0} \varphi_j \otimes e_j, \quad \varphi = \sum_{j=1}^{j_0} \varphi_j \otimes e_j^1.$$

As in the proof of Theorem 2.6 (Step 3), we have

$$-\int_0^T \int_\omega (u_n)_t \psi_t + \mu \int_0^T \int_\omega \nabla\theta_n \nabla\psi + \int_0^T \int_\omega \Delta u_n \Delta\psi = \int_0^T \int_\omega (F_1(u_{n-1}, \phi_{n-1}) + p)\psi - \int_\omega u_1 \psi(0) \quad (3.6)$$

and

$$\int_0^T \Big(-k \int_\omega \theta_n \varphi_t + \eta \int_\omega \nabla\theta_n \nabla\varphi - \mu \int_\omega \nabla u_n \nabla\varphi_t \Big) = -k \int_\omega \theta_0 \varphi(0) - \mu \int_\omega \nabla u_1 \nabla\varphi(0). \quad (3.7)$$

Letting $n \to \infty$ in (3.6) and (3.7) we deduce that, for all $\psi \in L^2([0,T], H_0^2(\omega))$, $\psi_t \in L^2([0,T], L^2(\omega))$, $\varphi \in L^2([0,T], H_0^1(\omega))$ and $\varphi_t \in L^2([0,T], L^2(\omega))$ with $\psi(T) = \varphi(T) = 0$, we have

$$-\int_0^T \int_\omega u_t \psi_t + \mu \int_0^T \int_\omega \nabla\theta \nabla\psi + \int_0^T \int_\omega \Delta u \Delta\psi = \int_0^T \int_\omega (F_1(u,\phi) + p)\psi - \int_\omega u_1 \psi(0),$$

and

$$\int_0^T \Big(-k \int_\omega \theta \varphi_t + \eta \int_\omega \nabla\theta \nabla\varphi - \mu \int_\omega \nabla u \nabla\varphi_t \Big) = -k \int_\omega \theta_0 \varphi(0) - \mu \int_\omega \nabla u_1 \nabla\varphi(0).$$

Hence, (u,θ) is a weak solution of the problem $(\mathcal{S}_1)$ with $f = F_1(u,\phi) + p$.

Summarizing, we have proved that (u,ϕ,θ) is a weak solution of the thermoelastic von-Karman evolution.

Step 4: We now prove the uniqueness. Assume that there exist two weak solutions (u^1, ϕ^1, θ^1) and (u^2, ϕ^2, θ^2) in $L^2([0,T], H_0^1(\omega) \times H_0^2(\omega) \times H_0^1(\omega))$ such that, for some $c > 0$ small enough, we have $\|u^1\| \le c$ and $\|u^2\| \le c$. Then $u^{12} =: u^1 - u^2$ and $\theta^{12} =: \theta^1 - \theta^2$ satisfy the following problem

$$(\mathcal{P}_3) \begin{cases} u_{tt}^{12} + \Delta^2 u^{12} = F(u^1, \phi^1, \theta^1) - F(u^2, \phi^2, \theta^2) & in \quad \omega \times [0,T], \\ k\theta_t^{12} - \eta\Delta\theta^{12} = \mu\Delta u_t^{12} & in \quad \omega \times [0,T], \\ u^{12} = \partial_\nu u^{12} = \theta^{12} = 0 & on \quad \Gamma \times [0,T], \\ u^{12}(x_1,x_2,0) = 0, (u^{12})_t(x_1,x_2,0) = 0 & in \quad \omega. \\ (\theta^{12})_t(x_1,x_2,0) = 0 & in \quad \omega. \end{cases}$$

It follows that $(u^1 - u^2, \theta^1 - \theta^2)$ is a solution of the problem $(\mathcal{P}_3)$. Proposition 2.5, Proposition 2.6 and Theorem 2.1 ensure that there exists $c_1 > 0$ such that

$$\begin{aligned} \left\|(u^1 - u^2, \theta^1 - \theta^2)\right\|_* &\le e^T \int_0^T \left|F_1(u^1,\phi^1) - F_1(u^2,\phi^2)\right|_{2,\omega}^2 \\ &\le c_1 e^T \int_0^T \left\|u^1 - u^2\right\|^2 \le c_1 e^T \int_0^T \left\|(u^1 - u^2, \theta^1 - \theta^2)\right\|_*. \end{aligned}$$

Since c_1 is small enough so $0 < c_3 = e^T c_1 < \frac{1}{T}$ and therefore

$$\int_0^T \left\|(u^1 - u^2, \theta^1 - \theta^2)\right\|_* \leq T c_3 \int_0^T \left\|(u^1 - u^2, \theta^1 - \theta^2)\right\|_*,$$

which, with $0 < Tc_3 < 1$, immediately yields $u^1 = u^2$, $\theta^1 = \theta^2$ and then $\phi^1 = \phi^2$.

In conclusion, the dynamic von-Karman equations coupled with thermal dissipation, without rotational inertia, has one and only one weak solution (u, ϕ, θ) in $L^2\Big([0,T], H_0^2(\omega) \times H_0^2(\omega) \times H_0^1(\omega)\Big)$. The proof of the theorem is finished. □

We end this section by stating the following result.

Proposition 3.2. *Let $(u, \phi, \theta) \in L^2\Big([0,T], H_0^2(\omega) \times H_0^2(\omega) \times H_0^1(\omega)\Big)$ be the unique solution of $(\mathcal{P}_0)$. Let $\phi_0 \in H_0^2(\omega)$ be the unique solution of $\Delta^2\phi_0 = -[u_0, u_0]$. Then the following equality*

$$\widetilde{E}\big(u(t), u_t(t), \phi\big) + k\,|\theta|_{2,\omega}^2 + 2\eta \int_0^t |\nabla\theta_t|_{2,\omega}^2 = \widetilde{E}_1(u_0, \overline{u}, \phi_0) + k\,|\theta_0|_{2,\omega}^2,$$

holds true for any $t \in [0,T]$, where we set

$$\widetilde{E}\big(u(t), u_t(t), \phi\big) =: |u_t|_{2,\omega}^2 + \|u\|^2 + \frac{1}{2}\int_\omega \Big(|\Delta\phi|^2 - 2\,[u, F_0]\,u - 4pu\Big)$$

and

$$\widetilde{E}_1(u_0, \overline{u}, \phi_0) =: |\overline{u}|_{2,\omega}^2 + \|u_0\|^2 + \frac{1}{2}\int_\omega \Big(|\Delta\phi_0|^2 - 2\,[u_0, F_0]\,u_0 - 4pu_0\Big).$$

Proof. By virtue of Theorem 2.6, for any $t \in [0,T]$ we have the following equality

$$\begin{aligned}\|u\|_0^2 + 2\eta\int_0^t |\nabla\theta|_{2,\omega}^2 + k\,|\theta|_{2,\omega}^2 &= \|u_0\|^2 + |\overline{u}|_{2,\omega}^2 + k\,|\theta_0|_{2,\omega}^2 \\ &\quad + 2\int_0^t\int_\omega F_1(u,\phi)u_t + 2\int_0^t\int_\omega p(x_1,x_2)u_t.\end{aligned} \tag{3.8}$$

First, let us observe that we have

$$\int_0^t\int_\omega p(x_1,x_2)u_t = \int_\omega p(x_1,x_2)u(t) - \int_\omega p(x_1,x_2)u_0.$$

Otherwise, with $\Delta^2\phi = [u,u]$, we have

$$\begin{aligned}\int_0^t\int_\omega F_1(u,\phi)u_t &= \int_0^t\int_\omega [u, \phi + F_0]\,u_t = \int_0^t\int_\omega [u,\phi]\,u_t + \int_0^t\int_\omega [u, F_0]\,u_t, \\ &= \frac{1}{2}\int_0^t\int_\omega \frac{d}{dt}([u,u]\,\phi) + \frac{1}{2}\int_0^t\int_\omega \frac{d}{dt}([u,F_0]\,u), \\ &= -\frac{1}{4}\int_\omega |\Delta\phi|^2 + \frac{1}{4}\int_\omega |\Delta\phi_0|^2 + \frac{1}{2}\int_\omega [u,u]\,F_0 - \frac{1}{2}\int_\omega [u_0,u_0]\,F_0.\end{aligned}$$

Substituting these into (3.8), we get

$$\begin{aligned}&\|u\|_0^2 + 2\eta\int_0^t |\nabla\theta|_{2,\omega}^2 + k\,|\theta|_{2,\omega}^2 + \frac{1}{2}\int_\omega |\Delta\phi|^2 - \int_\omega [u,u]\,F_0 - 2\int_\omega p(x_1,x_2)u \\ &= \|u_0\|^2 + |\overline{u}|_{2,\omega}^2 + k\,|\theta_0|_{2,\omega}^2 + \frac{1}{2}\int_\omega |\Delta\phi_0|^2 - \int_\omega [u_0,u_0]\,F_0 - 2\int_0^t\int_\omega p(x_1,x_2)u_0.\end{aligned}$$

The proof of the proposition is finished. □

4. **Numerical application**

In this section we will investigate a numerical resolution of our initial problem in the aim to illustrate the previous study.

4.1. **Preliminaries.** We take $\omega =]0,1[\times]0,1[\subset \mathbb{R}^2$ and let $T>0$. For solving numerically the problem $(\mathcal{P}_0)$, we use the finite difference method by considering a uniform mesh of width h. For this, let us denote by ω_h the set of all mesh points inside the domain ω with internal points: $(x_i, y_j) = (ih, jh),\ i,j = 1,...N-1,\ h = 1/(N+1),\ \Delta t = 1/T$. Otherwise, we denote by $\overline{\omega}_h$ the set of boundary mesh points and by u_h the finite-difference that approximates u. In [2], the author discussed a numerical study about the convergence and stability for the conservative finite difference schemes related to the dynamic von Karman plate equations.

In the aim to approximate numerically the unique weak solution of our problem, we use the discrete model of von-Karman evolution presented in [2, 9]:

$$(*)\begin{cases}\delta_t^2 u_{ij}^n + \mu(\delta_x^2+\delta_y^2)\theta_{ij}^n + \Delta_h^2 u_{ij}^n = [\ u_{ij}^n\ v_{ij}^n + F_{ij}\ ] + p_{ij} & \text{in } \ \omega_h,\\ k\delta_t\theta_{ij}^n - \eta(\delta_x^2+\delta_y^2)\theta_{ij}^n - \mu\delta_t(\delta_x^2+\delta_y^2)u_{ij}^n = 0 & \text{in } \ \omega_h,\\ \Delta_h^2 v_{ij}^n = -[\ u_{ij}^n\ u_{ij}^n\ ] & \text{in } \ \omega_h,\\ u_{ij}^0 = (\varphi_0)_{ij},\ \delta_t u_{ij}^0 = (\varphi_1)_{ij},\ \theta_{ij}^0 = (\theta_0)_{ij} & \text{in } \ \omega_h,\\ u_{ij}^n = v_{ij}^n = \theta_{ij}^n = 0 & \text{on } \ \overline{\omega_h},\\ \partial_\nu u_{ij}^n = \partial_\nu v_{ij}^n = 0 & \text{on } \ \overline{\omega_h},\end{cases}$$

with the following discrete differential operators:

$$\begin{aligned}
\delta_t^2 u_{ij}^n &= \frac{u_{ij}^{n+1} - 2u_{ij}^n + u_{ij}^{n-1}}{(\Delta t)^2},\\
\delta_t u_{ij}^n &= \frac{u_{ij}^{n+1} - u_{ij}^n}{\Delta t},\\
\Delta_h^2 u_{ij}^n &= h^{-4}\,[\ u_{ij-2} + u_{ij+2} + u_{i-2j} + u_{i+2j} - 8(u_{ij-1} + u_{ij+1} + u_{i-1j} + u_{i+1j})\\
&\qquad + 2(u_{i-1j-1} + u_{i-1j+1} + u_{i+1j-1} + u_{i+1j+1}) - 20u_{ij}\ ],\\
\delta_x^2 u_{ij}^n &= \frac{u_{i+1j}^n - 2u_{ij}^n + u_{i-1j}^n}{h^2},\\
\delta_y^2 u_{ij}^n &= \frac{u_{ij+1}^n - 2u_{ij}^n + u_{ij-1}^n}{h^2},\\
\delta_{xy}^2 u_{ij}^n &= \frac{u_{i+1j+1}^n - u_{i+1j-1}^n - u_{i-1j+1}^n + u_{i-1j-1}^n}{(2h)^2},\\
[\ u_{ij}^n,\ v_{ij}^n\ ] &= \delta_x^2 u_{ij}^n \delta_y^2 v_{ij}^n - 2\delta_{xy}^2 u_{ij}^n \delta_{xy}^2 v_{ij}^n + \delta_y^2 u_{ij}^n \delta_x^2 v_{ij}^n.
\end{aligned}$$

Summarizing the above, we have in fact transformed the above problem to the numerical resolution into 2 steps, as itemized below:

Step 1: We first utilize the numerical procedure of 13-point formula of finite difference discussed in [6]. This method is used for illustrating the weak solution of the next problem:

$$\begin{cases}\Delta^2 v = f_1 & in \quad \omega,\\ v = g_1 & on \quad \Gamma,\\ \partial_\nu v = g_2 & on \quad \Gamma.\end{cases}$$

Step 2: Afterwards, we adopt the discrete model of the von-Karman evolution $(*)$ for approaching the solution of the thermoelastic model coupled with the dynamic von-Karman evolution.

4.2. **Non-coupled approach.** In [6], the author discussed a numerical analysis of finite-difference method about the numerical resolution of the Biharmonic equation. Such method, which is known as the non-coupled method of 13-point, may be summarized by the following result:

Proposition 4.1. *The* 13*-point approximation of the Biharmonic equation for approaching the unique solution* v *of the problem* (P) *is defined by:*

$$L_h v_{ij} = h^{-4}\Big\{v_{ij-2} + v_{ij+2} + v_{i-2j} + v_{i+2j} - 8\big(v_{ij-1} + v_{ij+1} + v_{i-1j} + v_{i+1j}\big) + 2\big(v_{i-1j-1} + v_{i-1j+1} + v_{i+1j-1} + v_{i+1j+1}\big) - 20v_{ij}\Big\} = f_1\big(x_i, y_j\big),$$

for $i, j = 1, 2, ..., N-1$*, where we set* $v_{ij} = v(x_i, y_j)$*.*

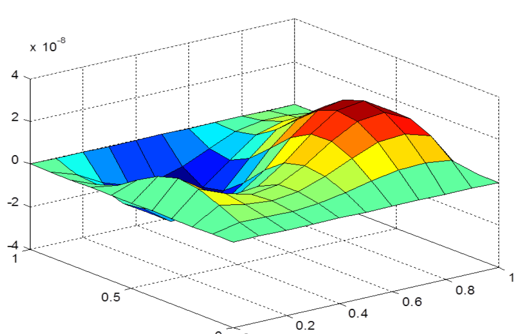
When the mesh point (x_i, y_j) is adjacent to the boundary $\overline{\omega}_h$, then the undefined values of v_h are conventionally calculated by the following approximation of $\partial_\nu v$:

$$\begin{aligned}
v_{i-2,j} &= \frac{1}{2}v_{i+1,j} - v_{ij} + \frac{3}{2}v_{i-1,j} - h(\partial_x v)_{i-1,j},\\
v_{i,j-2} &= \frac{1}{2}v_{i,j+1} - v_{ij} + \frac{3}{2}v_{i,j-1} - h(\partial_y v)_{i,j-1},\\
v_{i+2,j} &= \frac{1}{2}v_{i+1,j} - v_{ij} + \frac{3}{2}v_{i-1,j} - h(\partial_x v)_{i+1,j},\\
v_{i,j+2} &= \frac{1}{2}v_{i,j+1} - v_{ij} + \frac{3}{2}v_{i,j-1} - h(\partial_y v)_{i,j+1}.
\end{aligned}$$

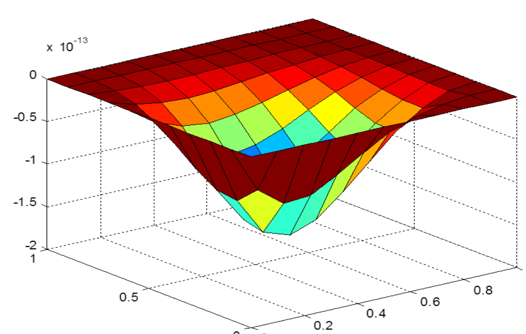
The following example illustrates the previous theoretical study.

Example 1. Let us consider the following body forces:

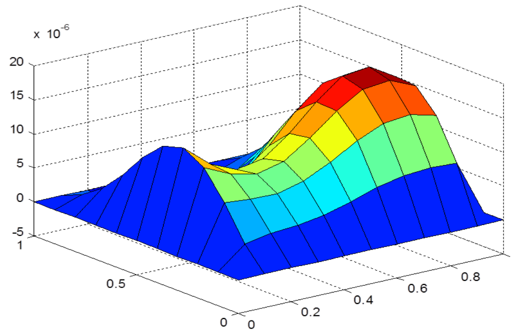
$$\begin{aligned}
p(x,y) &= 10^{-2}(x-1)^2(y-1)^2(e^{-x^2-y^2}),\\
u_1 &= 10^{-3}(y^3(x-4)^2)(e^{-x^2-y^2}),\\
\theta_0 &= 10x^2(x-y-1)(e^{-(x-1)^2-(y-1)^2}),\\
u_0 &= 10^{-3}x^2(y-3)^3(e^{-x^2-y^2}),\\
F_0 &= 10^{-3}x(e^{-x^2-y^2})\sin^2(\pi x).
\end{aligned}$$



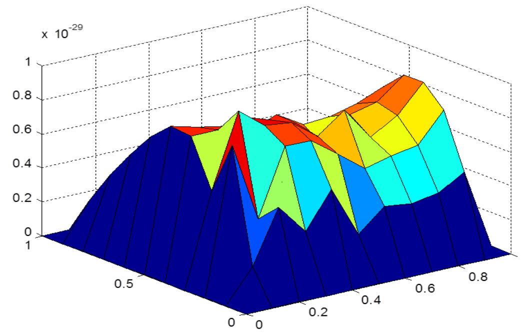
Displacement of plate, $T = 0.1s$



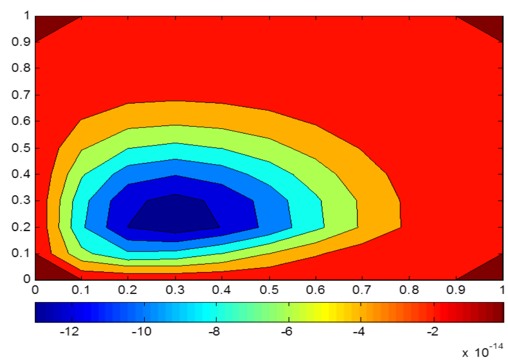
Displacement of plate, $T = 50s$



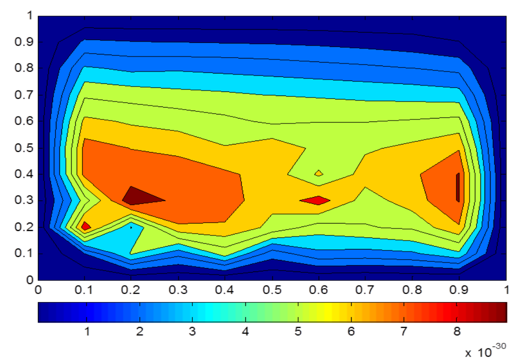
Thermal value, $T = 0.1s$



Thermal value, $T = 50s$



Contour displacement value $T = 50s$



Contour thermal value $T = 50s$

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