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ON A NEW GENERALIZED TSALLIS RELATIVE OPERATOR ENTROPY

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Abstract. In this paper, we present a generalization of Tsallis relative operator entropy defined for positive operators and we investigate some related properties. Some inequalities involving the generalized Tsallis relative operator entropy are pointed out as well.

1. Introduction

The relative entropy plays an important role in many areas. In the classical information theory, it serves as a notion to measure the difference between two probability distributions. For two discrete probability distributions $P = (p_1, p_2, \dots, p_n)$ and $Q = (q_1, q_2, \dots, q_n)$, the relative entropy $H(P|Q)$ is defined as follows [12]

$$H(P|Q) = \sum_{i=1}^{i=n} p_i \log \frac{p_i}{q_i}.$$

For $Q = (\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n})$, we get

$$H(P|Q) = \log n - S_s(P),$$

where $S_s(P) := -\sum_{i=1}^{i=n} p_i \log p_i$ stands for the famous Shannon entropy. It represents a fundamental tool that caused an enormous change in studying many fields like physical quantum systems and modern communication.

In [11], the authors provided a generalization for the entropy concept redefined by Tsallis in [17]. Namely, for a discrete probability distribution P of a random variable the Tsallis entropy is defined as follows

$$T_q(P) \equiv -\sum_{i=1}^{i=n} p_i^q \log_q (p_i),$$

where $\log_q$ refers to the q-logarithmic function defined by the following formula

$$\log_q(x) = \frac{x^{1-q} - 1}{1 - q},$$

for any nonnegative real numbers x and $q \neq 1$.

Given the growing diffusion of the use of entropy, many studies have been interested in generalizing this notion to positive operators. To give an overview, let us start by recalling some notions and fixing some notations that will be used in the rest of this article.

Let H be a complex Hilbert space endowed with an inner product $\langle ., . \rangle$. $\mathcal{B}(H)$ will stand for the C^*-algebra of all bounded linear operators acting on H. An operator $A \in \mathcal{B}(H)$ is called positive, in brief $A \geq 0$, if A is selfadjoint and $\langle Ax, x \rangle \geq 0$ for all $x \in H$. We denote by $\mathcal{B}^+(H)$ the closed cone of all

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positive operators in $\mathcal{B}(H)$ and $\mathcal{B}^{+*}(H)$ the open cone of all positive invertible operators in $\mathcal{B}(H)$. For $A, B \in \mathcal{B}(H)$ selfadjoint, we set $A \leq B$ to mean that $B - A \in \mathcal{B}^{+}(H)$.
The relative operator entropy $S(A|B)$ was introduced by Fujii and Kamei in [3] by the following expression

$$S(A|B) = A^{\frac{1}{2}}\left(\log A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)A^{\frac{1}{2}}.$$

Yanagi et al. provided in [19] a parameterized generalization of the operator $S(A|B)$. The authors introduced the Tsallis relative operator entropy $T_p(A \mid B)$ for two operators $A, B \in \mathcal{B}^{+*}(H)$ and $p \in (0,1]$ as follows

$$T_p(A \mid B) = \frac{A\sharp_p\, B - A}{p}. \tag{1.1}$$

The generalization is to be understood by the following result

$$\lim_{p\to 0} T_p(A|B) = S(A|B).$$

In (1.1), $A\sharp_p\, B$ stands for the well known as p-power mean defined as follows

$$A\sharp_p B := A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^p A^{\frac{1}{2}}.$$

We recall also the operator means p-weighted arithmetic and p-weighted harmonic, defined for any $A, B \in \mathcal{B}^{+*}(H)$ respectively by [2, 13, 16]

$$A\nabla_p\, B := (1-p)A + p\, B \text{ and } A!_p\, B := \left\{(1-p)A^{-1} + pB^{-1}\right\}^{-1},$$

where $p \in [0,1]$. When $p = \frac{1}{2}$ the subscript p will be omitted from the above notations.
In quantum systems, $T_p(A|B)$ is an operator variant of the Tsallis entropy [1, 9]. Many nice properties of $T_p(A|B)$ can be found for example in [4, 5, 6, 7].
In [16], Raïssouli et al provided an integral representation of $T_p(A|B)$ for $p \in (0,1]$ as follows

$$T_p(A|B) = \frac{\sin p\,\pi}{p\,\pi}\int_0^1 \left(\frac{t}{1-t}\right)^p \left(\frac{A!_t\, B - A}{t}\right)dt. \tag{1.2}$$

The definition of p-weighted geometric operator mean $A\natural_p\, B$ for any $p \in \mathbb{R}$ by the following formula [10]

$$A\natural_p B := A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^p A^{\frac{1}{2}},$$

allows to generalize (1.1) for any $p \in \mathbb{R}^*$ by setting,

$$T_p(A|B) = \frac{A\natural_p\, B - A}{p}.$$

In [9], the authors proved that for any unitary operator $U \in \mathcal{B}^{+*}(H)$,

$$T_p(U^*AU|U^*BU) = U^*T_p(A|B)U. \tag{1.3}$$

Fujii et Kamei [3] and [20] provided the following upper and lower bounds of Tsallis relative operator entropy,

$$A - AB^{-1}A \leq T_{-p}(A|B) \leq S(A|B) \leq T_p(A|B) \leq B - A.$$

By using the Hermite-Hadamard's inequality, Moradi et al. established in [15] for any $p \in (0,1]$ the following results,

$$A^{\frac{1}{2}}\left(\frac{A^{-\frac{1}{2}}BA^{-\frac{1}{2}}+I_H}{2}\right)^{p-1}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}-I_H\right)A^{\frac{1}{2}} \leqslant T_p(A \mid B)$$

$$\leqslant \frac{1}{2}\left(A\sharp_p B - A\natural_{p-1}B + B - A\right). \quad (1.4)$$

From (1.4), the following inequalities can be deduced [9, 20]

$$A - AB^{-1}A \leq T_p(A|B) \leq B - A.$$

The present work is centered on generalizing the operator $T_p(A|B)$ and on investigating some related properties. Some facets of our generalization are also highlighted.

The rest of this paper is organized as follows. In Section 2, we present a generalization for the Tsallis relative operator entropy and we determine some of its properties. In section 3, after establishing some Hermite-Hadamard type inequalities, we provide some inequalities which involve the generalized Tsallis relative operator.

2. Generalized Tsallis Relative Operator Entropy

We begin this section by providing the definition of a generalized Tsallis relative operator entropy. Then, we will deal with establishing some related properties.

Definition 2.1. Let $A, B \in \mathcal{B}^{+*}(H)$, $p \in \mathbb{R}^*$ and $0 \leq \nu, \mu \leq 1$. We define the generalized Tsallis relative operator entropy $\mathbb{T}_{(p,\mu,\nu)}(A|B)$ by

$$\mathbb{T}_{(p,\mu,\nu)}(A|B) = \frac{A\natural_p(A\nabla_\mu B) - A\natural_p(A\nabla_\nu B)}{p}. \quad (2.1)$$

For the particular case $\mu = 1$ and $\nu = 0$, we get $\mathbb{T}_{(p,1,0)}(A \mid B) = T_p(A \mid B)$. This confirms that $\mathbb{T}_{(p,\mu,\nu)}(A|B)$ represents effectively a generalization of $T_p(A|B)$.

Our first result concerning the generalized Tsallis relative operator entropy $\mathbb{T}_{(p,\mu,\nu)}(A|B)$ reads as follows.

Proposition 2.1. *Let $A, B \in \mathcal{B}^{+*}(H)$, $p \in \mathbb{R}^*$ and $0 \leq \nu, \mu \leq 1$. We have*

$$\lim_{p\to 0} \mathbb{T}_{(p,\mu,\nu)}(A|B) = S\left(A|A\nabla_\mu B\right) - S\left(A|A\nabla_\nu B\right).$$

Proof. Noticing that

$$\mathbb{T}_{(p,\mu,\nu)}(A|B) = \frac{A\natural_p(A\nabla_\mu B) - A}{p} - \frac{A\natural_p(A\nabla_\nu B) - A}{p},$$

the desired result can be deduced. □

Proposition 2.2. *Let $A, B \in \mathcal{B}^{+*}(H)$, $p \in \mathbb{R}^*$ and $0 \leq \nu, \mu \leq 1$. We have*

i) $\mathbb{T}_{(p,\mu,\nu)}(U^*AU|U^*BU) = U^*\mathbb{T}_{(p,\mu,\nu)}(A|B)U$, *for any unitary operator $U \in \mathcal{B}^{+*}(H)$.*

ii) $\mathbb{T}_{(p,\mu,\nu)}$ *is homogenous, i.e.*

$$\mathbb{T}_{(p,\mu,\nu)}(\alpha A|\alpha B) = \alpha\mathbb{T}_{(p,\mu,\nu)}(A|B) \textit{ for any } \alpha > 0.$$

Proof. Let us notice at first that

$$\mathbb{T}_{(p,\mu,\nu)}(A|B) = T_p(A|A\nabla_\mu B) - T_p(A|A\nabla_\nu B). \quad (2.2)$$

So,

$$\mathbb{T}_{(p,\mu,\nu)}(U^*AU|U^*BU) = T_p(U^*AU|(U^*AU)\nabla_\mu(U^*BU)) - T_p(U^*AU|(U^*AU)\nabla_\nu(U^*BU)).$$

Combining the formulas (2.2) and (1.3), we get

$$\begin{aligned}\mathbb{T}_{(p,\mu,\nu)}(U^*AU|U^*BU) &= T_p(U^*AU|U^*(A\nabla_\mu B)U) - T_p(U^*AU|U^*(A\nabla_\nu B)U)\\ &= U^*\left(T_p(A|A\nabla_\mu B) - T_p(A|A\nabla_\nu B)\right)U\\ &= U^*\mathbb{T}_{(p,\mu,\nu)}(A|B)U.\end{aligned}$$

Using (2.2) and the fact that the operator mean ∇ and T_p are homogenous, one can easily deduce the second assertion in the proposition. □

The following result provides an integral representation for the generalized Tsallis relative operator entropy.

Theorem 2.3. *Let $A, B \in \mathcal{B}^{+*}(H)$. For any $p \in \mathbb{R}\backslash\{-1\}$ and $a, b \in [0,1]$ with $a < b$, we have*

$$\mathbb{T}_{(p+1,b,a)}(A \mid B) = \int_a^b [A\natural_p(A\nabla_t B)]\left(A^{-1}B - I_H\right)dt. \tag{2.3}$$

Proof. Let us note that for all $x \in \mathbb{R}_+$, the following formula holds

$$\int_a^b (1-t+tx)^p(x-1)dt = \frac{(1-b+bx)^{p+1} - (1-a+ax)^{p+1}}{p+1}.$$

So, by theory of functional calculus and substituting x by $A^{\frac{-1}{2}}BA^{\frac{-1}{2}}$, we get

$$\int_a^b \left((1-t)I_H + tA^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^p\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}} - I_H\right)dt = \frac{\left((1-b)I_H + bA^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{p+1} - \left((1-a)I_H + aA^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{p+1}}{p+1}. \tag{2.4}$$

Noticing that $A^{-\frac{1}{2}}BA^{-\frac{1}{2}} - I_H = A^{\frac{1}{2}}\left(A^{-1}BA^{-\frac{1}{2}} - A^{-\frac{1}{2}}\right)$, the formula (2.4) is equivalent to

$$\int_a^b \left(A^{-\frac{1}{2}}((1-t)A + tB)A^{-\frac{1}{2}}\right)^p A^{\frac{1}{2}}\left(A^{-1}BA^{-\frac{1}{2}} - A^{-\frac{1}{2}}\right)dt = \frac{\left(A^{-\frac{1}{2}}((1-b)A + bB)A^{-\frac{1}{2}}\right)^{p+1} - \left(A^{-\frac{1}{2}}((1-a)A + aB)A^{-\frac{1}{2}}\right)^{p+1}}{p+1},$$

which leads to (2.3) by multiplying on its both sides by $A^{\frac{1}{2}}$. □

Remark 2.1. For $a = 0, b = 1$ and $p \in \mathbb{R}^*$, we obtain

$$T_p(A \mid B) = \int_0^1 [A\natural_{p-1}(A\nabla_t B)]\left(A^{-1}B - I_H\right)dt. \tag{2.5}$$

It is worth mentioning that the formula (2.5) provides an integral representation for $T_p(A \mid B)$ more general than the one given by (1.2) stated by the authors in [16, Definition 3.1] only for parameters $p \in (0,1)$.

The following results in the ongoing section deal with the monotonicity of $\mathbb{T}_{(p,\mu,\nu)}(A|B)$ according to each of the parameters μ, ν and p.

Proposition 2.4. *Let $A, B \in \mathcal{B}^{+*}(H)$, $p \neq 0$ and $\mu, \nu \in [0,1]$. We have*

i) If $A \leq B$ $(A \geq B)$ then $\mathbb{T}_{(p,\mu_1,\nu)}(A|B) \leq (\geq) \mathbb{T}_{(p,\mu_2,\nu)}(A|B)$ for $\nu \leq \mu_1 \leq \mu_2 \leq 1$.
ii) If $A \geq B$ $(A \leq B)$ then $\mathbb{T}_{(p,\mu,\nu_1)}(A|B) \leq (\geq) \mathbb{T}_{(p,\mu,\nu_2)}(A|B)$ for $0 \leq \nu_1 \leq \nu_2 \leq \mu$.

Proof. Let $\mu, \nu \in [0,1]$ and $x \geq 0$.
If $x \geq 1$ $(0 < x \leq 1)$, the function $t \mapsto \dfrac{(1-t+tx)^p - (1-\nu+\nu x)^p}{p}$ is increasing (decreasing) on $[\nu, 1]$.
Thus, for $\nu \leq \mu_1 \leq \mu_2 \leq 1$ it holds

$$\frac{(1-\mu_1+\mu_1 x)^p - (1-\nu+\nu x)^p}{p} \leq (\geq) \frac{(1-\mu_2+\mu_2 x)^p - (1-\nu+\nu x)^p}{p}.$$

So, by theory of functional calculus, after replacing x by $A^{-\frac{1}{2}} B A^{-\frac{1}{2}}$ and multiplying left and right by $A^{\frac{1}{2}}$, we get

$$\mathbb{T}_{(p,\mu_1,\nu)}(A|B) \leq (\geq)\, \mathbb{T}_{(p,\mu_2,\nu)}(A|B).$$

The proof of the second statement can be done in a similar way to that of i). □

To study he monotonicity of the map $p \longmapsto \mathbb{T}_{(p,\mu,\nu)}(A|B)$, we need the following lemma.

Lemma 2.5. *Let $0 \leq \nu \leq \mu \leq 1$, $p \in \mathbb{R}$ and $x > 0$. We have*

$$(p\log(1-\mu+\mu x) - 1)(1-\mu+\mu x)^p - (p\log(1-\nu+\nu x) - 1)(1-\nu+\nu x)^p \geq 0.$$

Proof. We define on $(0, \infty)$ the real function by $f_p(t) = (p\log(t) - 1)t^p$ and we put $\alpha = 1-\mu+\mu x$ and $\beta = 1-\nu+\nu x$.
If $\alpha = \beta$, that is $\mu = \nu$ or $x = 1$, the desired result is obvious. So, let us consider $\alpha \neq \beta$.
If $0 < x < 1$ then by noticing that $\alpha - \beta = (\mu - \nu)(x-1)$, we get $0 < \alpha < \beta < 1$.
Using the fact that f_p is continuous on $[\alpha, \beta]$ and differentiable on (α, β), we can deduce by virtue of Lagrange's mean value theorem that there exists $c \in (\alpha, \beta)$ such that

$$f_p(\alpha) - f_p(\beta) = (\alpha - \beta) f_p^{'}(c),$$

or equivalently

$$(p\log(\alpha) - 1)\alpha^p - (p\log(\beta) - 1)\beta^p = p^2 c^{p-1}(\alpha - \beta)\log c.$$

Since $c \in (\alpha, \beta)$ then $\log c < 0$ and consequently

$$p^2 c^{p-1}(\alpha - \beta)\log c \geq 0.$$

If $x \geq 1$, one can follow similar steps used for the previous case. Whence, the lemma is proved. □

Theorem 2.6. *Let $A, B \in \mathcal{B}^{+*}(H)$, $0 \leq \nu \leq \mu \leq 1$ and $p, q \in \mathbb{R}^*$ with $p \leq q$. We have*

$$\mathbb{T}_{(p,\mu,\nu)}(A|B) \leq \mathbb{T}_{(q,\mu,\nu)}(A|B). \tag{2.6}$$

Proof. Let us put for $x > 0$ and $p \in \mathbb{R}^*$

$$\alpha = 1-\mu+\mu x, \ \beta = 1-\nu+\nu x \ \text{ and } \ \phi_{\mu,\nu,x}(p) = \frac{\alpha^p - \beta^p}{p}.$$

We have,

$$\frac{d}{dp}\phi_{\mu,\nu,x}(p) = \frac{(p\log(\alpha) - 1)\alpha^p - (p\log(\beta) - 1)\beta^p}{p^2}.$$

By Lemma 2.5, we have $\dfrac{d}{dp}\phi_{\mu,\nu,x}(p) \geq 0$. That is $p \longmapsto \phi_{\mu,\nu,x}(p)$ is increasing on $\mathbb{R}^*$.
So, if $p \leq q$ then

$$\phi(p,\mu,\nu,x) \leq \phi(q,\mu,\nu,x),$$

which implies, by virtue of theory of functional calculus and after substituting x by $A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$, the following inequality

$$\frac{(A^{-\frac{1}{2}}\,(A\nabla_{\mu}\,B)\,A^{-\frac{1}{2}})^p-(A^{-\frac{1}{2}}\,(A\nabla_{\nu}\,B)\,A^{-\frac{1}{2}})^p}{p}\leq\frac{(A^{-\frac{1}{2}}\,(A\nabla_{\mu}\,B)\,A^{-\frac{1}{2}})^q-(A^{-\frac{1}{2}}\,(A\nabla_{\nu}\,B)\,A^{-\frac{1}{2}})^q}{q}.$$

Multiplying this last inequality by $A^{\frac{1}{2}}$, we get (2.6). □

Remark 2.2. For $\mu=1,\nu=0$ and $p,q\in[-1,0)\cup(0,1]$ with $p\leq q$, we get the well known inequality [8]

$$T_p(A|B)\leq T_q(A|B).$$

This confirms, once more again, the generalization character of $\mathbb{T}_{(p,\mu,\nu)}(A|B)$.

3. Inequalities involving $\mathbb{T}_{(p,\mu,\nu)}(A|B)$

In the current section, we aim to determine some estimations for $\mathbb{T}_{(p,\mu,\nu)}(A|B)$. Our first result is recited in the following proposition.

Proposition 3.1. *Let $A,B\in\mathcal{B}(H)^{+*}$, $p>0$ and $0\leq\nu,\mu\leq1$. We have*

$$\begin{aligned}\big[A\natural_p\,(A\nabla_{\nu}\,B)\,\big]A^{-1}\,\mathbb{T}_{(-p,\mu,\nu)}(A|B)\leq S(A|A\nabla_{\mu}B)-S(A|A\nabla_{\nu}B)\leq&\\ \big[A\natural_{-p}\,(A\nabla_{\nu}\,B)\,\big]A^{-1}\,\mathbb{T}_{(p,\mu,\nu)}(A|B).&\end{aligned}\tag{3.1}$$

If $p<0$, the inequalities (3.1) are reversed.

Proof. For $p>0$ and $y>0$, one can easily check by routine tools of real analysis that

$$\frac{y^{-p}-1}{-p}\leq\log y\leq\frac{y^p-1}{p}.$$

So, by setting $y=\dfrac{1-\mu+\mu x}{1-\nu+\nu x}>0$ for $x>0$, we obtain

$$\begin{aligned}(1-\nu+\nu x)^p\,\frac{(1-\mu+\mu x)^{-p}-(1-\nu+\nu x)^{-p}}{-p}\leq&\\ \log(1-\mu+\mu x)-\log(1-\nu+\nu x)\leq&\\ (1-\nu+\nu x)^{-p}\,\frac{(1-\mu+\mu x)^p-(1-\nu+\nu x)^p}{p}.&\end{aligned}\tag{3.2}$$

Thus, by theory of functional calculus, substitution of x by $A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$ and the following formula

$$\left(1-\nu+\nu A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^p=A^{-\frac{1}{2}}\left[A\natural_p\,(A\nabla_{\nu}B)\right]A^{-\frac{1}{2}},$$

allow us to state

$$\begin{aligned}\left(1-\nu+\nu A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^p\frac{\left(1-\mu+\mu A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{-p}-\left(1-\nu+\nu A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{-p}}{-p}\leq&\\ \log\left(1-\mu+\mu A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)-\log\left(1-\nu+\nu A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)\leq&\\ \left(1-\nu+\nu A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{-p}\frac{\left(1-\mu+\mu A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^p-\left(1-\nu+\nu A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^p}{p}.&\end{aligned}\tag{3.3}$$

Multiplying both sides of inequalities (3.3) by $A^{\frac{1}{2}}$, we deduce the inequalities (3.1).
If $p<0$, we apply the inequalities (3.1) for $-p>0$ to deduce the desired result. □

Remark 3.1. It is worth mentioning that by taking $\mu = 1, \nu = 0$ and $p \in (0,1]$ in (3.1), we get particularly the inequalities

$$T_{-p}(A|B) \leq S(A|B) \leq T_p(A|B),$$

established by Furuichi et al in [9, Proposition 3.1].

Proposition 3.2. *Let $A, B \in \mathcal{B}^{+*}(H)$, $q > 0$, $p \in [-q, q]$ and $0 \leq \nu \leq \mu \leq 1$. We have*

$$\begin{aligned} m_{p,q}\, A - \big[A\natural_q(A\nabla_\nu B)\big]A^{-1}\big[A\natural_{-q}(A\nabla_\mu B)\big] \leq \\ \big[A\natural_{-p}(A\nabla_\nu B)\big]A^{-1}\mathbb{T}_{(p,\mu,\nu)}(A|B) \leq \\ \big[A\natural_{-q}(A\nabla_\nu B)\big]A^{-1}\big[A\natural_q(A\nabla_\mu B)\big] + n_{p,q}\, A, \end{aligned} \tag{3.4}$$

with, $m_{p,q} := \frac{q^{\frac{p}{p+q}}-1}{p} + q^{\frac{-q}{p+q}}$ *and* $n_{p,q} := \dfrac{q^{\frac{p}{p-q}}-1}{p} - q^{\frac{q}{p-q}}$.

Proof. For $-q \leq p \leq q$ and $y > 0$, a simple study leads to the following inequalities

$$\frac{q^{\frac{p}{p+q}}-1}{p} + q^{\frac{-q}{p+q}} - y^{-q} \leq \frac{y^p-1}{p} \leq y^q + \frac{q^{\frac{p}{p-q}}-1}{p} - q^{\frac{q}{p-q}}.$$

Whence, choosing $y = \dfrac{1-\mu+\mu x}{1-\nu+\nu x} > 0$ for $x > 0$, we deduce

$$\begin{aligned} \frac{q^{\frac{p}{p+q}}-1}{p} + q^{\frac{-q}{p+q}} - (1-\nu+\nu x)^q(1-\mu+\mu x)^{-q} \\ \leq (1-\nu+\nu x)^{-p}\frac{(1-\mu+\mu x)^p-(1-\nu+\nu x)^p}{p} \\ \leq (1-\nu+\nu x)^{-q}(1-\mu+\mu x)^q + \frac{q^{\frac{p}{p-q}}-1}{p} - q^{\frac{q}{p-q}}. \end{aligned} \tag{3.5}$$

Changing x by $A^{-\frac12}BA^{-\frac12}$ and multiplying both sides of inequalities (3.5) by $A^{\frac12}$, we deduce the inequalities (3.4). □

Remark 3.2. Proposition 3.2 provides a generalization of the Proposition 3.4 stated in [9]. In fact, taking $q = \mu = 1, \nu = 0$ and $p \in (0,1]$ in (3.4), we get for all $A, B \in \mathcal{B}^{+*}(H)$

$$A - AB^{-1}A \leq T_p(A|B) \leq B - A.$$

For further results, the following Hermite-Hadamard type inequalities will be very useful.

Theorem 3.3. *Let f be a convex function on an open interval $I \subseteq \mathbb{R}$. For all $[x,y] \subseteq I$ and for each $\lambda \in [0,1]$, we have*

$$2\mu f\left((1-\lambda)x+\lambda y\right) \leq \int_{\lambda-\mu}^{\lambda+\mu} f((1-t)x+ty)dt \leq 2\mu\left[(1-\lambda)f(x)+\lambda f(y)\right], \tag{3.6}$$

where $\mu = min\{\lambda, 1-\lambda\}$.
If f is a concave function on I then the inequalities in (3.6) are reversed.

Proof. If f is a convex function on I, we get

$$\begin{aligned} f\left((1-\lambda)x+\lambda y\right) &\leq \frac12\Big[f\left((1-\lambda-t)x+(\lambda+t)y\right)+f\left((1-\lambda+t)x+(\lambda-t)y\right)\Big] \\ &\leq (1-\lambda)f(x)+\lambda f(y). \end{aligned}$$

Integrating this inequalities over $t \in [0,\mu]$, we obtain

$$\mu f\left((1-\lambda)x+\lambda y\right) \leq \frac{1}{2}\int_0^{\mu} f\left((1-\lambda-t)x+(\lambda+t)y\right)dt + \frac{1}{2}\int_0^{\mu} f\left((1-\lambda+t)x+(\lambda-t)y\right)dt \leq \mu\left((1-\lambda)f(x)+\lambda f(y)\right).$$

Using appropriately the changes of the variables $u=\lambda+t$ and $u=\lambda-t$, it yields

$$\mu f\left((1-\lambda)x+\lambda y\right) \leq \frac{1}{2}\int_{\lambda-\mu}^{\lambda+\mu} f\left((1-t)x+ty\right)dt \leq \mu\left((1-\lambda)f(x)+\lambda f(y)\right),$$

which ends the proof. $\square$

Remark 3.3. For $\mu=\lambda=\dfrac{1}{2}$, we find the following well known Hermite-Hadamard inequalities

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(t)dt \leq \frac{f(a)+f(b)}{2}. \tag{3.7}$$

It is important to note that the inequalities (3.7) could also be deduced from the generalization pointed out separately in [14] and [18] which differs from the one stated in (3.6). An investigation on the extension of (3.6) to positive operators and the establishment of some applications will be an interesting topic for future work.

The following theorem will be very useful to establish some inequalities involving generalized Tsallis relative operator entropy.

Theorem 3.4. *Let $A,B \in \mathcal{B}^{+*}(H)$ and $\lambda \in [0,1]$. For all $p<0$ with $p \neq -1$ or $p>1$, we have*

$$2\mu\left(A\natural_p(A\nabla_\lambda B)\right) \leq \int_{\lambda-\mu}^{\lambda+\mu} A\natural_p\left(A\nabla_t B\right)dt \leq 2\mu\left[A\nabla_\lambda(A\natural_p B)\right], \tag{3.8}$$

where $\mu=min\{\lambda,1-\lambda\}$.
If $p\in[0,1]$, the inequalities in (3.8) are reversed.

Proof. Consider on $(0,+\infty)$ the function defined by $f(t)=t^p$, $p\in(-\infty,0)\cup(1,+\infty)$.
Since f is a convex function on $(0,+\infty)$, taking $x=1$ in (3.6) we get the following inequalities

$$2\mu\left(1-\lambda+\lambda y\right)^p \leq \int_{\lambda-\mu}^{\lambda+\mu}\left(1-t+ty\right)^p dt \leq 2\mu\left(1-\lambda+\lambda y^p\right). \tag{3.9}$$

By theory of functional calculus and replacing y by $A^{\frac{-1}{2}}BA^{\frac{-1}{2}}$, we have

$$2\mu\left((1-\lambda)I_H+\lambda A^{\frac{-1}{2}}BA^{\frac{-1}{2}}\right)^p \leq \int_{\lambda-\mu}^{\lambda+\mu}\left((1-t)I_H+tA^{\frac{-1}{2}}BA^{\frac{-1}{2}}\right)^p dt \leq 2\mu\left[(1-\lambda)I_H+\lambda\left(A^{\frac{-1}{2}}BA^{\frac{-1}{2}}\right)^p\right]. \tag{3.10}$$

Multiplying both sides of inequalities (3.10) by $A^{\frac{1}{2}}$, we deduce the desired result. $\square$

In the following theorem, we will state another main result.

Theorem 3.5. *Let $A,B\in\mathcal{B}^{+*}(H)$ be two selfadjoint operators, with $A\leq B$. For any $\lambda\in(0,1]$ and $\mu=min\{\lambda,1-\lambda\}$, we have*

$$\frac{2\mu}{\lambda}\left[A\natural_{p+1}(A\nabla_\lambda B)-A\natural_p(A\nabla_\lambda B)\right] \leqslant \mathbb{T}_{(p+1,\lambda+\mu,\lambda-\mu)}(A\mid B) \leqslant 2\mu\left[B\nabla_\lambda(A\natural_{p+1}B)-A\nabla_\lambda(A\natural_p B)\right], \tag{3.11}$$

for all $p<0$ with $p\neq -1$ or $p>1$. If $p\in[0,1]$, the inequalities (3.11) *are reversed.*

Proof. According to the condition $A\leq B$, we can set $C=A^{\frac{-1}{2}}\left(A^{\frac{-1}{2}}BA^{\frac{-1}{2}}-I_H\right)^{\frac{1}{2}}A^{\frac{1}{2}}$. For $p<0$ with $p\neq -1$ or $p>1$ and by the use of the Theorem 3.4, we get

$$2\mu\left[A\natural_p(A\nabla_\lambda B)\right]\leqslant\int_{\lambda-\mu}^{\lambda+\mu}\left[A\natural_p\left(A\nabla_t B\right)\right]d\,t\leqslant 2\mu\left[A\nabla_\lambda(A\natural_p B)\right].$$

Thus,

$$2\mu\, C^*\left[A\natural_p(A\nabla_\lambda B)\right]C\leqslant\int_{\lambda-\mu}^{\lambda+\mu}C^*\left[A\natural_p\left(A\nabla_t B\right)\right]Cd\,t\leqslant 2\mu\, C^*\left[A\nabla_\lambda(A\natural_p B)\right]C.$$

By setting $D=A^{\frac{-1}{2}}BA^{\frac{-1}{2}}$, we have

$$\begin{aligned}C^*\left[A\natural_p(A\nabla_\lambda B)\right]C&=A^{\frac{1}{2}}\left(D-I_H\right)^{\frac{1}{2}}\left(A^{\frac{-1}{2}}(A\nabla_\lambda B)A^{\frac{-1}{2}}\right)^p\left(D-I_H\right)^{\frac{1}{2}}A^{\frac{1}{2}}\\&=A^{\frac{1}{2}}\left(D-I_H\right)^{\frac{1}{2}}\left((1-\lambda)I_H+D\right)^p\left(D-I_H\right)^{\frac{1}{2}}A^{\frac{1}{2}}\\&=A^{\frac{1}{2}}\left((1-\lambda)I_H+D\right)^p\left(D-I_H\right)A^{\frac{1}{2}}\\&=A^{\frac{1}{2}}\left(A^{\frac{-1}{2}}(A\nabla_\lambda B)A^{\frac{-1}{2}}\right)^p\left(A^{\frac{-1}{2}}BA^{\frac{-1}{2}}-I_H\right)A^{\frac{1}{2}}\\&=A^{\frac{1}{2}}\left(A^{\frac{-1}{2}}(A\nabla_\lambda B)A^{\frac{-1}{2}}\right)^pA^{\frac{1}{2}}A^{\frac{-1}{2}}\left(A^{\frac{-1}{2}}B-A^{\frac{1}{2}}\right)\\&=\left[A\natural_p(A\nabla_\lambda B)\right]\left(A^{-1}B-I_H\right).\end{aligned}$$

Furthermore, noticing that for any $\lambda\in(0,1]$,

$$A^{-1}B=\frac{1}{\lambda}\left[A^{\frac{-1}{2}}\left(A^{\frac{-1}{2}}(A\nabla_\lambda B)A^{\frac{-1}{2}}\right)A^{\frac{1}{2}}-(1-\lambda)I_H\right],$$

it yields

$$\begin{aligned}C^*\left[A\natural_p(A\nabla_\lambda B)\right]C&=\frac{1}{\lambda}\left[A\natural_{p+1}(A\nabla_\lambda B)-(1-\lambda)A\natural_p(A\nabla_\lambda B)\right]-A\natural_p(A\nabla_\lambda B)\\&=\frac{1}{\lambda}\left[A\natural_{p+1}(A\nabla_\lambda B)-A\natural_p(A\nabla_\lambda B)\right].\end{aligned}$$

On the other hand, we have

$$\begin{aligned}C^*\left[A\nabla_\lambda(A\natural_p B)\right]C&=A^{\frac{1}{2}}\left(D-I_H\right)^{\frac{1}{2}}\left[(1-\lambda)I_H+\lambda D^p\right]\left(D-I_H\right)^{\frac{1}{2}}A^{\frac{1}{2}}\\&=A^{\frac{1}{2}}\left(D-I_H\right)^{\frac{1}{2}}\left[(1-\lambda)I_H+\lambda D^p\right]\left(D-I_H\right)^{\frac{1}{2}}A^{\frac{1}{2}}\\&=A^{\frac{1}{2}}\left[(1-\lambda)I_H+\lambda D^p\right]\left(D-I_H\right)^{\frac{1}{2}}\left(D-I_H\right)^{\frac{1}{2}}A^{\frac{1}{2}}\\&=A^{\frac{1}{2}}\left[(1-\lambda)I_H+\lambda D^p\right]\left(D-I_H\right)A^{\frac{1}{2}}\\&=\left[A\nabla_\lambda(A\natural_p B)\right]\left(A^{-1}B-I_H\right)\\&=(1-\lambda)B+\lambda(A\natural_p B)A^{-1}B-A\nabla_\lambda(A\natural_p B).\end{aligned}$$

Noticing that $A^{-1}B=A^{\frac{-1}{2}}\left(A^{\frac{-1}{2}}BA^{\frac{-1}{2}}\right)A^{\frac{1}{2}}$, we obtain

$$\begin{aligned}C^*\left[A\nabla_\lambda(A\natural_p B)\right]C&=(1-\lambda)B+\lambda(A\natural_{p+1}B)-A\nabla_\lambda(A\natural_p B)\\&=B\nabla_\lambda(A\natural_{p+1}B)-A\nabla_\lambda(A\natural_p B).\end{aligned}$$

Finally, by Theorem 2.3 we have

$$\int_{\lambda-\mu}^{\lambda+\mu}C^*\left[A\natural_p\left(A\nabla_t B\right)\right]Cdt=\int_{\lambda-\mu}^{\lambda+\mu}\left[A\natural_p\left(A\nabla_t B\right)\right]\left(A^{-1}B-I_H\right)dt=\mathbb{T}_{(p+1,\lambda+\mu,\lambda-\mu)}.$$

For $p \in [0,1]$, using the inverses of inequalities (3.8) and following the same steps used for the proof of (3.11), one can deduce the result. $\square$

Remark 3.4. Inequalities (3.11) provides a generalization for (1.4), in the sense that for $p \in (0,1]$ and $\lambda = \frac{1}{2}$ in (3.11) we find (1.4).

The following result provides also an extension for the inequalities (1.4).

Corollary 3.6. *Let $A, B \in \mathcal{B}(H)^{+*}$ with $A \leq B$. For any $p \in \mathbb{R}^*\backslash[1,2]$, we have*

$$\begin{aligned} A^{\frac{1}{2}} & \left[\left(\frac{I_H + A^{\frac{-1}{2}} B A^{\frac{-1}{2}}}{2}\right)^p - \left(\frac{I_H + A^{\frac{-1}{2}} B A^{\frac{-1}{2}}}{2}\right)^{p-1}\right] A^{\frac{1}{2}} \\ & \leqslant 2A^{\frac{1}{2}} \left[\left(\frac{I_H + A^{\frac{-1}{2}} B A^{\frac{-1}{2}}}{2}\right)^p - \left(\frac{I_H + A^{\frac{-1}{2}} B A^{\frac{-1}{2}}}{2}\right)^{p-1}\right] A^{\frac{1}{2}} \leqslant T_p(A \mid B) \\ & \leqslant \frac{1}{2}\left(A \natural_p B - A \natural_{p-1} B + B - A\right). \end{aligned} \tag{3.12}$$

Proof. Let $p \in \mathbb{R}^*\backslash[1,2]$. By inequalities (3.11) when taking $\lambda = \dfrac{1}{2}$ and replacing p by $p-1$, we find

$$2\left[A\natural_p(A\nabla B) - A\natural_{p-1}(A\nabla B)\right] \leq \mathbb{T}_{(p,1,0)}(A \mid B) \leq B\nabla(A\natural_p B) - A\nabla(A\natural_{p-1} B),$$

or equivalently

$$\begin{aligned} 2A^{\frac{1}{2}} \left[\left(A^{\frac{-1}{2}}\left(\frac{B+A}{2}\right)A^{\frac{-1}{2}}\right)^p - \left(A^{\frac{-1}{2}}\left(\frac{B+A}{2}\right)A^{\frac{-1}{2}}\right)^{p-1}\right] A^{\frac{1}{2}} & \leqslant \mathbb{T}_{(p,1,0)}(A \mid B) \\ & \leqslant \frac{1}{2}\left(B + A\natural_p B - A - A\natural_{p-1} B\right). \end{aligned}$$

Using the relation $\mathbb{T}_{(p,1,0)}(A \mid B) = T_p(A \mid B)$, we deduce the inequalities (3.12). $\square$

Corollary 3.7. *Let $A, B \in \mathcal{B}^{+*}(H)$ be two selfadjoint operators with $A \leq B$. For any $\lambda \in \left(0, \frac{1}{2}\right]$ and $p \in \mathbb{R}^* \setminus [1,2]$, we have*

$$\begin{aligned} 2[A\natural_p(A\nabla_\lambda B) - A\natural_{p-1}(A\nabla_\lambda B)] - \frac{2\lambda}{p}(B-A) & \leqslant T_p(A \mid A\nabla_{2\lambda} B) \\ & \leqslant 2\lambda\left[B\nabla_\lambda(A\natural_p B) - A\nabla_\lambda(A\natural_{p-1} B)\right] - \frac{2\lambda}{p}(B-A). \end{aligned} \tag{3.13}$$

For $p \in [1,2]$, the inequalities reverse (3.13) are reversed.

Proof. As $\lambda \in \left(0, \frac{1}{2}\right]$ then $\mu = \lambda$. So, applying the inequalities (3.11) combined with the following formula

$$\mathbb{T}_{(p,2\lambda,0)} = T_p\left(A|A\nabla_{2\lambda}B\right) + \frac{2\lambda}{p}(B-A),$$

we get (3.13). $\square$

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