DYNAMICS OF A DISCRETE PREDATOR-PREY SYSTEM WITH FEAR EFFECT AND DENSITY DEPENDENT BIRTH RATE OF THE PREY SPECIES

DEBASIS MUKHERJEE

Abstract. This paper analyses a discrete predator-prey system with fear effect and density dependent birth rate of the prey species. The fixed points of the system are determined and their stability is examined. The criterion for Neimark-Sacker bifurcation and flip bifurcation is developed. The chaotic orbit at an unstable fixed point can be stabilized by applying the state feedback control method. Numerically, we illustrate our analytical findings and observe the complex behaviour of the system that leads to stable state to chaotic one.

1. INTRODUCTION

In recent years, it is observed that the predator-prey interaction is not only governed by direct killing of prey by the predator, but also the indirect effect such as fear caused by the predator. The fear factor influences the birth rate of prey [1]. Based on the fact of fear effect on the prey's growth rate, several research works are explored [1, 2, 3, 4, 5, 6, 7, 8, 9, 10]. In predator-prey interaction, it is commonly assumed that the birth rate of the prey species is constant. But in a real ecological system, a birth rate of the prey species is dependent on the density of prey. In [11], the authors considered density dependent birth rate of the prey species and discussed the dynamical behaviour of the predator-prey system. Aforesaid studies are mainly confined on continuous predator-prey models with two variables. Although, discrete time models are more appropriate than the continuous system when the populations have nonoverlapping generations and virtually remain constant over a generation. From a biological point of view, a discrete time model is applied to investigate the taxonomic group of organisms and species with the passage of time. There are some biological situations where a discrete time system is applicable. For example, fish populations reproduce at specific time moments or for insect populations, for which nonoverlapping generations are occurring in real ecosystems. Other examples include monocarpic plants and semelparous animals which have nonoverlapping populations and their births take place in usual breeding seasons. Moreover, dynamics of discrete time predator-prey system can exhibit a richer set of patterns than those found in continuous systems [12, 13, 14, 15]. Also discrete time models can exhibit chaotic dynamics [12, 13]. So chaos control becomes an interesting topic of research in discrete dynamical system. We will show chaos control by the state feedback control strategy.

In [15], the authors observed flip bifurcation and Neimark-Sacker bifurcation in a discrete predator prey system with Holling type III functional response. Santra et al. [16] analysed a discrete predator-prey model with Crowley-Martin functional response where prey population takes refuge. They showed the effects of refuge on the stability of the system in the discrete domain. Furthermore, they obtained period doubling bifurcation and Neimark-Sacker bifurcation. Elettreby et al. [14] addressed the complex behaviour of a discrete prey-predator model considering mixed functional response of Holling

type I and III. Din [13] investigated the complex nature and chaos prevention in a discrete model of prey-predator interaction and found period doubling and Neimark-Sacker bifurcation for larger range of bifurcation parameter. Seno [17] remarked that the dynamics of discrete prey-predator system is consistent with continuous counterpart. Agiza et al. [12] discussed the dynamics of a discrete-time prey-predator model with Holling type II response function. They derived bifurcation diagrams, phase portraits and Lyapunov exponents for various system parameters. They also computed the fractional dimension of a strange attractor of the model.

In this paper, we propose a discrete predator-prey system with fear effect and density dependent birth rate of the prey species. We study the existence and stability of different fixed points. After then, we identify the system parameters that give Neimark-Sacker bifurcation and flip bifurcation.

The paper is formatted as follows. In Section 2, we present a discrete model of predator-prey interaction with fear effect and the density dependent birth rate of the prey species. Dynamical analysis of different fixed points is described in Section 3. Chaos control is shown in Section 4. In Section 5, the behaviour of the system is demonstrated when values of parameters are changed. A short discussion is given in Section 6.

2. Discrete model

Now, we present the following discrete time predator-prey model:

$$
\begin{cases}
x_{n+1} = x_n \left[\dfrac{r}{(b + cx_n)(1 + ky_n)} - \alpha - \beta x_n - \dfrac{py_n}{1 + hx_n} \right], \\
y_{n+1} = y_n \left[-d + \dfrac{pqx_n}{1 + hx_n} \right]
\end{cases}
\tag{2.1}
$$

where x_n and y_n represent population densities of prey and predator respectively, and $r, b, c, k, \alpha, \beta, p, q,$ h, d are positive constants. Here r denotes the birth rate of the prey which is affected by the fear factor $1/(1 + ky_n)$ where k is the level of fear factor. The birth rate r is modified by the density of prey in the form of Beverton-Holt function [18] as $r/(b + cx_n)$ where, b and c are positive parameters. α is the natural mortality rate of prey. β represents the intraspecific competition among the prey species. p, h, q, d represent consumption rate, handling time, q conversion efficiency and the death rate of the predator respectively.

3. Fixed points and their nature

In this section, we determine the fixed points and their dynamics. Evidently, System (2.1) has three fixed points $E_0 = (0, 0), E_1 = (\bar{x}, 0)$ and $E^* = (x^*, y^*)$ where

$$
\bar{x} = \frac{-\{c(1 + \alpha) + \beta b\} + \sqrt{\{c(1 + \alpha) + \beta b\}^2 - 4\beta c\{b(1 + \alpha) - r\}}}{2\beta c},
$$

and

$$
\begin{cases}
x^* = \dfrac{1 + d}{pq - h(1 + d)}, \\
y^* = \dfrac{-D + \sqrt{D^2 - 4pk(b + cx^*)(1 + hx^*)\{(b + cx^*)(\alpha + \beta x^* + 1) - r\}}}{2p(b + cx^*)}
\end{cases}
\tag{3.1}
$$

where $D = (b + cx^*)\{p + k(1 + hx^*)(\alpha + \beta x^* + 1)\}$. Now E_1 is feasible if $b(1 + \alpha) < r$; E^* is feasible if $pq > h(1 + d)$ and $(b + cx^*)(\alpha + \beta x^* + 1) < r$.

To determine the nature of the fixed points, we compute the Jacobian matrix at each fixed point. The Jacobian matrix at an arbitrary fixed point (x, y) is given by

$$J(x,y) = \begin{pmatrix} \frac{br}{(b+cx)^2(1+ky)} - \alpha - 2\beta x - \frac{py}{(1+hx)^2} & -x\{\frac{rk}{(b+cx)(1+ky)^2} + \frac{p}{1+hx}\} \\ \\ \frac{pqy}{(1+hx)^2} & -d + \frac{pqx}{1+hx} \end{pmatrix}$$

Proposition 1. The fixed point $E_0 = (0,0)$ of system (2.1) is stable if $b(\alpha - 1) < r < b(\alpha + 1)$ and $0 < d < 1$.

Proof. The characteristic equation at E_0 is

$$\begin{vmatrix} \frac{r}{b} - \alpha - \lambda & 0 \\ 0 & -d - \lambda \end{vmatrix} = 0.$$

Thus the eigenvalues are $\lambda_1 = \frac{r}{b} - \alpha$ and $\lambda_2 = -d$. Then the fixed point E_0 is locally asymptotically stable if $|\lambda_i| < 1, i = 1, 2$. Now $|\lambda_1| = \left|\frac{r}{b} - \alpha\right| < 1$ then $\alpha - 1 < \frac{r}{b} < \alpha + 1$. Also, $|\lambda_2| = |-d| < 1$ then $-1 < d < 1$. As d is the death rate of predator, this implies that $0 < d < 1$. This completes the proof.

Proposition 2. Assume that $b(1 + \alpha) < r$ holds. Then, the fixed point $E_1 = (\bar{x}, 0)$ of system (2.1) is stable if the following inequalities are fulfilled:

$$\alpha + 2\beta\bar{x} - 1 < \frac{br}{(b + c\bar{x})^2} < \alpha + 2\beta\bar{x} + 1 \text{ and } d - 1 < \frac{pq\bar{x}}{1 + h\bar{x}} < d + 1.$$

Proof. The characteristic equation at E_1 is

$$\begin{vmatrix} \frac{br}{(b+c\bar{x})^2} - \alpha - 2\beta\bar{x} - \lambda & -\bar{x}\left(\frac{rk}{b+c\bar{x}} + \frac{p}{1+h\bar{x}}\right) \\ \\ 0 & -d + \frac{pq\bar{x}}{1+h\bar{x}} - \lambda \end{vmatrix} = 0.$$

Hence, the eigenvalues are

$$\lambda_1 = \frac{br}{(b + c\bar{x})^2} - \alpha - 2\beta\bar{x}, \quad \lambda_2 = -d + \frac{pq\bar{x}}{1 + h\bar{x}}.$$

The fixed point E_1 is locally stable if $|\lambda_i| < 1, i = 1, 2$. Now $|\lambda_1| < 1$ is equivalent to

$$\alpha + 2\beta\bar{x} - 1 < \frac{br}{(b + c\bar{x})^2} < \alpha + 2\beta\bar{x} + 1$$

and $|\lambda_2| < 1$ is equivalent to

$$d - 1 < \frac{pq\bar{x}}{1 + h\bar{x}} < d + 1.$$

This completes the proof.

We remark that the above stability conditions imply that the predator goes to extinction while prey is there.

Proposition 3. Assume that $pq > h(1 + d)$ and $(b + cx^*)(\alpha + \beta x^* + 1) < r$ hold. Then, the fixed point $E^* = (x^*, y^*)$ of system (2.1) is stable if the following inequalities are fulfilled:

$$\frac{pqx^*y^*}{(1 + hx^*)^2}\left[\frac{rk}{(b + cx^*)(1 + ky^*)^2} + \frac{p}{1 + hx^*}\right] + \frac{br}{(b + cx^*)^2(1 + ky^*)} < 1 + \alpha + 2\beta x^* + \frac{py^*}{(1 + hx^*)^2}, \quad (3.2)$$

$$\alpha + 2\beta x^* + \frac{py^*}{(1 + hx^*)^2} - \frac{br}{(b + cx^*)^2(1 + ky^*)} < 1 + \frac{pqx^*y^*}{2(1 + hx^*)^2}\left[\frac{rk}{(b + cx^*)(1 + ky^*)^2} + \frac{p}{1 + hx^*}\right]. \quad (3.3)$$

Proof. The characteristic equation at E^* is

$$\begin{vmatrix} \frac{br}{(b+cx^*)^2(1+ky^*)} - \alpha - 2\beta x^* - \frac{py^*}{(1+hx^*)^2} - \lambda & -x^*\left[\frac{rk}{(b+cx^*)(1+ky^*)^2} + \frac{p}{1+hx^*}\right] \\ \frac{pqy^*}{(1+hx^*)^2} & 1 - \lambda \end{vmatrix} = 0,$$

which is now written in the form $\lambda^2 - \eta\lambda + \gamma = 0$, where

$$\eta = \frac{br}{(b+cx^*)^2(1+ky^*)} - \alpha - 2\beta x^* - \frac{py^*}{(1+hx^*)^2} + 1,$$

and

$$\gamma = \frac{br}{(b+cx^*)^2(1+ky^*)} - \alpha - 2\beta x^* - \frac{py^*}{(1+hx^*)^2} + \frac{pqx^*y^*}{(1+hx^*)^2}\left[\frac{rk}{(b+cx^*)(1+ky^*)^2} + \frac{p}{1+hx^*}\right].$$

For stability of E^*, we use Jury Criterion which is given by $|\eta| < 1 + \gamma < 2$. This condition has two parts, namely (i) $\gamma < 1$, and (ii) $-1 - \gamma < \eta < 1 + \gamma$. By the formula for γ given above, Part (i) is precisely (3.2). For Part (ii), the left inequality is nothing by (3.3), while the right inequality reduces to

$$0 < \frac{pqx^*y^*}{(1+hx^*)^2}\left[\frac{rk}{(b+cx^*)(1+ky^*)^2} + \frac{p}{1+hx^*}\right]$$

which is always true when the interior fixed point exists. From the above analysis, we infer that E^* is stable under the conditions of the theorem, and the proof is completed.

3.1. **Bifurcation around the interior fixed point.** In discrete context, the Neimark-Sacker bifurcation is the counterpart of the Hopf bifurcation that takes place in continuous systems. It was explored by Neimark [19] and alone by Sacker [20]. Hopf bifurcation generates limit cycles in the phase plane in the continuous models. Alternately, Neimark-Sacker bifurcation produces dynamically invariant cycles. Subsequently, we may get isolated periodic orbits as well as trajectories that cover the invariant circle densely. Biologically, Neimark-Sacker bifurcation implies that all the populations can oscillate around some mean values.

Flip bifurcation is another type of bifurcation which is also recognized as period doubling bifurcation and it occurs when a small changes in bifurcation parameters give rise to a new system that bifurcate twice the period as the original system. This bifurcation indicates the loss of stability of a periodic orbit.

System (2.1) has at most one unique fixed point E^*, hence the system does not admit fold bifurcation. So we are interested in examining the Neimark-Sacker bifurcation and flip bifurcation in the sequel.

Proposition 4. System (2.1) admits Neimarck-Sacker bifurcation at E^* if the following conditions are satisfied:

$$\frac{br}{(b+cx^*)^2(1+ky^*)} + \frac{pqx^*y^*}{(1+hx^*)^2}\left[\frac{rk}{(b+cx^*)(1+ky^*)^2} + \frac{p}{1+hx^*}\right] = \alpha + 2\beta x^* + \frac{py^*}{(1+hx^*)^2} + 1, \tag{3.4}$$

and

$$\frac{pqx^*y^*}{(1+hx^*)^2}\left[\frac{rk}{(b+cx^*)(1+ky^*)^2} + \frac{p}{1+hx^*}\right] < 4. \tag{3.5}$$

Proof.. If the Jacobian matrix $J(E^*)$ has two complex conjugate eigenvalues with modulus 1, Neimark-Sacker bifurcation appears [21]. This requires that $\det(J(E^*)) = \gamma = 1$ and $-2 < \operatorname{tr}(J(E^*)) = \eta < 2$. Replacing η and γ (see the proof of Proposition 3), the first condition ($\gamma = 1$) is precisely (3.4); and the second condition on η is (3.5). This completes the proof.

Proposition 5. System (2.1) admits a flip bifurcation at E^* if the following conditions are satisfied:

$$2\left[1 + \frac{br}{(b+cx^*)^2(1+ky^*)}\right] + \frac{pqx^*y^*}{(1+hx^*)^2}\left[\frac{rk}{(b+cx^*)(1+ky^*)^2} + \frac{p}{1+hx^*}\right]$$
$$= 2\left[\alpha + 2\beta x^* + \frac{py^*}{(1+hx^*)^2}\right]. \tag{3.6}$$

Proof. System (2.1) admits flip bifurcatiopn when a single eigenvalue is -1. Thus the condition for flip bifurcation can be written in the form $1 + \eta + \gamma = 0$ where γ and η are as in the proof of Proposition 3. This condition is precisely (3.6), and hence completes the proof.

4. CHAOS CONTROL

Chaos control is a technique of stabilization by means of small perturbation which are used to unstable periodic orbits for a given system. Sometimes bifurcation and chaotic behaviour are really undesirable phenomena in discrete dynamical systems, because there may be an extinction of population due to chaos. So controlling chaos is an important issue. There are different methods for controlling chaos, e.g., feedback control strategy, hybrid control technique and pole-placement method. By applying these methods, one can retard or remove the chaotic behaviour due to appearance of bifurcation in the dynamical systems and rebuilt the stability of the system. In this section, we use mainly the state feedback control technique [13] to stabilize a chaotic orbit at an unstable fixed point of system (2.1).

Consider the following controlled system related to (2.1):

$$\begin{cases} x_{n+1} = x_n\left[\dfrac{r}{(b+cx_n)(1+ky_n)} - \alpha - \beta x_n - \dfrac{py_n}{1+hx_n}\right] - u(x_n, y_n), \\ y_{n+1} = y_n\left[-d + \dfrac{pqx_n}{1+hx_n}\right] \end{cases} \tag{4.1}$$

where $u(x_n, y_n) = c_1(x_n - x^*) + c_2(y_n - y^*)$ is a feedback controlling force with c_1 and c_2 being the feedback gains and (x^*, y^*) being the unique fixed point of system (2.1). The Jacobian matrix of system (4.1) evaluated at (x^*, y^*) is given by

$$J(x^*, y^*) = \begin{pmatrix} m_{11} - c_1 & m_{12} - c_2 \\ m_{21} & m_{22} \end{pmatrix}$$

where

$$m_{11} = \frac{br}{(b+cx^*)^2(1+ky^*)} - \alpha - 2\beta x^* - \frac{py^*}{(1+hx^*)^2},$$

$$m_{12} = -x^*\left(\frac{rk}{(b+cx^*)(1+ky^*)^2} + \frac{p}{1+hx^*}\right),$$

$$m_{21} = \frac{pqy^*}{(1+hx^*)^2},$$

$$m_{22} = 1.$$

The characteristic equation of the variational matrix $J(x^*, y^*)$ is

$$\lambda^2 - (m_{11} + m_{22} - c_1)\lambda + m_{22}(m_{11} - c_1) - m_{21}(m_{12} - c_2) = 0 \tag{4.2}$$

Suppose λ_1 and λ_2 are the roots of equation (4.2), then we have

$$\lambda_1 + \lambda_2 = m_{11} + m_{22} - c_1, \quad \lambda_1\lambda_2 = m_{22}(m_{11} - c_1) - m_{21}(m_{12} - c_2). \tag{4.3}$$

The lines of marginal stability can be derived by the equations $\lambda_1 = \pm 1$ and $\lambda_1\lambda_2 = 1$. These restrictions ensure that the eigenvalues λ_1 and λ_2 have moduli equal to 1.

First suppose that $\lambda_1\lambda_2 = 1$. Then from (4.3), we find

$$l_1: \quad c_1m_{22} - c_2m_{21} = m_{11}m_{22} - m_{12}m_{21} - 1.$$

Next assume that $\lambda_1 = 1$. Then from (4.3), we obtain

$$l_2: \quad c_1(1 - m_{22}) + c_2m_{21} = m_{11} + m_{22} - m_{11}m_{22} + m_{12}m_{21}.$$

Lastly, assume that $\lambda_1 = -1$ and from. (4.3), we get

$$l_3: \quad c_1(1 + m_{22}) - c_2m_{21} = m_{11} + m_{22} + 1 + m_{11}m_{22} - m_{12}m_{21}.$$

The stable eigenvalues lie within a triangular region bounded by the lines l_1, l_2 and l_3 in the c_1-c_2 plane.

5. Numerical Simulation

In this section, we present some numerical simulation to illustrate the usefulness of the obtained results as well as for giving direction to find desirable bifurcations and chaos of the discrete time system (2.1).

In Fig. 1, we select the parameter values $r = 4.5, k = 1, b = 1, c = 1, p = 4, q = 1, h = 1, \alpha = 0.1, \beta = 0.1$. We draw the bifurcation diagram with respect to the parameter d in the interval (1.5, 2.8). As d increases, we observe a transition phase from stability to bifurcation within a limit cycle, to a periodic window and ultimately to chaos.

In Fig. 2, we select the parameter values $r = 4.5, k = 1, b = 1, c = 1, p = 4, q = 1, h = 1, \alpha = 0.1, \beta = 0.01$. Here β is decreased from 0.1 to 0.01 from previously chosen parameters in Fig. 1. We draw the bifurcation diagram with respect to the parameter d in the interval (1.5, 2.8). As d increases, we observe a transition phase from stability to bifurcation within a limit cycle, to a periodic window and ultimately to chaos. Here we observe flip bifurcation.

In Fig. 3, we select the parameter values $r = 4.5, b = 1, c = 1, p = 1.8, q = 1, h = 1, \alpha = 0.1, \beta = 0.01$. We draw the bifurcation diagram with respect to the parameter k in the interval (2.1, 2.8). As k increases, we observe a transition phase from stability to bifurcation within a limit cycle, to a periodic window and ultimately to chaos.

In Fig. 4, we select the parameter values $r = 4.5, k = 1, b = 1, c = 1, p = 4, q = 1, h = 1, d = 1, \alpha = 0.1, \beta = 0.01$. and the initial value is (0.1, 0.1). With the above choice of parameters, we find chaotic behaviour of the system. To avoid the chaotic dynamics, feedback gains $c_1 = 0.3$ and $c_2 = -1.2$. are chosen. The chaotic orbit is stabilized at the fixed point (1, 0.306246).

6. Discussion

In this article, we have studied the qualitative behaviour of a discrete predator-prey model with fear effect and density dependent birth rate of the prey species. The predator functional response is taken as Holling type II. Prey's birth rate is assumed to be as Beverton-Holt type function [18]. We have mainly identified the system parameters that affect the dynamics of the system. We have observed two boundary fixed points and a unique interior fixed point. Stability analysis of these fixed points is examined by Jury technique. The criterion for Neimark-Sacker bifurcation and flip bifurcation are used for examining the bifurcations around the positive fixed point. It is identified that the parameter d, the death rate of predator in the system is more relevant for the appearance of flip bifurcation and Neimark-Sacker bifurcation whenever it is varied in some appropriate interval. We have also found Neimark-Sacker bifurcation by varying the parameter k. In investigating bifurcation, we have noted
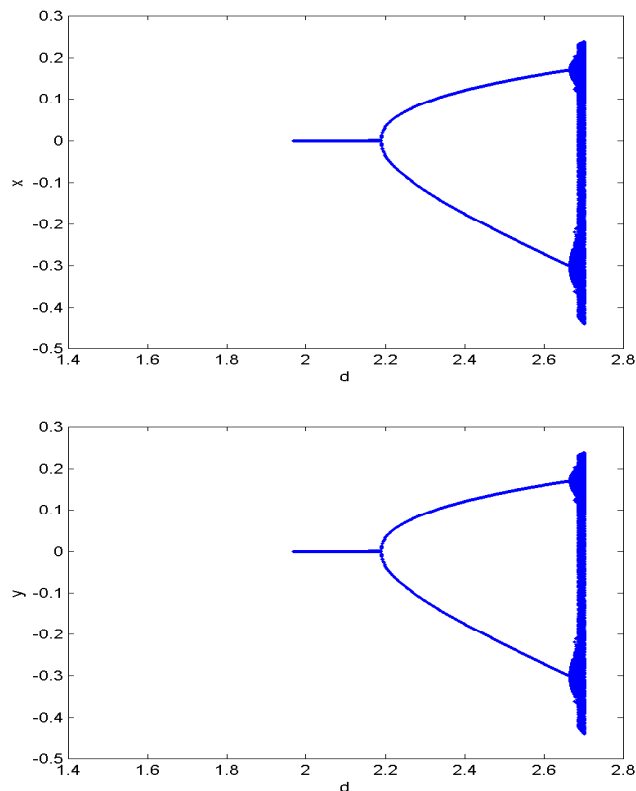
FIGURE 1.   Bifurcation diagram for prey and predator populations with d for fixed
values $r = 4.5, k = 1, b = 1, c = 1, p = 4, q = 1, h = 1, \alpha = 0.1$ and $\beta = 0.1$.

that β, the intraspecific competition among the prey species has an important role. It is checked that
smaller values of β may result flip bifurcation while larger values for β may result for Neimark-Sacker
bifurcation. In [1], the authors studied system (2.1) with $b = 1$, and $c = 0$ and remarked that the
cost of fear affect the existence of Hopf bifurcation as well as the direction of Hopf bifurcation in the
continuous model. But in our discrete time model (2.1), we observed Neimark-Sacker bifurcation and
chaotic behaviour of the system varying the fear factor. Recently, Kundu et al. [22] analysed similar
type of system with $b = 1, c = 0$ and $h = 0$ without obtaining different type of bifurcations and they
also observed that the system with fear effect becomes stable from chaotic dynamics by increasing fear
factor which is not so in our system (see Fig. 3).

The conditions of Proposition 1, shows that when the intrinsic growth rate of prey lies in a certain
interval and the death rate of predator remains below a certain threshold value, both the populations
go to extinction. If the restrictions of Proposition 2 are satisfied, the predator population goes to
extinction while prey population can sustain there. The stable coexistence of all the populations are
possible when all the conditions of Proposition 3 hold. The conditions of Proposition 4 suggests that
Neimark-Sacker bifurcation is possible for system (2.1). But it is difficult to interpret biologically these
conditions. Numerical simulations indicates that E^* is stable for $d < 2.2$ and loses its stability at
$d = 2.2$ and the system undergoes Neimark-Sacker bifurcation when the death rate d exceeds the value
2.2 (see Fig. 1). We have observed that when intraspecific competition among the prey species is low
and the death rate of the predator exceeds the value 2.05, system admits flip bifurcation (see Fig. 2)
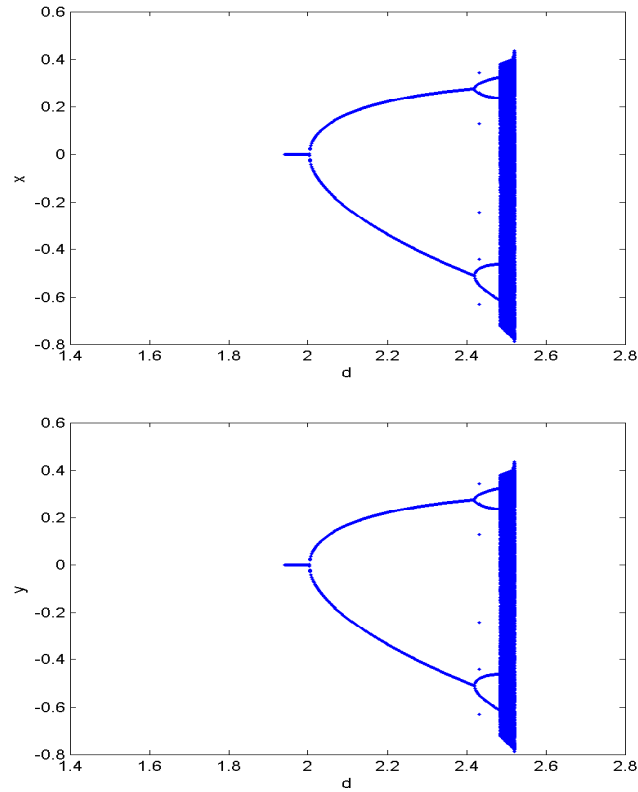
FIGURE 2. Bifurcation diagram for prey and predator populations with d for fixed values $r = 4.5, k = 1, b = 1, c = 1, p = 4, q = 1, h = 1, \alpha = 0.1$ and $\beta = 0.01$.

follows from the Proposition 5. We also note that when the fear factor k crosses the critical value 2.15, the system undergoes Neimark-Sacker bifurcation (see Fig. 3).

The chaotic nature of the system is nicely controlled by the state feedback control strategy (see Fig. 4). Numerical simulation exhibits that feedback control mechanism can dominate chaos to unstable fixed point strongly and ultimately stability of the system is achieved.
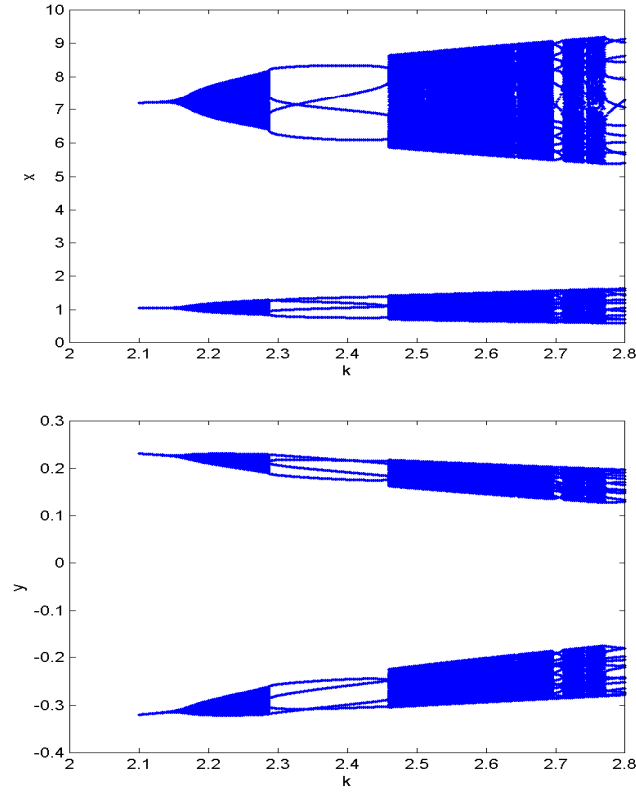
FIGURE 3. Bifurcation diagram for prey and predator populations with k for fixed values $r = 4.5, d = 1.6, b = 1, c = 1, p = 1.8, q = 1, h = 1, \alpha = 0.1$ and $\beta = 0.01$.
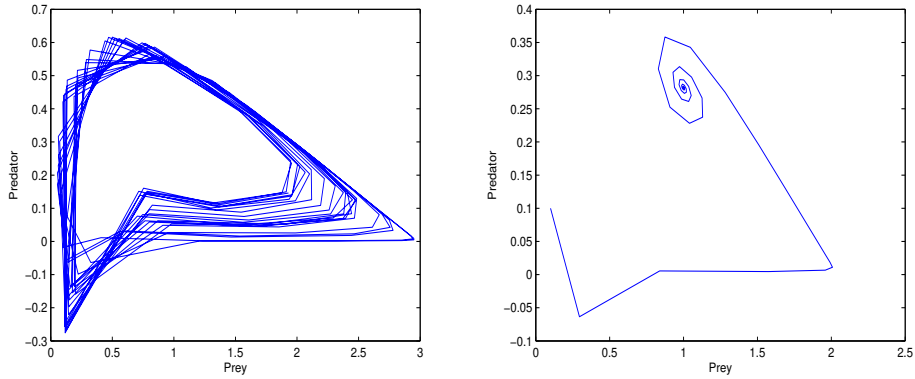

FIGURE 4. Phase diagram of system (2.1) for $r = 4.5, k = 1, b = 1, c = 1, d = 1, p = 4, q = 1, h = 1, \alpha = 0.1$ and $\beta = 0.01$. with initial values $(x_0, y_0) = (0.1, 0.1)$ in the left panel and controlled system (4) for $c_1 = 0.3$ and $c_2 = -1.2$ in the right panel.

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DEPARTMENT OF MATHEMATICS, VIVEKANANDA COLLEGE, THAKURPUKUR, KOLKATA-700063, INDIA.

Email address: mukherjee1961@gmail.com