

Existence and Uniqueness of Fixed Points in Cone Metric Spaces for ω -distance

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Abstrak. Di dalam penelitian ini, fungsi jarak yang digunakan adalah fungsi jarak- ω . Beberapa teorema, pembuktian teorema, dan contoh tentang ruang metrik *cone* dibahas pada penelitian ini. Jika diberikan ruang metrik *cone* lengkap dengan fungsi jarak- ω , *cone* normal P di X , fungsi kontraksi $f: X \rightarrow X$ dengan bentuk $\omega(f(x), f(y)) \leq \alpha\omega(x, y) + \beta[\omega(x, f(x)) + \omega(y, f(y))] + \gamma[\omega(x, f(y)) + \omega(y, f(x))]$ untuk setiap $x, y \in X$, dan α, β, γ adalah bilangan real tak negatif dimana $\alpha + 2\beta + 2\gamma < 1$, maka fungsi f memiliki titik tetap tunggal di X .


Abstract. In this study, the distance function used is the distance function- ω . Several theorems, proof of theorem, and example of cone metric space are discussed in this study. If given a complete cone metric space with distance function, the cone is normal at contraction function $f: X \rightarrow X$ with $\omega(f(x), f(y)) \leq \alpha\omega(x, y) + \beta[\omega(x, f(x)) + \omega(y, f(y))] + \gamma[\omega(x, f(y)) + \omega(y, f(x))]$ for every $x, y \in X$, and α, β, γ is a non-negative real number where $\alpha + 2\beta + 2\gamma < 1$, then function f has unique fixed point at X .

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1. Introduction

Fixed point has many useful for solving linear equation, ordinary differential equation, partial differential equation, intergral equation. The famous fixed point theorem is Banach fixed point theorem. According to [1], the Banach fixed point guarantee the existence and uniqueness of fixed point for function in complete space and contractive function.

In this paper we discuss some fixed point theorems in cone metric space with ω -distance. According to [2], cone metric space is generalization of metric space. Range of cone metric space is Banach space.

We use real Banach space and ω -distance for mertic. According to [3], a ω -distance is a function in metric spaces with three condition: symmetry, lower semicontinuous function, and relationship ω -distance with metric itself.

2. Preliminaries

Definition 1. [4] Let V is non-empty set with addition and scalar (real number) multiplication operation. Addition operation: $\bar{u}, \bar{v} \in V, \bar{u} + \bar{v} \in V$; and scalar multiplication operation: $k \in \mathbb{R}, \bar{u} \in V, k\bar{u} \in V$. V is called vector space if satisfy

- V1. $\forall \bar{u}, \bar{v} \in V, \bar{u} + \bar{v} \in V$;
- V2. $\forall \bar{u}, \bar{v}, \bar{w} \in V, (\bar{u} + \bar{v}) + \bar{w} = \bar{u} + (\bar{v} + \bar{w})$;
- V3. $\exists \bar{0} \in V, \forall \bar{u} \in V, \bar{0} + \bar{u} = \bar{u} + \bar{0} = \bar{u}$;
- V4. $\forall \bar{u} \in V, \exists -\bar{u} \in V, \bar{u} + (-\bar{u}) = -\bar{u} + \bar{u} = \bar{0}$;
- V5. $\forall \bar{u}, \bar{v} \in V, \bar{u} + \bar{v} = \bar{v} + \bar{u}$;
- V6. $\forall k \in \mathbb{R}, \forall \bar{u} \in V, k\bar{u} \in V$;
- V7. $\forall k \in \mathbb{R}, \forall \bar{u}, \bar{v} \in V, k(\bar{u} + \bar{v}) = k\bar{u} + k\bar{v}$;
- V8. $\forall k, h \in \mathbb{R}, \forall \bar{u} \in V, (k + h)\bar{u} = k\bar{u} + h\bar{u}$;
- V9. $\forall k, h \in \mathbb{R}, \forall \bar{u} \in V, k(h\bar{u}) = kh(\bar{u})$;
- V10. $\exists 1 \in \mathbb{R}, \forall \bar{u} \in V, 1 \cdot \bar{u} = \bar{u}$.

Definition 2. [5] Let V is vector space over the field \mathbb{F} . Function $\|\cdot\|: V \rightarrow \mathbb{R}$ is called norm of V if satisfy

- N1. $\|x\| \geq 0$ for all $x \in V$;
- N2. If $x \in V$ dan $\|x\| = 0$ then $x = 0$;
- N3. $\|\alpha x\| = |\alpha| \|x\|$ for all $x \in V$ and $\alpha \in \mathbb{F}$;
- N4. $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in V$.

A normed space is a vector space V together with a norm $\|\cdot\|$.

Definition 3. [6] All complete normed vector space is called Banach space.

Definition 4. [7] A metric on a set X is a function $d: X \times X \rightarrow \mathbb{R}$ that satisfies the following properties:

- M1. $d(x, y) \geq 0$ for all $x, y \in X$. (positivity)
 - M2. $d(x, y) = 0$ if and only if $x = y$. (definiteness)
 - M3. $d(x, y) = d(y, x)$ for all $x, y \in X$. (symmetry)
 - M4. $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$. (triangle inequality)
- A metric space (X, d) is a set X together with a metric d on X .

Definition 5. [8] Let \mathbb{E} is real Banach space and P a subset of \mathbb{E} . A set P is called a cone if only if:

- i P is closed, nonempty, and $P \neq 0$;
- ii $ax + by \in P$ for all $x, y \in P$ dan $a, b \in \mathbb{R}^+ \cup \{0\}$;

iii $P \cap (-P) = 0$.

Further, if $P \subseteq \mathbb{E}$ is cone, then we define partial ordering “ \preceq ” with respect to P by $x \preceq y$ if only if $y - x \in P$. And then $x < y$ is interpreted $x \preceq y$ and $x \neq y$. Whereas $x < y$ is interpreted $y - x \in \text{int } P$ (interior of P).

Definition 6. [9] The cone P is called normal if there is a number $M > 0$ such that for all $x, y \in \mathbb{E}$, $0 \leq x \leq y$ implies

$$\|x\| \leq M\|y\|. \tag{1}$$

The least positive M satisfying (1) is called the normal constant of P .

Definition 7. [2] A cone metric on non-empty set X is a function $d: X \times X \rightarrow \mathbb{E}$ that satisfies the following properties:

C1. $0 < d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if only if $x = y$;

C2. $d(x, y) = d(y, x)$ for all $x, y \in X$;

C3. $d(x, y) \leq d(x, z) + d(z, y)$ for all x, y dan $z \in X$.

A cone metric spaces (X, d) is a set X together with a metric d on X .

Definition 8. [10] Let (X, d) be a cone metric space, $\langle x_n \rangle$ be a sequence in X . If for any $c \in \mathbb{E}$ with $0 < c$, there is N such that for all $n > N$, $d(x, x_n) < c$, then $\langle x_n \rangle$ is called a convergent sequence to a point $x \in X$.

Definition 9. [2] Let (X, d) be a cone metric space, $\langle x_n \rangle$ be a sequence in X . If for any $c \in \mathbb{E}$ with $0 < c$, there is N such that for all $m, n > N$, $d(x_n, x_m) < c$, then $\langle x_n \rangle$ is called a Cauchy sequence in X .

Definition 10. [11] Let (X, d) be a cone metric space, $\langle x_n \rangle$ be a sequence in X . (X, d) is a complete cone metric space if every Cauchy sequence is convergent.

Definition 11. [12] A function ψ from a metric space (X, d) to \mathbb{R} is called lower semicontinuous if for every $y \in X$,

$$\liminf_{x \rightarrow y} \psi(x) \geq \psi(y).$$

Example 1. Let a function

$$f(x) = \begin{cases} 2, & x > 2 \\ 1, & x \leq 2 \end{cases}$$

The function $f(x)$ is lower semicontinuous at $x = 2$.

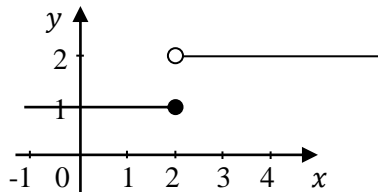


Figure 1. Example of lower semicontinuous function

Definition 12. [13] Let (X, d) be a metric space and a function $f: X \rightarrow X$. A point $x \in X$ is called a fixed point of function f if $x = f(x)$.

Definition 13. [14] Let metric space (X, d) . Function $f: X \rightarrow X$ is said contraction function if there is a real number c where $0 \leq c < 1$ such that:

$$d(f(x), f(y)) \leq cd(x, y), \forall x, y \in X.$$

Theorem 1. [15] (Banach’s Fixed Point). Let (X, d) is complete metric space. If function $f: X \rightarrow X$ is contraction function in X , then function f has unique fixed point.

Defintion 14. [16] A function $\omega: X \times X \rightarrow [0, \infty)$ is a ω -distance on X if it satisfies the following conditions for any $x, y, z \in X$:

$$\omega_1. \omega(x, z) \leq \omega(x, y) + \omega(y, z)$$

$\omega_2.$ The function $\omega(x, \cdot): X \rightarrow [0, \infty)$ is lower semicontinuous

$\omega_3.$ For any $\varepsilon > 0$, there exists $\delta > 0$ such that $\omega(z, x) \leq \delta$ and $\omega(z, y) \leq \delta$ imply $d(x, y) \leq \varepsilon$.

Of course, the metric d is a ω -distance on X .

3. Result and Discussion

Theorem 2. Let complete cone metric space (X, d) with ω -distance. Let normal cone P in X and function $f: X \rightarrow X$ that satisfy

$$\begin{aligned} \omega(f(x), f(y)) \leq & \alpha\omega(x, y) + \beta[\omega(x, f(x)) + \omega(y, f(y))] \\ & + \gamma[\omega(x, f(y)) + \omega(y, f(x))] \end{aligned} \quad (2)$$

for all $x, y \in X$ where α, β, γ are non-negative real numbers such that $\alpha + 2\beta + 2\gamma < 1$, then f has unique fixed point in X .

Proof:

Let a sequence $\langle x_n \rangle$ in cone metric space (X, d) . A sequence $\langle x_n \rangle$ satisfies this property:

$$x_n = f(x_{n-1}) = f(f(x_{n-2})) = f^2(x_{n-2}) = \dots = f^n(x_0)$$

where $n \in \mathbb{N}$.

According (2), we get

$$\begin{aligned} \omega(f(x_0), f^2(x_0)) & \leq \alpha\omega(x_0, f(x_0)) + \beta[\omega(x_0, f(x_0)) + \omega(f(x_0), f^2(x_0))] \\ & \quad + \gamma[\omega(x_0, f^2(x_0)) + \omega(f(x_0), f(x_0))] \\ & = \alpha\omega(x_0, f(x_0)) + \beta\omega(x_0, f(x_0)) + \beta\omega(f(x_0), f^2(x_0)) \\ & \quad + \gamma\omega(x_0, f^2(x_0)) + \gamma\omega(f(x_0), f(x_0)) \\ & = \alpha\omega(x_0, f(x_0)) + \beta\omega(x_0, f(x_0)) + \beta\omega(f(x_0), f^2(x_0)) \\ & \quad + \gamma\omega(x_0, f^2(x_0)) + 0 \\ & = \alpha\omega(x_0, f(x_0)) + \beta\omega(x_0, f(x_0)) + \beta\omega(f(x_0), f^2(x_0)) \\ & \quad + \gamma\omega(x_0, f^2(x_0)). \end{aligned} \quad (3)$$

We use triangle inequality and we get

$$\omega(x_0, f^2(x_0)) \leq \omega(x_0, f(x_0)) + \omega(f(x_0), f^2(x_0))$$

so that (3) become

$$\begin{aligned} \omega(f(x_0), f^2(x_0)) & \leq \alpha\omega(x_0, f(x_0)) + \beta\omega(x_0, f(x_0)) + \beta\omega(f(x_0), f^2(x_0)) \\ & \quad + \gamma\omega(x_0, f^2(x_0)) \\ & \leq \alpha\omega(x_0, f(x_0)) + \beta\omega(x_0, f(x_0)) + \beta\omega(f(x_0), f^2(x_0)) \\ & \quad + \gamma[\omega(x_0, f(x_0)) + \omega(f(x_0), f^2(x_0))] \\ & = \alpha\omega(x_0, f(x_0)) + \beta\omega(x_0, f(x_0)) + \beta\omega(f(x_0), f^2(x_0)) \\ & \quad + \gamma\omega(x_0, f(x_0)) + \gamma\omega(f(x_0), f^2(x_0)) \\ & = \alpha\omega(x_0, f(x_0)) + \beta\omega(x_0, f(x_0)) + \gamma\omega(x_0, f(x_0)) \\ & \quad + \beta\omega(f(x_0), f^2(x_0)) + \gamma\omega(f(x_0), f^2(x_0)) \\ & = (\alpha + \beta + \gamma)\omega(x_0, f(x_0)) + (\beta + \gamma)\omega(f(x_0), f^2(x_0)) \end{aligned}$$

or we can write

$$\begin{aligned} \omega(f(x_0), f^2(x_0)) & \leq (\alpha + \beta + \gamma)\omega(x_0, f(x_0)) \\ & \quad + (\beta + \gamma)\omega(f(x_0), f^2(x_0)) \\ \omega(f(x_0), f^2(x_0)) - (\beta + \gamma)\omega(f(x_0), f^2(x_0)) & \leq (\alpha + \beta + \gamma)\omega(x_0, f(x_0)) \\ [1 - (\beta + \gamma)]\omega(f(x_0), f^2(x_0)) & \leq (\alpha + \beta + \gamma)\omega(x_0, f(x_0)) \end{aligned}$$

$$\omega(f(x_0), f^2(x_0)) \leq \frac{\alpha + \beta + \gamma}{1 - (\beta + \gamma)} \omega(x_0, f(x_0)).$$

Suppose $K = \frac{\alpha + \beta + \gamma}{1 - (\beta + \gamma)}$ according to definition 8 about contraction function, then

$$\omega(f(x_0), f^2(x_0)) \leq K\omega(x_0, f(x_0)) \tag{4}$$

where $0 \leq K = \frac{\alpha + \beta + \gamma}{1 - (\beta + \gamma)} < 1$.

According (4), we get

$$\omega(f^n(x_0), f^{n+1}(x_0)) \leq K\omega(f^{n-1}(x_0), f^n(x_0)) \leq \dots \leq K^n\omega(x_0, f(x_0))$$

where $0 \leq K < 1$. Let $m > n \geq N \in \mathbb{N}$, according to triangle inequality then

$$\begin{aligned} \omega(f^n(x_0), f^m(x_0)) &\leq \omega(f^n(x_0), f^{n+1}(x_0)) + \omega(f^{n+1}(x_0), f^{n+2}(x_0)) + \dots \\ &\quad + \omega(f^{m-2}(x_0), f^{m-1}(x_0)) + \omega(f^{m-1}(x_0), f^m(x_0)) \\ &\leq K^n\omega(x_0, f(x_0)) + K^{n+1}\omega(x_0, f(x_0)) + \dots \\ &\quad + K^{m-2}\omega(x_0, f(x_0)) + K^{m-1}\omega(x_0, f(x_0)) \\ &= (K^n + K^{n+1} + \dots + K^{m-2} + K^{m-1})\omega(x_0, f(x_0)) \\ &= K^n(1 + K + K^2 + \dots + K^{m-n-1})\omega(x_0, f(x_0)) \\ &= K^n \left(\sum_{i=0}^{m-n-1} K^i \right) \omega(x_0, f(x_0)) \\ &\leq K^n \left(\sum_{i=0}^{\infty} K^i \right) \omega(x_0, f(x_0)) \\ &= K^n \left(\frac{1}{1-K} \right) \omega(x_0, f(x_0)) \\ &= \left(\frac{K^n}{1-K} \right) \omega(x_0, f(x_0)). \end{aligned}$$

Suppose $\omega(x_0, f(x_0)) = c$, then

$$\omega(f^n(x_0), f^m(x_0)) \leq \left(\frac{K^n}{1-K} \right) \omega(x_0, f(x_0)) = \frac{cK^n}{1-K}.$$

Choose $N \in \mathbb{N}$ with

$$N < {}^K \log \frac{\varepsilon(1-K)}{c}$$

such that for all $m, n \geq N$, we get

$$\omega(f^n(x_0), f^m(x_0)) \leq \frac{cK^n}{1-K} \leq \frac{cK^N}{1-K} < \frac{c}{1-K} \cdot \frac{\varepsilon(1-K)}{c} = \varepsilon.$$

So, sequence $\langle x_n \rangle$ is Cauchy sequence.

Because space (X, d) is complete cone metric space, then sequence $\langle x_n \rangle$ is convergent to point $x \in X$ or we can write $\langle x_n \rangle \rightarrow x$. And then, we will proof that point x is fixed point of function f . According to triangle inequality and (2), then

$$\begin{aligned} \omega(f(x), x) &\leq \omega(f(x), f(x_n)) + \omega(f(x_n), x) \\ &\leq \alpha\omega(x, x_n) + \beta[\omega(x, f(x)) + \omega(x_n, f(x_n))] \\ &\quad + \gamma[\omega(x, f(x_n)) + \omega(x_n, f(x))] + \omega(f(x_n), x) \\ &= \alpha\omega(x, x_n) + \beta\omega(x, f(x)) + \beta\omega(x_n, f(x_n)) \\ &\quad + \gamma\omega(x, f(x_n)) + \gamma\omega(x_n, f(x)) + \omega(f(x_n), x) \end{aligned}$$

or we can write

$$\begin{aligned} \omega(f(x), x) &\leq \alpha\omega(x, x_n) + \beta\omega(x, f(x)) + \beta\omega(x_n, f(x_n)) \\ &\quad + \gamma\omega(x, f(x_n)) + \gamma\omega(x_n, f(x)) + \omega(f(x_n), x) \\ \omega(f(x), x) &\leq \alpha\omega(x, x_n) + \beta\omega(f(x), x) + \beta\omega(x_n, f(x_n)) \\ &\quad + \gamma\omega(x, f(x_n)) + \gamma\omega(x_n, f(x)) + \omega(f(x_n), x) \end{aligned}$$

$$\begin{aligned}
 \omega(f(x), x) - \beta\omega(f(x), x) &\leq \alpha\omega(x, x_n) + \beta\omega(x_n, f(x_n)) + \gamma\omega(x, f(x_n)) \\
 &\quad + \gamma\omega(x_n, f(x)) + \omega(f(x_n), x) \\
 (1 - \beta)\omega(f(x), x) &\leq \alpha\omega(x, x_n) + \beta\omega(x_n, f(x_n)) + \gamma\omega(x, f(x_n)) \\
 &\quad + \gamma\omega(x_n, f(x)) + \omega(f(x_n), x) \\
 (1 - \beta)\omega(f(x), x) &\leq \alpha\omega(x, x_n) + \beta\omega(x_n, f(x_n)) + \gamma\omega(x_n, f(x)) \\
 &\quad + \gamma\omega(x, f(x_n)) + \omega(f(x_n), x) \\
 (1 - \beta)\omega(f(x), x) &\leq \alpha\omega(x, x_n) + \beta\omega(x_n, f(x_n)) + \gamma\omega(x_n, f(x)) \\
 &\quad + \gamma\omega(f(x_n), x) + \omega(f(x_n), x) \\
 (1 - \beta)\omega(f(x), x) &\leq \alpha\omega(x, x_n) + \beta\omega(x_n, f(x_n)) + \gamma\omega(x_n, f(x)) \\
 &\quad + (\gamma + 1)\omega(f(x_n), x) \\
 \omega(f(x), x) &\leq \frac{\alpha}{(1 - \beta)}\omega(x, x_n) + \frac{\beta}{(1 - \beta)}\omega(x_n, f(x_n)) \\
 &\quad + \frac{\gamma}{(1 - \beta)}\omega(x_n, f(x)) + \frac{(\gamma + 1)}{(1 - \beta)}\omega(f(x_n), x) \\
 \omega(f(x), x) &\leq \frac{\alpha}{(1 - \beta)}\omega(x, x_n) + \frac{\beta}{(1 - \beta)}\omega(x_n, x_{n+1}) \\
 &\quad + \frac{\gamma}{(1 - \beta)}\omega(x_n, f(x)) + \frac{(\gamma + 1)}{(1 - \beta)}\omega(x_{n+1}, x)
 \end{aligned}$$

because sequence $\langle x_n \rangle$ is convergent for $n \rightarrow \infty$, then:

$$\begin{aligned}
 \omega(f(x), x) &\leq \frac{\alpha}{(1 - \beta)}\omega(x, x_n) + \frac{\beta}{(1 - \beta)}\omega(x_n, x_{n+1}) \\
 &\quad + \frac{\gamma}{(1 - \beta)}\omega(x_n, f(x)) + \frac{(\gamma + 1)}{(1 - \beta)}\omega(x_{n+1}, x) \\
 &= \frac{\alpha}{(1 - \beta)}(0) + \frac{\beta}{(1 - \beta)}(0) + \frac{\gamma}{(1 - \beta)}(0) \\
 &\quad + \frac{(\gamma + 1)}{(1 - \beta)}(0) \\
 &= 0.
 \end{aligned} \tag{5}$$

Because $0 \leq \omega(f(x), x)$, according (5) then

$$\omega(f(x), x) = 0,$$

or we can write $x = f(x)$. So, point x is fixed point of function f .

Furthermore, we will proof uniqueness of a fixed point. Suppose x and y are fixed points of function f , such that

$$x = f(x) \text{ and } y = f(y)$$

then

$$\begin{aligned}
 \omega(x, y) &= \omega(f(x), f(y)) \\
 &\leq \alpha\omega(x, y) + \beta[\omega(x, f(x)) + \omega(y, f(y))] + \gamma[\omega(x, f(y)) + \omega(y, f(x))] \\
 &= \alpha\omega(x, y) + \beta[\omega(x, x) + \omega(y, y)] + \gamma[\omega(x, y) + \omega(y, x)] \\
 &= \alpha\omega(x, y) + \beta[0 + 0] + \gamma[\omega(x, y) + \omega(y, x)] \\
 &= \alpha\omega(x, y) + \beta[0] + \gamma[\omega(x, y) + \omega(y, x)] \\
 &= \alpha\omega(x, y) + 0 + \gamma[\omega(x, y) + \omega(y, x)] \\
 &= \alpha\omega(x, y) + \gamma[\omega(x, y) + \omega(y, x)] \\
 &= \alpha\omega(x, y) + \gamma\omega(x, y) + \gamma\omega(y, x) \\
 &= \alpha\omega(x, y) + \gamma\omega(x, y) + \gamma\omega(x, y) \\
 &= \alpha\omega(x, y) + 2\gamma\omega(x, y) \\
 &= (\alpha + 2\gamma)\omega(x, y)
 \end{aligned}$$

because $\alpha + 2\beta + 2\gamma < 1$ and $\alpha + 2\gamma < 1$, so that

$$\omega(x, y) \leq (\alpha + 2\gamma)\omega(x, y)$$

is contradiction. So, point x is unique fixed point of function f in X .

Example 2. Let set $\mathbb{E} = \mathbb{R}^2$, $P = \{(x, y) \in \mathbb{E} : x, y \geq 0\} \subset \mathbb{R}^2$, $X = \mathbb{R}$, and $d: X \times X \rightarrow \mathbb{E}$ such that $d(x, y) = (|x - y|, \alpha|x - y|)$ where $\alpha \geq 0$ is a constant, then (X, d) is cone metric space.

Example 3. Let metric space (X, d) and function $\omega: X \times X \rightarrow [0, \infty)$. Function $\omega(x, y) = c$ for every $x, y \in X$ is ω -distance on X (c is positive real number). Function ω is not metric space because $\omega(x, x) = c \neq 0$ for any $x \in X$.

4. Conclusion

Cone metric space is generalization of metric space. Range of cone metric in cone metric space is Banach space. In this research, we use ω -distance for metrik. a ω -distance is a function in metric spaces with three condition: symmetry, lower semicontinuous function, and relationship ω -distance with metric itself.

Fixed point has many useful for solving linear equation, ordinary differential equation, partial differential equation, integral equation. The famous fixed-point theorem is Banach fixed point theorem. The Banach fixed point guarantee the existence and uniqueness of fixed point for function in complete space and contractive function. Let complete cone metric space with ω -distance. Let normal cone P in X and function $f: X \rightarrow X$ that satisfy $\omega(f(x), f(y)) \leq \alpha\omega(x, y) + \beta[\omega(x, f(x)) + \omega(y, f(y))] + \gamma[\omega(x, f(y)) + \omega(y, f(x))]$ for all $x, y \in X$ where α, β, γ are non negative real numbers such that $\alpha + 2\beta + 2\gamma < 1$, then f has unique fixed point in X .

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