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On the Semi-Local Convergence of a Third Order Scheme for Solving Nonlinear Equations

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Abstract. The semi-local convergence analysis of a third order scheme for solving nonlinear equation in Banach space has not been given under Lipschitz continuity or other conditions. Our goal is to extend the applicability of the Cordero-Torregrosa scheme in the semi-local convergence under conditions on the first Fréchet derivative of the operator involved. Majorizing sequences are used for proving our results. Numerical experiments testing the convergence criteria are given in this study.

1. Introduction

Cordero and Torregrosa in [10] considered the third order scheme, defined for $n = 0, 1, 2, \dots,$ by

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1}F(x_n) \\ x_{n+1} &= x_n - 3M_n^{-1}F(x_n), \end{aligned} \tag{1.1}$$

for solving the nonlinear equation

$$F(x) = 0, \tag{1.2}$$

where $M_n = 2F'\left(\frac{3x_n+y_n}{4}\right) - F'\left(\frac{x_n+y_n}{2}\right) + 2F'\left(\frac{x_n+3y_n}{4}\right)$. Here $F : D \subset E \longrightarrow E_1$ is an operator acting between Banach spaces E and E_1 with $D \neq \emptyset$. In general a closed form solution for (1.2) is not possible, so iterative schemes are used for approximating a solution x_* of (1.2) (see [1–27]).

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The local convergence of the this scheme in the special case when $E = E_1 = \mathbb{R}$ was shown to be of order three using Taylor expansion and assumptions on the fourth order derivative of F, which is not on these schemes [10]. So, the assumptions on the fourth derivative reduce the applicability of these schemes [1–27].

For example: Let $E = E_1 = \mathbb{R}$, $D = [-0.5, 1.5]$. Define λ on D by

$$\lambda(t) = \begin{cases} t^3 \log t^2 + t^5 - t^4 & if\ t \neq 0 \\ 0 & if\ t = 0. \end{cases}$$

Then, we get $f(1) = 0$, and

$$\lambda'''(t) = 6 \log t^2 + 60t^2 - 24t + 22.$$

Obviously $\lambda'''(t)$ is not bounded on D. So, the convergence of scheme (1.1) is not guaranteed by the previous analyses in [1–27].

In this study we introduce a majorant sequence and use general continuity conditions to extend the applicability of scheme (1.1). Our analysis includes error bounds and results on uniqueness of x_* based on computable Lipschitz constants not given before in [1–27] and in other similar studies using Taylor series. Our idea is very general. So, it applies on other schemes too.

The rest of the study is set up as follows: In Section 2 we present results on majorizing sequences. Sections 3,4 contain the semi-local and local convergence, respectively, where in Section 4 the numerical experiments are presented. Concluding remarks are given in the last Section 5.

2. Majorizing Sequences

Scalar sequences are developed that majorize scheme (1.1). Let $K_0 > 0, K > 0$ and $\eta > 0$ be given constants. Define sequences $\{t_n\}, \{s_n\}$ by

$$\begin{aligned} t_0 &= 0,\ s_0 = \eta \\ t_{n+1} &= s_n + \frac{2K(s_n - t_n)(t_{n+1} - t_n)}{9(1 - K_0 t_n)(1 - p_n)}, \\ s_{n+1} &= t_{n+1} + \frac{K(t_{n+1} - t_n + s_n - t_n)(t_{n+1} - t_n)}{2(1 - K_0 t_{n+1})}, \end{aligned} \tag{2.1}$$

where $p_n = \frac{5K_0}{6}(s_n + t_n)$. Notice that t_{n+1} is given implicitly in the first substep of sequence (2.1). It we solve for t_{n+1}, we get its explicit form

$$t_{n+1} = \frac{9s_n(1 - K_0 t_n)(1 - p_n) - 2t_n K(s_n - t_n)}{9(1 - K_0 t_n)(1 - p_n) - 2K(s_n - t_n)}.$$

But for the convergence analysis in Theorem 3.1 we prefer t_{n+1} in its implicit form.

Next, we present sufficient conditions for the convergence scheme (1.1).

LEMMA 2.1. *Suppose that*

$$5(t_n + s_n) < \frac{6}{K_0}. \tag{2.2}$$

for all $n = 0, 1, 2, \ldots$. Then, sequences $\{t_n\}$ is nondecreasing and bounded from above by $T^* = \frac{3}{5K_0}$ and as such it converge to its unique least upper $T \in [0, T^*]$.

Proof. It follows from the definition (2.1) of sequences $\{t_n\}$ and (2.2) that this sequence is nondecreasing and bounded from above by T^*, and as such it converges to T.

□

The next result shows the convergence of sequence $\{t_n\}$, under stronger but easier to verify conditions than (2.2). But first we need to introduce some functions and parameters. Define functions g_1 and g_2 on the interval $(0, 1)$ by

$$g_1(t) = 4K(1+t)t - 4K(1+t) + 9K_0 t,$$

and

$$g_2(t) = K(2+t)(1+t)t - K(2+t)(1+t) + 2K_0 t^3.$$

Then, we get $g_1(0) = -4K$, $g_1(1) = 9K_0$, $g_2(0) = -2K$ and $g_2(1) = 2K_0$.

Hence, functions g_1 and g_2 have roots in $(0, 1)$. Denote the minimal such roots by α_1 and α_2, respectively. Set $a = \frac{2K(t_1-t_0)}{9(1-K_0t)(1-p_0)}$, $b = \frac{K(t_1-t_0+s_0-t_0)(t_1-t_0)}{2\eta(1-K_0t_1)}$, $\bar{c} = \min\{a, b\}$, $c = \max\{a, b\}, \alpha_3 = \min\{\alpha_1, \alpha_2\}$ and $\alpha = \max\{\alpha_1, \alpha_2\}$.

Then, we can show the second result on majorizing sequences for method (1.2).

LEMMA 2.2. *Suppose*

$$0 < \bar{c} \le c \le \alpha_3 \le \alpha \le 1 - \frac{10}{3}K_0\eta. \tag{2.3}$$

Then, sequence $\{t_n\}$ is nondecreasing, bounded from above by $T = \frac{\eta}{1-\alpha}$ and as such it converges to its unique least upper bound $t^* \in [0, T]$.

Proof. Items

$$0 \le \frac{2K(t_{k+1}-t_k)}{9(1-K_0t_k)(1-p_k)} \le \alpha, \tag{2.4}$$

$$0 \le \frac{K(t_{k+1}-t_k+s_k-t_k)(t_{k+1}-t_k)}{2(1-K_0t_{k+1})} \le \alpha(s_k - t_k), \tag{2.5}$$

$$0 \le \frac{1}{1-p_k} \le 2 \tag{2.6}$$

and

$$t_k \le s_k \le t_{k+1} \tag{2.7}$$

are shown using induction on k. These estimates are true for $k = 0$ by (2.3). Suppose these hold for all k smaller than $n-1$. By induction hypotheses and (1.2), we have $0 \le s_k - t_k \le \alpha(s_{k-1} - t_{k-1}) \le \ldots \le \alpha^k \eta$,

$$t_{k+1} - t_k = (t_{k+1} - s_k) + (s_k - t_k) \le (1+\alpha)(s_k - t_k)$$

and

$$t_{k+1} \leq \frac{(1-\alpha^{k+2})\eta}{1-\alpha} < T.$$

Evidently, (2.4) holds if

$$\frac{4K(1+\alpha)\alpha^{k-1}\eta}{9(1-K_0\frac{1-\alpha^{k+1}}{1-\alpha}\eta} \leq \alpha, \tag{2.8}$$

where we used (2.6). Define recurrent polynomials $f_k^{(1)}$ on the interval $(0,1)$ by

$$f_n^{(1)}(t) = 4K(1+t)t^{k-1}\eta + 9K_0(1+t+\ldots+t^{k-1})\eta - 9. \tag{2.9}$$

Then, estimate (2.8) holds if

$$f_n^{(1)}(t) \leq 0 \text{ at } t = \alpha_1. \tag{2.10}$$

We need a relationship between two consecutive polynomials $f_k^{(1)}$:

$$\begin{aligned} f_{k+1}^{(1)}(t) &= 4K(1+t)t^k\eta + 3K_0(1+t+\ldots+t^k)\eta - 9 + f_k^{(1)}(t) \\ &\quad -4K(1+t)t^{k-1} + 3K_0(1+t+\ldots+t^{k-1})\eta + 9 \\ &= f_k^{(1)}(t) + g_1(t)t^{k-1}\eta. \end{aligned} \tag{2.11}$$

In particular, one gets $f_{k+1}^{(1)}(\alpha_1) = f_k^{(1)}(\alpha_1)$ since by the definition of α_1 and g_1, $g_1(\alpha_1) = 0$. Define function

$$f_\infty^{(1)}(t) = \lim_{k\longrightarrow\infty} f_k^{(1)}(t). \tag{2.12}$$

Then, (2.10) holds if

$$f_\infty^{(1)}(t) \leq 0 \text{ at } t = \alpha_1. \tag{2.13}$$

But by (2.9) and (2.12) one gets

$$f_\infty^{(1)}(t) = \frac{9K_0\eta}{1-t} - 9, \tag{2.14}$$

so (2.13) holds if $f_\infty^{(1)}(t) \leq 0$ at $t = \alpha_1$ which is true by (2.3).

Similarly, (2.5) holds if

$$\frac{K(2+\alpha)(1+\alpha)\alpha^k\eta}{2(1-K_0\frac{1-\alpha^{k+2}}{1-\alpha}\eta)} \leq \alpha. \tag{2.15}$$

Define polynomials $f_k^{(2)}(t)$ on the interval $(0,1)$ by

$$f_k^{(2)}(t) = K(2+t)(1+t)t^{k-1}\eta + 2K_0(1+t+\ldots+t^{k+1})\eta - 2. \tag{2.16}$$

Then, (2.15) holds if

$$f_k^{(2)}(t) \leq 0 \text{ at } t = \alpha_2. \tag{2.17}$$

We get

$$\begin{aligned} f_{k+1}^{(2)}(t) &= K(2+t)(1+t)t^k\eta + 2K_0(1+t+\ldots+t^{k+2})\eta - 2 + f_k^{(2)}(t) \\ &\quad -K(2+t)(1+t)t^{k-1}\eta - 2K_0(1+t+\ldots+t^{k+1})\eta + 2 \\ &= f_k^{(2)}(t) + g_2(t)t^{k-1}\eta, \end{aligned} \tag{2.18}$$

and

$$f_{k+1}^{(2)}(\alpha_2) = f_k^{(2)}(\alpha_2). \tag{2.19}$$

Define function

$$f_\infty^{(2)}(t) = \lim_{k\longrightarrow\infty} f_k^{(2)}(t). \tag{2.20}$$

Then, (2.17) holds if

$$f_\infty^{(2)}(t) \le 0 \text{ at } t = \alpha_2. \tag{2.21}$$

By (2.16) and (2.20), we get

$$f_\infty^{(2)}(t) = \frac{K_0\eta}{1-t} - 1,$$

so (2.21) holds by (2.3). Moreover, estimate (2.6) certainly holds if $2p_k = \frac{5K_0}{3}(s_k+t_k) < \frac{5K_0}{3}(\frac{\eta}{1-\alpha} + \frac{\eta}{1-\alpha}) = \frac{10K_0\eta}{3(1-\alpha)} < 1$, which is true by (2.3). Furthermore, estimate (2.7) holds by (2.4)-(2.6) and the definition of sequence $\{t_k\}$. Hence the induction for estimates (2.4)-(2.7) is completed. It follows that sequence $\{t_k\}$ is nondecreasing and bounded from above by T^*, and such it converges to T. $\square$

If one desires iterates to be given explicitly in (2.1), then define instead sequence $\{t_n\}$ as follows

$$\begin{aligned} t_0 &= 0,\ s_0 = \eta \\ t_{n+1} &= s_n + \frac{2K(1+K_0t_n)(s_n-t_n)^2}{3(1-K_0t_n)(1-p_n)} \\ s_{n+1} &= t_{n+1} + \frac{2K(t_{n+1}-t_n+s_n-t_n)(t_{n+1}-t_n)}{2(1-K_0t_{n+1})}. \end{aligned} \tag{2.22}$$

Moreover, define recurrent polynomial on the interval $[0,1)$ by

$$f_n^{(1)}(t) = \frac{4K}{3}t^{n-1}\eta + \frac{4KK_0}{3}t^{n-1}(1+t+\ldots+t^n)\eta^2 + K_0(1+t+\ldots+t^n)\eta - 1.$$

This time we have

$$f_{n+1}^{(1)}(t) = f_n^{(1)}(t) + g_n^{(1)}(t)t^{n-1}\eta, \tag{2.23}$$

where

$$g_n^{(1)}(t) = \frac{4KK_0}{3}t^{n+2}\eta + \frac{4KK_0}{3}t^{n+1}\eta + \frac{4}{3}Kt - \frac{4K}{3}(1-K_0\eta).$$

We get $g_n^{(1)}(0) = -\frac{4K}{3}(1 - K_0\eta) < 0$ for $K_0\eta < 1$, and $g_1^{(1)}(1) = 4KK_0\eta > 0$. Denote by r_n the smallest solution of $g_n^{(1)}(t)$,respectively. Notice that these solutions are increasing as n increases, since $g_n^{(1)}(t) \leq g_{n-1}^{(1)}(t)$. Hence, it follows by (2.23) that

$$f_{n+1}^{(1)}(t) \leq f_n^{(1)}(t) + g_1^{(1)}(t)t^{n-1}\eta.$$

In particular for $\alpha_1 = r_1$, we get

$$f_{n+1}^{(1)}(t) \leq f_n^{(1)}(t) \text{ at } t = \alpha_1.$$

Hence,

$$f_n^{(1)}(t) \leq 0$$

holds if

$$f_1^{(1)}(t) \leq 0 \text{ at } t = \alpha_1.$$

But

$$f_1^{(1)}(t) = \frac{4K}{3}\eta + \frac{4}{3}KK_0\eta^2 + K_0\eta - 1.$$

Define $b = \frac{2K(s_0 - t_0)}{3}$. Then, we arrive at the following convergence results for majorizing sequence (2.2).

LEMMA 2.3. *Suppose*

$$5(t_n + s_n) < \frac{6}{K_0},$$

where $\{t_n\}$ is the sequence defined by (2.22). Then, the conclusions of Lemma 2.2 hold for this sequence.

LEMMA 2.4. *Suppose*

$$0 < \bar{c} \leq c \leq \alpha_3 \leq \alpha \leq 1 - \frac{10K_0}{3}\eta \tag{2.24}$$

and

$$\left(\frac{4K}{3} + \frac{4}{3}K_0K\eta + K_0\right)\eta \leq 1. \tag{2.25}$$

Then, the conclusions of Lemma 2.2 hold for sequence $\{t_n\}$ given by (2.22).

REMARK 2.5. *The solutions α_1 and α_2 in Lemma 2.2 depend only on K_0 and K. Similarly α_2 in Lemma 2.4 depends on K_0 and K_1. But α_1 depends K_0, K and η. To avoid this dependence pick any $\gamma \in (0, 1]$ and set $\gamma = K_0\eta$. Define functions $\bar{g}_n^{(1)}(t)$ on $[0, 1)$ by*

$$\bar{g}_n^{(1)}(t) = \frac{4K\gamma}{3}t^{n+1} + \frac{4K\gamma}{3}t^{n=1} + \frac{4}{3}Kt - \frac{4K}{3}(1 - \gamma).$$

Then, according to the proof of Lemma 2.2 we can set $\alpha_1 = \bar{r}_1$, where $\bar{r}_1$ is the smallest solution in $(0, 1)$ of equation $\bar{g}_1^{(1)}(t) = 0$ assured also to exist. Finally, notice that the first condition shows implicitly and the second explicitly the smallness of η.

3. Semi-Local Convergence

The following sufficient convergence criteria (A) are used. Suppose:

(A1) There exist $x_0 \in D$ and $\eta > 0$ such that $F'(x_0)^{-1}$ exists and

$$\|F'(x_0)^{-1}F(x_0)\| \leq \eta.$$

(A2)

$$\|F'(x_0)^{-1}(F'(w) - F'(x_0))\| \leq K_0\|w - x_0\|$$

for all $w \in D$. Set $D_0 = D \cap U(x_0, \frac{1}{K_0})$.

(A3)

$$\|F'(x_0)^{-1}(F'(w) - F'(v)\| \leq K\|w - v\|$$

for all $v \in D_0$ and $w = v - F'(v)^{-1}F(v)$. Denote by L the constant, if (A3) holds for all $u, v \in D_0$, and by L_1 the constant for all $u, v \in D$. It follows that $K \leq L \leq L_1$. In practice we shall use whichever of K or L is easier to compute (see also the numerical section).

(A4) Hypotheses of Lemma 2.1 or Lemma 2.2 hold.

and

(A5) $U[x_0, t^*] \subset D$ (or $U[x_0, T] \subseteq D$).

Next, the semi-local convergence of scheme (1.1) is developed based on conditions (A) and the aforementioned notation.

THEOREM 3.1. *Suppose conditions (A) hold. Then, the following items hold*

$$\{x_n\} \in U(x_0, t^*) \tag{3.1}$$

and

$$\|x^* - x_n\| \leq t^* - t_n, \tag{3.2}$$

where $x^* = \lim_{n\longrightarrow\infty} x_n \in U[x_0, t^*]$ *and* $F(x^*) = 0$.

Proof. Mathematical induction is used to show

$$\|y_k - x_k\| \leq s_k - t_k \tag{3.3}$$

and

$$\|x_{k+1} - y_k\| \leq t_{k+1} - s_k. \tag{3.4}$$

It follows from (A1) and (1.1) that

$$\|y_0 - x_0\| = \|F'(x_0)^{-1}F(x_0) \leq \eta =\leq s_0 - t_0 = \eta \leq T, \tag{3.5}$$

so $y_0 \in U(x_0, t^*)$ and (3.3) hold for $k = 0$. Let $z \in U(x_0, t^*)$. In view of (A2), one has

$$\|F'(x_0)^{-1}(F'(z) - F'(x_0)) \leq K_0\|z - x_0\| \leq K_0t^* < 1,$$

so $F'(z)^{-1} \in L(E_1, E)$ and

$$\|F'(z)^{-1}F'(x_0)\| \leq \frac{1}{1-K_0\|z-x_0\|}. \tag{3.6}$$

by a result due to Banach [14] on linear invertible operators. Operator M_k can be shown to be invertible. Indeed, by the definition of operator M_k, (2.2) and (A2) we obtain

$$\begin{aligned}
\|(3F'(x_0))^{-1}(M_k - 3F'(x_0))\| &\leq \frac{1}{3}[2\|F'(x_0)^{-1}\left(F'\left(\frac{3x_k+y_k}{4}\right) - F'(x_0)\right)\| \\
&\quad +\|F'(x_0)^{-1}\left(F'\left(\frac{x_k+y_k}{2}\right) - F'(x_0)\right)\| \\
&\quad +2\|F'(x_0)^{-1}\left(F'\left(\frac{x_k+3y_k}{4}\right) - F'(x_0)\right) \\
&\leq \frac{1}{3}(2K_0\|\frac{3x_k+y_k}{4} - x_0\| + K_0\|\frac{x_k+y_k}{2} - x_0\| \\
&\quad +2K_0\|\frac{x_k+3y_k}{4} - x_0\|) \\
&\leq \frac{1}{3}(2K_0\frac{3t_k+s_k}{4} + K_0\frac{s_k+t_k}{2} + 2K_0\frac{t_k+3s_k}{4}) \\
&= \frac{5K_0}{6}(t_k+s_k) = p_k < 1,
\end{aligned}$$

so M_k is invertible and

$$\|M_k^{-1}F'(x_0)\| \leq \frac{1}{3(1-p_k)}, \tag{3.7}$$

and x_{k+1} is well defined by the second substep of method (1.1). Then, we can write by method (1.1) that

$$\begin{aligned}
x_{k+1} &= x_k - F'(x_k)^{-1}F(x_k) + (F'(x_k)^{-1} - 3M_k^{-1})F(x_k) \\
&= y_k - \frac{1}{3}F'(x_k)^{-1}(M_k - 3F'(x_k))M_k^{-1}(x_{k+1} - x_k).
\end{aligned} \tag{3.8}$$

We need the estimate,

$$\begin{aligned}
M_k - 3F'(x_k) &= 2F'\left(\frac{3x_k+y_k}{4}\right) - F'\left(\frac{x_k+y_k}{2}\right) \\
&\quad +2F'\left(\frac{x_k+3y_k}{4}\right) - 3F'(x_k) \\
&= \left(F'\left(\frac{3x_k+y_k}{4}\right) - F'\left(\frac{x_k+y_k}{2}\right)\right) \\
&\quad +\left(F'\left(\frac{3x_k+y_k}{4}\right) - F'(x_k)\right) + 2\left(F'\left(\frac{x_k+3y_k}{4}\right) - F'(x_k)\right),
\end{aligned}$$

so by (A3)

$$\begin{aligned}\|F'(x_0)^{-1}(M_k-3F'(x_k))\| &\leq K\|\frac{3x_k+y_k}{4}-\frac{2x_k+2y_k}{4}\| \\ & K\|\frac{3x_k+y_k}{4}-\frac{4x_k}{4}\|+2K\|\frac{x_k+3y_k}{4}-\frac{4x_k}{4}\| \\ &= 2K\|y_k-x_k\|\leq 2K(s_k-t_k). \end{aligned} \tag{3.9}$$

Using (1.1), (3.6) (for $z=x_k$) and (3.7)-(3.9)

$$\|x_{k+1}-y_k\| \leq \frac{2K(s_k-t_k)(t_{k+1}-t_k)}{9(1-K_0t_k)(1-p_k)}=t_{k+1}-s_k. \tag{3.10}$$

We also have

$$\|x_{k+1}-x_0\|\leq\|x_{k+1}-y_k\|+\|y_k-x_0\|\leq t_{k+1}-s_k+s_k-t_0=t_{k+1}\leq t^*, \tag{3.11}$$

so $x_{k+1}\in U(x_0,t^*)$. We can write by method (1.1)

$$\begin{aligned} F(x_{k+1}) &= F(x_{k+1})-F(x_k)-\frac{1}{3}M_k(x_{k+1}-x_k) \\ &= \int_0^1(F'(x_k+\theta(x_{k+1}-x_k))d\theta-\frac{1}{3}M_k)(x_{k+1}-x_k). \end{aligned} \tag{3.12}$$

One can obtain the estimate

$$\begin{aligned} & \int_0^1(F'(x_k+\theta(x_{k+1}-x_k))d\theta-\frac{2}{3}F'\left(\frac{3x_k+y_k}{4}\right) \\ & +\frac{1}{3}F'\left(\frac{x_k+y_k}{2}\right)-\frac{2}{3}F'\left(\frac{x_k+4y_k}{4}\right) \\ = & \int_0^1 F'(x_k+\theta(x_{k+1}-x_k))d\theta-F'(x_k)) \\ & +\frac{2}{3}(F'(x_k)-F'\left(\frac{3x_k+y_k}{4}\right))+\frac{1}{3}(F'(x_k)-F'\left(\frac{x_k+3y_k}{4}\right) \\ & +\frac{1}{3}(F'\left(\frac{x_k+y_k}{2}\right)-F'\left(\frac{x_k+3y_k}{4}\right)), \end{aligned} \tag{3.13}$$

so

$$\begin{aligned} & \|F'(x_0)^{-1}\int_0^1(F'(x_k+\theta(x_{k+1}-x_k))d\theta-\frac{1}{3}M_k)\| \\ \leq & K\left[\frac{\|x_{k+1}-x_k\|}{2}+\frac{\|y_k-x_k\|}{6}+\frac{\|y_k-x_k\|}{4}+\frac{\|y_k-x_k\|}{12}\right] \\ \leq & K(\frac{t_{k+1}-t_k}{2}+\frac{s_k-t_k}{6}+\frac{s_k-t_k}{4}+\frac{s_k-t_k}{12}) \\ = & \frac{K}{2}(t_{k+1}-t_k+s_k-t_k). \end{aligned} \tag{3.14}$$

It follows from method (1.1), (3.6) (for $z=x_{k+1}$), (3.11) and (2.10) that

$$\begin{aligned}
\|y_{k+1} - x_{k+1}\| &\leq \|(F'(x_{k+1})^{-1}F'(x_0)F'(x_0)^{-1}F(x_{k+1})\| \\
&\leq \frac{K(t_{k+1} - t_k + s_k - t_k)(t_{k+1} - t_k)}{2(1 - K_0 t_{k+1})} \\
&= s_{k+1} - t_{k+1}, \qquad (3.15)
\end{aligned}$$

showing (3.3). Moreover, we get

$$\begin{aligned}
\|y_{k+1} - x_0\| &\leq \|y_{k+1} - x_{k+1}\| + \|x_{k+1} - x_0\| \\
&\leq s_{k+1} - t_{k+1} + t_{k+1} - t_0 = s_{k+1} \leq t^*, \qquad (3.16)
\end{aligned}$$

so $y_{k+1} \in U(x_0, t^*)$. The induction for (3.3) and (3.6) is completed. It follows from(3.3), (3.6), (3.10) and (3.16) that sequence $\{x_n\}$ is fundamental in Banach space E, and as such it converges to $x_* \in U[x_0, t^*]$. Using (3.9) and letting $k \longrightarrow \infty$ in $\|F'(x_0)^{-1}F(x_{k+1})\| \leq \frac{K}{2}(t_{k+1} - t_k + s_k - t_k)$, we obtain $F(x_*) = 0$.

□

Next, a uniqueness of the solution x_* result is presented.

PROPOSITION 3.2. *Suppose:*
(1) The element $x_* \in U(x_*, s^*)$ is a simple solution of (1.2), and (A2) holds.
(2) There exists $\delta \geq s^*$ so that

$$K_0(s^* + \delta) < 2. \qquad (3.17)$$

Set $D_1 = D \cap U[x_*, \delta]$. Then, x_* is the unique solution of equation (1.2) in the domain D_1.

Proof. Let $q \in D_1$ with $F(q) = 0$. Define $S = \int_0^1 F'(q + \theta(x_* - q))d\theta$. Using (H2) and (3.17) one obtains

$$\begin{aligned}
\|F'(x_0)^{-1}(S - F'(x_0))\| &\leq K_0 \int_0^1 ((1-\theta)\|q - x_0\| + \theta\|x^* - x_0\|)d\theta \\
&\leq \frac{K_0}{2}(s^* + \delta) < 1,
\end{aligned}$$

so $q = x_*$, follows from the invertability of S and the identity $S(q - x_*) = F(q) - F(x_*) = 0 - 0 = 0$.

□

REMARK 3.3. *(i) Point T given in closed form can repalce t^* in Theorem 3.1.*
(ii) We used majorizing sequence $\{t_n\}$ given by (2.1) and Lemma 2.2 to prove Theorem 3.1. But we can also use majorizing sequence $\{t_n\}$ given by (2.22) and Lemma 2.3 to arrive at the conclusions of the Theorem 3.1. Simply notice that in the proof of this theorem we got using the second substep of

scheme (1.1), (3.8) and (3.9) estimate (3.10) leading to the definition of the first substep of sequence (2.1). But we can use the first substep of scheme (1.1) to write instead of (3.8) that

$$x_{k+1} = y_k - F'(x_k)^{-1}(M_k - 3F'(x_k))M_k^{-1}F(x_k)(y_k - x_k)$$

leading to

$$\|x_{k+1} - y_k\| \le \frac{2K(1 + K_0 t_k)(s_k - t_k)^2}{3(1 - K_0 t_k)(1 - p_k)} = t_{k+1} - s_k,$$

where, we also used

$$\begin{aligned} \|F'(x_0)^{-1}F(x_k)\| &= \|F'(x_0)^{-1}((F'(x_k) - F(x_0)) + F'(x_0))\| \\ &\le 1 + K_0\|x_k - x_0\| \le 1 + K_0 t_k. \end{aligned}$$

Hence, we arrive at the second semi-local convergence rsult for scheme (1.1).

THEOREM 3.4. *Suppose:conditions (A) hold with (A4) replaced by (A4)' Hypotheses of Lemma 2.3 or Lemma 2.4 hold. Then, the conclusions of Theorem 3.1 hold with (2.22) replacing (2.1).*

In practice we shall use the theorem providing the best results.

4. Numerical Experiments

Lipschitz parameters are determinded and convegence criteria are tested for some numerical experiments.

EXAMPLE 4.1. *Define scalar function*

$$\zeta(t) = \xi_0 t + \xi_1 + \xi_2 \sin \xi_3 t, \; x_0 = 0,$$

where ξ_j, $j = 0, 1, 2, 3$ are parameters. Then, clearly for ξ_3 large and ξ_2 small, $\frac{K_0}{L_1}$ can be small (arbitrarily). In particular, notice that $\frac{K}{L_1} \longrightarrow 0$.

EXAMPLE 4.2. *Let $E = E_1 = C[0, 1]$ and $D = U[0, 1]$. It is well known that the boundary value problem [12].*

$$\varsigma(0) = 0, \varsigma(1) = 1,$$

$$\varsigma'' = -\varsigma - \sigma\varsigma^2$$

can be given as a Hammerstein-like nonlinear integral equation

$$\varsigma(s) = s + \int_0^1 Q(s, t)(\varsigma^3(t) + \sigma\varsigma^2(t))dt$$

where σ is a parameter. Then, define $F : D \longrightarrow E_1$ by

$$[F(x)](s) = x(s) - s - \int_0^1 Q(s, t)(x^3(t) + \sigma x^2(t))dt.$$

Choose $\varsigma_0(s) = s$ *and* $D = U(\varsigma_0, \rho_0)$. *Then, clearly* $U(\varsigma_0, \rho_0) \subset U(0, \rho_0 + 1)$, *since* $\|\varsigma_0\| = 1$. *Suppose* $2\sigma < 5$. *Then, conditions (A) are satisfied for*

$$K_0 = \frac{2\sigma + 3\rho_0 + 6}{8},\ L = \frac{\sigma + 6\rho_0 + 3}{4},$$

and $\eta = \frac{1+\sigma}{5-2\sigma}$. *Notice that* $K_0 < L$.

EXAMPLE 4.3. *Let us consider a scalar function* ψ *defined on the set* $D = U[x_0, 1-q]$ *for* $q \in (0, \frac{1}{2})$, *by*

$$\psi(x) = x^3 - q.$$

Choose $x_0 = 1$. *Then, we obtain the estiamtes*

$$\begin{aligned} |\psi'(x_0)^{-1}(\psi'(x) - \psi'(x_0))| &= |x^2 - x_0^2| \\ &\leq |x + x_0||x - x_0| \leq (|x - x_0| + 2|x_0|)|x - x_0| \\ &= (1 - q + 2)|x - x_0| = (3 - q)|x - x_0|, \end{aligned}$$

for all $x \in D$, *so* $K_0 = 3 - q$, $D_0 = U(x_0, \frac{1}{K_0}) \cap D = U(x_0, \frac{1}{K_0})$,

$$\begin{aligned} |\psi'(x_0)^{-1}(\psi'(y) - \psi'(x)| &= |y^2 - x^2| \\ &\leq |y + x||y - x| \leq (|y - x_0 + x - x_0 + 2x_0)||y - x| \\ &= (|y - x_0| + |x - x_0| + 2|x_0|)|y - x| \\ &\leq (\frac{1}{K_0} + \frac{1}{K_0} + 2)|y - x| = 2(1 + \frac{1}{K_0})|y - x|, \end{aligned}$$

for all $x, y \in D_0$, *so* $L = 2(1 + \frac{1}{K_0})$,

$$\begin{aligned} |\psi'(x_0)^{-1}(\psi'(y) - \psi'(x)| &= (|y - x_0| + |x - x_0| + 2|x_0|)|y - x| \\ &\leq (1 - q + 1 - q + 2)|y - x| = 2(2 - q)|y - x|, \end{aligned}$$

for all $x, y \in D$ *and* $L_1 = 2(2 - q)$. *Notice that for all* $q \in (0, \frac{1}{2})$

$$K_0 < L < L_1.$$

Next, set $y = x - \psi'(x)^{-1}\psi(x)$, $x \in D$. *Then, we have*

$$y + x = x - \psi'(x)^{-1}\psi(x) + x = \frac{5x^3 + q}{3x^2}.$$

Define fundtion $\bar{\psi}$ *on the interval* $D = [q, 2 - q]$ *by*

$$\bar{\psi}(x) = \frac{5x^3 + q}{3x^2}.$$

Then, we get by this definition that

$$\bar{\psi}'(x) = \frac{15x^4 - 6xq}{9x^4} = \frac{5(x-q)(x^2+xq+q^2)}{3x^3},$$

where $p = \sqrt[3]{\frac{2q}{5}}$ *is the critical point of function* $\bar{\psi}$. *Notice that* $q < p < 2-q$. *It follows that this function is decreasing on the interval* (q, p) *and increasing on the interval* $(q, 2-q)$, *since* $x^2 + xq + q^2 > 0$ *and* $x^3 > 0$. *So, we can set*

$$K_1 = \frac{5(2-q)^2 + q}{9(2-q)^2}, \eta = \frac{1-q}{3}$$

and

$$K_1 < K_0.$$

But if $x \in D_0 = [1 - \frac{1}{K_0}, 1 + \frac{1}{K_0}]$, *then*

$$K = \frac{5\varrho^3 + q}{9\varrho^2},$$

where $\varrho = \frac{4-q}{3-q}$ *and* $K < K_1$ *for all* $q \in (0, \frac{1}{2})$.

Next, we verify conditions (2.2), (2.3), (2.24) and (2.25).

Then for $q = 0.95$, $\frac{6}{K_0} = 2.9268$ *and*

n	1	2	3	4	5
t_n	0.1683	0.1694	0.1694	0.1694	0.1694

$\alpha_1 = 0.1643 = \alpha_3$, $\alpha_2 = 0.6588 = \alpha$, $a = 0.0030 = \bar{c}$, $b = 0.0136 = c$,
$1 - \frac{10K_0\eta}{3} = 0.8861$, *and* $(\frac{4K}{3} + \frac{4}{3}K_0K\eta + K_0)\eta = 0.0521 < 1$. *Hence, conditions (2.2),(2.3), (2.24) and (2.25) hold.*

5. Conclusion

The semi-local convergence of scheme (1.1) with order three is extended using general conditions on F' and recurrent majorizing sequences.

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