Eur. J. Math. Anal. 2 (2022) 11
doi: 10.28924/ada/ma.2.11

# On the $\alpha-\psi-$ Contractive Mappings in $C^*$-Algebra Valued b-Rectangular Metric Spaces and Fixed Point Theorems

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Abstract. This present paper extends a version of $\alpha-\psi-$contraction in $C^*$-algebra valued rectangular b-metric spaces and establishing the existence and uniqueness of fixed point for them. Non-trivial examples are further provided to support the hypotheses of our results.

## 1. Introduction

A $C^*$-algebra valued metric spaces were introduced by Ma et al. [6] as a generalization of metric spaces they proved certain fixed point theorems, by giving the definition of $C^*$-algebra valued contractive mapping analogous to Banach contraction principle. Many mathematicians worked on this interesting space.

Various fixed point results were established on such spaces, see [1–3] and references therein.

Combining conditions used for definitions of $C^*$-algebra valued metric and generalized metric spaces, G Kalapana and Tasneem [4] announced the notions of $C^*$-algebra valued metric space and establish nice results of fixed point on such space.

In this paper, inspired by the work done in [9], we introduce the notion of $\alpha-\psi-$contraction and establish some new fixed point theorems for mappings in the setting of complete $C^*$-algebra valued rectangular b- metric spaces.

Moreover, an illustrative examples is presented to support the obtained results.

Received: 18 Dec 2021.
*Key words and phrases.* fixed point; $C^*$-algebra valued metric spaces; $\alpha-\psi-$ contraction; $\alpha-\psi-C^*$ valued contraction.

## 2. PRELIMINARIES

Throughout this paper, we denote $\mathbb{A}$ by an unital (i.e ,unity element I) $C^*$-algebra with linear involution $*$, such that for all $x, y \in \mathbb{A}$,

$$(xy)^* = y^*x^*, \text{and } x^{**} = x.$$

We call an element $x \in \mathbb{A}$ a positive element, denote it by $x \succeq \theta$

if $x \in \mathbb{A}_h = \{x \in \mathbb{A} : x = x^*\}$ and $\sigma(x) \subset \mathbb{R}_+$,where $\sigma(x)$ is the spectrum of $x$.Using positive element ,we can define a partial ordering $\preceq$ on $\mathbb{A}_h$ as follows :

$$x \preceq y \text{ if and only if } y - x \succeq \theta$$

where $\theta$ means the zero element in $\mathbb{A}$.

we denote the set $x \in \mathbb{A} : x \succeq \theta$ by $\mathbb{A}_+$ and $|x| = (x^*x)^{\frac{1}{2}}$.

and $\mathbb{A}'$ will denote the set $\{a \in \mathbb{A}_+; ab = ba, \forall b \in \mathbb{A}\}$

**Lemma 2.1.** *[8] Suppose that $\mathbb{A}$ is a unital $C^*$-algebra with a unit $I$,*

(1) *for any $x \in \mathbb{A}_+$ we have $x \preceq I \iff \|x\| \leq 1$*
(2) *If $a \in \mathbb{A}_+$ with $\|a\| < \frac{1}{2}$ then $I - a$ is unvertible and $\|a(1 - a)^{-1}\| < 1$*
(3) *Suppose that $a, b \in \mathbb{A}_+$ and $ab = ba$, then $ab \succeq \theta$*
(4) *Let $a \in \mathbb{A}'$, if $b, c \in \mathbb{A}$, with $b \succeq c \succeq \theta$, and $I - a \in \mathbb{A}'_+$ is invertible operator, then $(I - a)^{-1}b \succeq (I - a)^{-1}c$*

**Definition 2.2.** [4] Let $X$ be a non-empty set and $b \in A$ such that $b \succeq I$. supposa the mapping $d : X \times X \to \mathbb{A}_+$ satisfies:

(i) $d(x, y) = \theta$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$ for all distinct points $x, y \in X$;
(iii) $d(x, y) \preceq b[d(x, u) + d(u, v) + d(v, y)]$ for all $x, y \in X$ and for all distinct points $u, v \in X - \{x, y\}$.

Then $(X, \mathbb{A}_+, d)$ is called a $C^*$-algebra valued rectangular $b-$metric space.

**Example 2.3.** Let $X = \mathbb{R}$ and $\mathbb{A} = M_2(\mathbb{R})$. Define $d(x, y) = diag(|x - y|, 2|x - y|)$ where $x, y \in \mathbb{R}$. It is easy to verify $d$ is a $C^*-$ algebra-valued rectangular $b-$ metric and $(X, M_2(\mathbb{R}), d)$ is a copmlete $C^*$-algebra valued rectangular $b-$metric space.

**Definition 2.4.** [9] If $\psi : A \to B$ is a linear mapping in $C^*$-algebra, it is said to be positive if $\psi(A^+) \subseteq B^+$. In this case $\psi(A_h) \subseteq B_h$, and the restriction map $\psi : A_h \to B_h$ is increasing.

**Definition 2.5.** [9] Suppose that $A$ and $B$ are $C^*$-algebra .
A mapping $\psi : A \to B$ is said to be $C^*$- homomorphism if :

(i) $\psi(ax + by) = a\psi(x) + b\psi(y)$ for all $a, b \in \mathbb{C}$ and $x, y \in A$
(ii) $\psi(xy) = \psi(x)\psi(y)$ for all $x, y \in A$

(iii) $\psi(x^*) = \psi(x)^*$ for all $x \in A$

(iv) $\psi$ maps the unit in $A$ to the unit in $B$.

**Definition 2.6.** [9] Let $A$ and $B$ be $C^*$-algebra spaces and let $\psi : A \to B$ be a homomorphism then $\psi$ is called an $*-$ homomorphism if it is one to one $*-$ homomorphism.
A $C^*$-algebra $A$ is $*-$isomorphic to a $C^*$-algebra $B$ if there exists $*-$ isomorphism of $A$ onto $B$.

**Definition 2.7.** [9] Let $\Psi$ be the set of positive functions $\psi : A^+ \to A^+$ satisfying the following conditions :

(i) $\psi$ is continous and nondecrasing

(ii) $\psi(a) = \theta$ if and only if $a = \theta$

(iii) $lim_{n\longrightarrow\infty}\psi^n(a) = \theta$, $(a \succ \theta)$ ,$\sum_{n=1}^{\infty} \psi^n(a) < \infty$

(iv) The series $\sum_{k=1}^{\infty} b^k\psi^k(a) < \infty$ for $a \succ \theta$ is increasing and continuous at $\theta$.

**Corollary 2.8.** *[9] Every $C^*-$ homomorphism is contractive and hence bounded.*

**Lemma 2.9.** *Every $*-$ homomorphism is positive.*

**Definition 2.10.** [9] Let $X$ be a nonempty set and $\alpha : X \times X \to \mathbb{A}'_+$ be a function, wesay that the self map $T$ is $\alpha-$ admissible if
$(x, y) \in X \times X, \alpha(x, y) \succeq I \Rightarrow \alpha(Tx, Ty) \succeq I$,where $I$ the unit of $\mathbb{A}$.

**Definition 2.11.** [9] Let $(X, \mathbb{A}, d)$ be a $C^*$-algebra valued $b-$ metric space and $T : X \to X$ is mapping, we say that $T$ is an $\alpha-\psi-$ contractive mapping if there exist two functions $\alpha : X \times X \to \mathbb{A}_+$ and $\psi \in \Psi$ such that

$$\alpha(x, y)d(Tx, Ty) \preceq \psi(d(x, y)), \text{ for all } x, y \in X$$

## 3. Main result

In [9] introduced the concept of $\alpha - \psi-$ contractive mappings in a unital $C^*$-algebra valued $b-$ metric space. In this paper we will develop the definitions in case of unital $C^*$-algebra valued rectangular $b-$ metric space and give some Banach fixed point theorems.

**Definition 3.1.** Let $(X, \mathbb{A}, d)$ be a $C^*$-algebra valued $b-$ rectangular metric space and $T : X \to X$ is mapping, we say that $T$ is an $\alpha - \psi-$ contractive mapping if there exist two functions $\alpha : X \times X \to \mathbb{A}_+$ and $\psi \in \Psi$ such that

$$\alpha(x, y)d(Tx, Ty) \preceq \psi(d(x, y)), forallx, y \in X \tag{3.1}$$

**Theorem 3.2.** *Let $(X, \mathbb{A}, d)$ be a complete $C^*$-algebra valued rectangular $b-$ metric space and let $T : X \to X$ be a $\alpha, \psi-$ contractive mapping satisfying the following conditions:*

(i) *$T$ is $\alpha-$ admissible*

(ii) *There exists* $x_0 \in X$ *such that* $\alpha(x_0, Tx_0) \succeq I$

(iii) *for all* $x, y \in X$ *,there exists* $z \in X$ *such that* $\alpha(x, z) \succeq I$ *and* $\alpha(y, z) \succeq I$

(iv) $T$ *is continuous*

*Then,* $T$ *has a unique fixed point in* $X$*.*

*Proof.* Let $x_0 \in X$ such that $\alpha(x_0, Tx_0) \succeq I$ and define a sequence $\{x_n\} \in X$ such that $x_{n+1} = Tx_n$, $\forall n \in \mathbb{N}$. Suppose that there exists $n \in \mathbb{N}$ such that $x_n = Tx_n$. Then $x_n$ is a fixed point of $T$ and the proof is finished.

Hence, we assume that $x_n \neq Tx_{n+1}$, $\forall n \in \mathbb{N}$, since $T$ is $\alpha-$admissible, we get

$$\alpha(x_0, x_1) = \alpha(x_0, Tx_0) \succeq I \Rightarrow \alpha(Tx_0, T^2x_0) = \alpha(x_1, x_2) \succeq I.$$

Continuing this process, we have

$$\alpha(x_n, x_{n+1}) \succeq I \quad \forall n \in \mathbb{N}. \tag{3.2}$$

By 3.1 and 3.2, we get

$$\begin{aligned} d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) &\preceq \alpha(x_{n-1}, x_n)d(Tx_{n-1}, Tx_n) \\ &\preceq \psi(d(x_{n-1}, x_n)) \\ &\preceq \\ &\preceq \psi^n(d(x_0, x_1)). \end{aligned}$$

For $m \geq 1$ and $p \geq 1$, it follows that

$d(x_{m+p}, x_m) \preceq b[d(x_{m+p}, x_{m+p-1}) + d(x_{m+p-1}, x_{m+p-2}) + d(x_{m+p-2}, x_m)]$

$\preceq bd(x_{m+p}, x_{m+p-1}) + bd(x_{m+p-1}, x_{m+p-2}) + b[b[d(x_{m+p-2}, x_{m+p-3}) + d(x_{m+p-3}, x_{m+p-4}) + d(x_{m+p-4}, x_m)]]$

$= bd(x_{m+p}, x_{m+p-1}) + bd(x_{m+p-1}, x_{m+p-2}) + b^2d(x_{m+p-2}, x_{m+p-3}) + b^2d(x_{m+p-3}, x_{m+p-4}) + b^2d(x_{m+p-4}, x_m)$

$\preceq bd(x_{m+p}, x_{m+p-1}) + bd(x_{m+p-1}, x_{m+p-2}) + b^2d(x_{m+p-2}, x_{m+p-3}) + b^2d(x_{m+p-3}, x_{m+p-4}) + .... + b^{\frac{p-1}{2}}d(x_{m+3}, x_{m+2}) + b^{\frac{p-1}{2}}d(x_{m+2}, x_{m+1}) + b^{\frac{p-1}{2}}d(x_{m+1}, x_m)$

$\preceq b\psi^{m+p-1}(d(x_0, x_1)) + b\psi^{m+p-2}(d(x_0, x_1)) + ... + b^{\frac{p-1}{2}}d(x_0, x_1)$

Since $b \preceq I$, using definition 2.6 we have

$d(x_m, x_{m+p}) \preceq b\psi^{m+p-1}(d(x_0, x_1)) + b\psi^{m+p-2}(d(x_0, x_1)) + ... + b^{\frac{p-1}{2}}d(x_0, x_1) \to \theta$ as $n \to +\infty$

Therefore $\{x_n\}$ is a Cauchy sequence in $X$. By the completeness of $(X, \mathbb{A}, d)$ there exists an $x \in X$ such that

$$lim_{n\to\infty}x_n = lim_{n\to\infty}Tx_{n-1} = x.$$

From continuity of $T$ and by uniqueness of the limit, we get $Tx = x$, ie. $x$ is a fixed point of $T$.

Now suppose that $y \neq x$ is another fixed point of $T$.

From $(iii)$, there exists $z \in X$ such that $\alpha(x,z) \succeq I$ and $\alpha(y,z) \succeq I$.

Since $T$ is $\alpha-$ admissible, we have

$$\alpha(x, T^n z) \succeq I \text{ and } \alpha(y, T^n z) \succeq I \text{ for all } n \in \mathbb{N}$$

Using (1), we obtain

$$d(x, T^n z) = d(Tx, T(T^{n-1}z))$$

$$\preceq \alpha(x, T^{n-1}z) d(Tx, T(T^{n-1}z))$$

$$\preceq \psi^n(d(x,z)) \to \theta \text{ as } n \to \infty.$$

Thus, $T^n z = x$. Similary $T^n z = y$ as $n \to \infty$ So, the uniqueness of the limit we obtain $x = y$. □

**Example 3.3.** Let $X = \mathbb{R}$ and $\mathbb{A} = M_2(\mathbb{R})$ as given in Example 2.3, define $T : X \to X$, by $Tx = \dfrac{x}{3}$ and $\alpha : X \times X \to M_2(\mathbb{R})$ such that

$\alpha(x,y) = \begin{pmatrix} |x-y| & 0 \\ 0 & 0 \end{pmatrix}$ thus, $T$ is $\alpha-$ admissible,

and $\psi : M_2(\mathbb{R})^+ \to M_2(\mathbb{R})^+$ , $\psi(a) = \begin{pmatrix} a^2 & 0 \\ 0 & a^2 \end{pmatrix}$ $\forall a \in (\mathbb{R})^+$.

This is clear that $T$ is $\alpha - \psi -$ contractive mapping and satisfies

$\alpha(x,y)d(Tx,Ty) \preceq \psi(d(x,y))$, for all $x, y \in X$

**Theorem 3.4.** *Let $(X, \mathbb{A}, d)$ be a complete $C^*$-algebra valued rectangular $b-$ metric space and let $T : X \to X$ be a $\alpha, \psi-$ contractive mapping of Kannan type ie,*

$$\alpha(x,y)d(Tx,Ty) \preceq \psi(d(Tx,x) + d(Ty,y)) \tag{3.3}$$

*for all $x, y \in X$ Where $\psi \in \Psi$ and $\alpha : X \times X \to \mathbb{A}_+$*

*and the following conditions holds:*

(i) *$T$ is $\alpha-$ admissible*

(ii) *There exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \succeq I$*

(iii) *$T$ is continuous*

*Then, $T$ has a fixed point in $X$.*

*Proof.* By (3.3), we obtain

$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) \preceq \alpha(x_{n-1}, x_n) d(Tx_{n-1}, Tx_n)$

$$\preceq \psi(d(Tx_{n-1}, x_{n-1}) + d(Tx_n, x_n)) = \psi(d(x_n, x_{n-1}) + d(x_{n+1}, x_n))$$

$$= \psi(d(x_n, x_{n-1})) + \psi(d(x_{n+1}, x_n))$$

$$(I - \psi)(d(x_n, x_{n-1})) \preceq \psi(d(x_n, x_{n-1}))$$

from Lemma 2.1 and Definition 2.6, we obtain

$$d(x_n, x_{n+1}) \preceq (I-\psi)^{-1}\psi(d(x_n, x_{n-1})) = \Phi(d(x_n, x_{n-1})) \text{ where}$$
$$\Phi = (I-\psi)^{-1}\psi$$

Therefore

$$d(x_n, x_{n+1}) \preceq \Phi^n(d(x_0, x_1)) \forall n \in \mathbb{N}$$

For any $m \geq 1$ and $p \geq 1$ similary in Theorem 3.1 we have

$d(x_m, x_{m+p}) \preceq b\psi^{m+p-1}(d(x_0, x_1)) + b\psi^{m+p-2}(d(x_0, x_1)) + \ldots + b^{\frac{p-1}{2}} d(x_0, x_1) \to \theta$ as $n \to +\infty$.

Thus $\{x_n\}$ is a Cauchy sequence in $X$. By the completeness of $(X, \mathbb{A}, d)$, there exists $x \in X$ such that

$$lim_{n\to\infty} x_n = lim_{n\to\infty} T x_{n-1} = x.$$

the continuity of $T$ gives that $x$ is a fixed point of $T$.

To prove that $x$ is the unique fixed point, we suppose that $y \in X$ is another fixed point of $T$. Then

$$\begin{aligned}\theta &\preceq d(x, y) = d(Tx, Ty)\\ &\preceq \alpha(x, y) d(Tx, Ty)\\ &\preceq \psi(d(Tx, x) + d(Ty, y))\\ &= \psi(d(x, x) + d(y, y)) = \theta\end{aligned}$$

Hence $x = y$.Therefore the fixed point is unique. □

**Theorem 3.5.** *Let $(X, \mathbb{A}, d)$ be a complete $C^*$-algebra valued rectangular $b-$ metric space and let $T : X \to X$ be a $\alpha, \psi-$ contractive mapping of Banach-Kannan type ie,*

$$\alpha(x, y) d(Tx, Ty) \preceq \psi(d(x, y) + d(Tx, x) + d(Ty, y)) \tag{3.4}$$

*for all $x, y \in X$ Where $\psi \in \Psi$ and $\alpha : X \times X \to \mathbb{A}_+$ such that $\psi(1-\psi)^{-1} \preceq \dfrac{1}{2I}$,*

*and the following conditions holds:*

(i) *$T$ is $\alpha-$ admissible*

(ii) *There exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \succeq I$*

(iii) *$T$ is continuous*

*Then, $T$ has a fixed point in $X$*

*Proof.* Using (3.4), we get

$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) \preceq \alpha(x_{n-1}, x_n) d(Tx_{n-1}, Tx_n)$

$\preceq \psi(d(x_{n-1}, x_n) + d(Tx_{n-1}, x_{n-1}) + d(Tx_n, x_n))$

$= \psi(d(x_{n-1}, x_n)2I + d(x_n, x_{n+1})) \Rightarrow (I-\psi)(d(x_n, x_{n+1})) \preceq 2I\psi(d(x_n, x_{n-1}))$

$\Rightarrow d(x_n, x_{n+1}) \preceq 2I(I-\psi)^{-1}\psi(d(x_n, x_{n-1})) \preceq \Phi(d(x_n, x_{n-1}))$.

Where

$$\varphi = 2I(I-\psi)^{-1}\psi.$$

Then

$$d(x_n, x_{n+1}) \preceq \Phi^n(d(x_0, x_1)).$$

We refer to the proof of the Theorem 3.1 we get that $x$ is a fixed point of $T$. Now, if $y \neq x$ is another fixed point of $T$, we have

$$\begin{aligned}\theta \preceq d(x,y) &= d(Tx,Ty)\\ &\preceq \alpha(x,y)d(Tx,Ty)\\ &\preceq \psi(d(x,y)+d(Tx,x)+d(Ty,y))\\ &= \psi(d(x,y)+d(x,x)+d(y,y)) = \psi(d(x,y)).\end{aligned}$$

So $d(x,y)=\theta$ ; ie $x=y$. □

## 4. Applications

As application of $\alpha-\psi$ contractive in unital $C^*$-algebra valued rectangular $b-$ metric spaces, existence and uniqueness results for a type of operator equation is given.

**Example 4.1.** Suppose that $H$ is a Hilbert space, $B(H)$ is the set of linear bounded operators on $H$. Let $A_1, A_2, ..., A_n, ... \in B(H)$

which satisfy $\sum_{n=1}^{\infty}\|A_n\| < 1$ and $Q \in B(H)_+$.

Then the operator equation $X - \sum_{n=1}^{\infty} A_n^* X A_n = Q$ has a unique solution in $B(H)$.

*Proof.* Set $a = (\sum_{n=1}^{\infty}\|A_n\|)^p$ with $p \geq 1$, then $\|a\| < 1$.Without loss of generality, one can suppose that $a > 0$.

Choose a positive operator $M \in B(H)$.For $X, Y \in B(H)$ and $p \geq 1$, set

$d(X,Y) = \|X-Y\|^p M$.

Then $d(X,Y)$ is a $C^*$-algebra valued rectangular $b-$ metric.

Suppose that $X, Y, Z, W \in B(H)$ we have

$\|X-Y\|^p \preceq 2^P(\|X-Z\|^p + \|Z-W\|^p + \|W-Y\|^p)$.

Which implies that $d(X,Y) \preceq A[d(X,Z)+d(Z,W)+d(W,Y)]$

Where $A = 2^p I$. Consider the map

$T : B(H) \to B(H)$ such that $T(X) = \sum_{n=1}^{\infty} A_n^* X A_n + Q$.

Then

$$\begin{aligned}d(T(X),T(Y)) &= \|T(X),T(Y)\|^p M\\ &= \|\textstyle\sum_{n=1}^{\infty} A_n^*(X-Y)A_n\|^p M\\ &\preceq \textstyle\sum_{n=1}^{\infty}\|A_n\|^{2p}\|X-Y\|^p M\\ &\preceq a^2 d(X,Y)\end{aligned}$$

Let $\alpha : B(H) \times B(H) \to B(H)^+$ defined by $\alpha(X, Y) = (X, Y)I$

and $\psi : B(H)^+ \to B(H)^+$ defined by $\psi(X) = X$.

We get $\alpha(X, Y)d(TX, TY) \preceq \psi(d(X, Y))$.

Using Theorem 3.1, there exists a unique fixed point $X$ in $B(H)$. □

## 5. Acknowledgments

It is our great pleasure to thank the referee for his careful reading of the paper and for several helpful suggestions.

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