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Group Analysis of Equal-Width Equation

Joseph Owuor Owino

Faculty of Applied Sciences and Technology, School of Mathematics and Actuarial Science, Department of Pure and Applied Mathematics, The Technical University of Kenya, Kenya

Correspondence: josephowuorowino@gmail.com

Abstract. We study a third-order nonlinear equal width equation, which has been used for simulation of a one-dimensional wave propagation in a non-linear medium with dispersion process, by symmetry analysis. First, Lie point symmetries are obtained and used to reduce reduce the equal width equation thereby constructing exact solutions. Traveling waves are constructed using of a linear combination of space and time translation symmetries. We have used the multiplier technique to compute conservation laws.

1. Introduction

The Equal width Equation [1] is given by,

$$\Delta \equiv u_t + \alpha u u_x + \beta u_{txx} = 0, \tag{1.1}$$

where t and x represents time and spatial independent variables ; α and β are the nonlinearity and the dispersion parameters respectively. Equation (1.1) was first studied by Morrison [2] and describes nonlinear dispersive waves, particularly those generated in a shallow water channel. Several techniques have been employed to compute solutions of Equation (1.1). A case in point, is in [3], where a Petrov-Galerkin approach applied quadratic B-spline finite element. In [4], the researchers applied least-squares approach in the construction of numerical solutions. We present a group analysis approach in this paper by first giving the preliminaries.

2. Preliminaries

This section is a prelude to the sequel.

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Local Lie groups. [5] We will consider the transformations

$$T_\epsilon : \quad \bar{x}^i = \varphi^i(x^i, u^\alpha, \epsilon), \quad \bar{u}^\alpha = \psi^\alpha(x^i, u^\alpha, \epsilon), \tag{2.1}$$

in the Euclidean space $\mathbb{R}^n$ of $x = x^i$ independent variables and $\mathbb{R}^m$ of $u = u^\alpha$ dependent variables. The continuous parameter ϵ ranges from a neighbourhood $\mathcal{N}' \subset \mathcal{N} \subset \mathbb{R}$ of $\epsilon = 0$ for φ^i and ψ^α differentiable and analytic in the parameter ϵ.

Definition 2.1. Let $\mathcal{G}$ be a set of transformations in (2.1) . Then $\mathcal{G}$ is a local Lie group if:

(i). Given $T_{\epsilon_1}, T_{\epsilon_2} \in \mathcal{G}$, for $\epsilon_1, \epsilon_2 \in \mathcal{N}' \subset \mathcal{N}$, then
$T_{\epsilon_1} T_{\epsilon_2} = T_{\epsilon_3} \in \mathcal{G},\ \epsilon_3 = \phi(\epsilon_1, \epsilon_2) \in \mathcal{N}$ (Closure).

(ii). There exists a unique $T_0 \in \mathcal{G}$ if and only if $\epsilon = 0$ such that $T_\epsilon T_0 = T_0 T_\epsilon = T_\epsilon$(Identity).

(iii). There exists a unique $T_{\epsilon^{-1}} \in \mathcal{G}$ for every transformation $T_\epsilon \in \mathcal{G}$,
where $\epsilon \in \mathcal{N}' \subset \mathcal{N}$ and $\epsilon^{-1} \in \mathcal{N}$ such that
$T_\epsilon T_{\epsilon^{-1}} = T_{\epsilon^{-1}} T_\epsilon = T_0$ (Inverse).

Remark 2.2. The condition (i) is sufficient for associativity of $\mathcal{G}$.

Prolongations. Consider the system,

$$\Delta_\alpha \left(x^i, u^\alpha, u_{(1)}, \dots, u_{(\pi)}\right) = \Delta_\alpha = 0, \tag{2.2}$$

where u^α are dependent variables with partial derivatives $u_{(1)} = \{u_i^\alpha\}$,
$u_{(2)} = \{u_{ij}^\alpha\}, \dots, u_{(\pi)} = \{u_{i_1 \dots i_\pi}^\alpha\}$, of the first, second, ..., up to the πth-orders. We shall denote by

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \dots, \tag{2.3}$$

the total differentiation operator with respect to the variables x^i and δ_i^j, the Kronecker delta. Then

$$D_i(x^j) = \delta_i^j,\ ',\ u_i^\alpha = D_i(u^\alpha),\ \ u_{ij}^\alpha = D_j(D_i(u^\alpha)),\ \ \dots, \tag{2.4}$$

where u_i^α defined in (2.4) are differential variables [6].

(1) **Prolonged groups** Let $\mathcal{G}$ given by

$$\bar{x}^i = \varphi^i(x^i, u^\alpha, \epsilon), \quad \varphi^i\Big|_{\epsilon=0} = x^i,\ \ \bar{u}^\alpha = \psi^\alpha(x^i, u^\alpha, \epsilon), \quad \psi^\alpha\Big|_{\epsilon=0} = u^\alpha, \tag{2.5}$$

where $\Big|_{\epsilon=0}$ means evaluated on $\epsilon = 0$.

Definition 2.3. The construction of $\mathcal{G}$ in (2.5) is equivalent to the computation of infinitesimal transformations

$$\begin{aligned} \bar{x}^i &\approx x^i + \xi^i(x^i, u^\alpha)\epsilon, \quad \varphi^i\Big|_{\epsilon=0} = x^i, \\ \bar{u}^\alpha &\approx u^\alpha + \eta^\alpha(x^i, u^\alpha)\epsilon, \quad \psi^\alpha\Big|_{\epsilon=0} = u^\alpha, \end{aligned} \tag{2.6}$$

obtained from (2.1) by a Taylor series expansion of $\varphi^i(x^i, u^\alpha, \epsilon)$ and $\psi^i(x^i, u^\alpha, \epsilon)$ in ϵ about $\epsilon = 0$ and keeping only the terms linear in ϵ, where

$$\xi^i(x^i, u^\alpha) = \frac{\partial \varphi^i(x^i, u^\alpha, \epsilon)}{\partial \epsilon}\Big|_{\epsilon=0}, \quad \eta^\alpha(x^i, u^\alpha) = \frac{\partial \psi^\alpha(x^i, u^\alpha, \epsilon)}{\partial \epsilon}\Big|_{\epsilon=0}. \tag{2.7}$$

Remark 2.4. By using the symbol of infinitesimal transformations, X, (2.6) becomes

$$\bar{x}^i \approx (1 + X)x^i, \quad \bar{u}^\alpha \approx (1 + X)u^\alpha, \tag{2.8}$$

where

$$X = \xi^i(x^i, u^\alpha)\frac{\partial}{\partial x^i} + \eta^\alpha(x^i, u^\alpha)\frac{\partial}{\partial u^\alpha}, \tag{2.9}$$

is the generator $\mathcal{G}$ in (2.5).

Remark 2.5. The change of variables formula

$$D_i = D_i(\varphi^j)\bar{D}_j, \tag{2.10}$$

is employed to construct transformed derivatives from (2.1). The $\bar{D}_j$ is total differentiation $\bar{x}^i$. As a result

$$\bar{u}_i^\alpha = \bar{D}_i(\bar{u}^\alpha), \;\; \bar{u}_{ij}^\alpha = \bar{D}_j(\bar{u}_i^\alpha) = \bar{D}_i(\bar{u}_j^\alpha). \tag{2.11}$$

If we apply the change of variable formula given in (2.10) on $\mathcal{G}$ given by (2.5), we get

$$D_i(\psi^\alpha) = D_i(\varphi^j), \;\; \bar{D}_j(\bar{u}^\alpha) = \bar{u}_j^\alpha D_i(\varphi^j). \tag{2.12}$$

If we expand (2.12), we obtain

$$\left(\frac{\partial \varphi^j}{\partial x^i} + u_i^\beta \frac{\partial \varphi^j}{\partial u^\beta}\right)\bar{u}_j^\beta = \frac{\partial \psi^\alpha}{\partial x^i} + u_i^\beta \frac{\partial \psi^\alpha}{\partial u^\beta}. \tag{2.13}$$

The $\bar{u}_i^\alpha$ can be written as functions of $x^i, u^\alpha, u_{(1)}$, meaning that,

$$\bar{u}_i^\alpha = \Phi^\alpha(x^i, u^\alpha, u_{(1)}, \epsilon), \quad \Phi^\alpha\Big|_{\epsilon=0} = u_i^\alpha. \tag{2.14}$$

Definition 2.6. The transformations in (2.5) and (2.14) give the first prolongation group $\mathcal{G}^{[1]}$.

Definition 2.7. Infinitesimal transformation of the first derivatives is

$$\bar{u}_i^\alpha \approx u_i^\alpha + \zeta_i^\alpha \epsilon, \quad \text{where} \quad \zeta_i^\alpha = \zeta_i^\alpha(x^i, u^\alpha, u_{(1)}, \epsilon). \tag{2.15}$$

Remark 2.8. In terms of infinitesimal transformations, $\mathcal{G}^{[1]}$ is given by (2.6) and (2.15).

(2) **Prolonged generators**

Definition 2.9. By the relation (2.12) on $\mathcal{G}^{[1]}$ from 2.6, we obtain [7]

$$D_i(x^j+\xi^j\epsilon)(u_j^\alpha+\zeta_j^\alpha\epsilon)=D_i(u^\alpha+\eta^\alpha\epsilon),\quad \text{which gives} \tag{2.16}$$

$$u_i^\alpha+\zeta_j^\alpha\epsilon+u_j^\alpha\epsilon D_i\xi^j=u_i^\alpha+D_i\eta^\alpha\epsilon, \tag{2.17}$$

and thus

$$\zeta_i^\alpha=D_i(\eta^\alpha)-u_j^\alpha D_i(\xi^j), \tag{2.18}$$

is the first prolongation formula.

Remark 2.10. Analogously, one constructs higher order prolongations [7],

$$\zeta_{ij}^\alpha=D_j(\zeta_i^\alpha)-u_{i\kappa}^\alpha D_j(\xi^\kappa),\quad \ldots,\quad \zeta_{i_1,\ldots,i_\kappa}^\alpha=D_{i_\kappa}(\zeta_{i_1,\ldots,i_{\kappa-1}}^\alpha)-u_{i_1,i_2,\ldots,i_{\kappa-1}j}^\alpha D_{i_\kappa}(\xi^j). \tag{2.19}$$

Remark 2.11. The prolonged generators of the prolongations $\mathcal{G}^{[1]},\ldots,\mathcal{G}^{[\kappa]}$ of the group $\mathcal{G}$ are

$$X^{[1]}=X+\zeta_i^\alpha\frac{\partial}{\partial u_i^\alpha},\quad \ldots, X^{[\kappa]}=X^{[\kappa-1]}+\zeta_{i_1,\ldots,i_\kappa}^\alpha\frac{\partial}{\partial\zeta_{i_1,\ldots,i_\kappa}^\alpha},\ \kappa\geq 1, \tag{2.20}$$

for the group generator X in (2.9).

Group invariants.

Definition 2.12. A function $\Gamma(x^i,u^\alpha)$ is said to be an invariant of $\mathcal{G}$ of in (2.1) if

$$\Gamma(\bar{x}^i,\bar{u}^\alpha)=\Gamma(x^i,u^\alpha). \tag{2.21}$$

Theorem 2.13. *A function $\Gamma(x^i,u^\alpha)$ is an invariant of the group $\mathcal{G}$ given by (2.1) if and only if it solves the following first-order linear PDE: [8]*

$$X\Gamma=\xi^i(x^i,u^\alpha)\frac{\partial\Gamma}{\partial x^i}+\eta^\alpha(x^i,u^\alpha)\frac{\partial\Gamma}{\partial u^\alpha}=0. \tag{2.22}$$

From Theorem (2.13), we have the following result.

Theorem 2.14. *The Lie group $\mathcal{G}$ in (2.1) [9] has precisely $n-1$ functionally independent invariants and one can take as the basic invariants, the left-hand sides of the first integrals*

$$\psi_1(x^i,u^\alpha)=c_1,\ldots,\psi_{n-1}(x^i,u^\alpha)=c_{n-1}, \tag{2.23}$$

of the characteristic equations for (2.22):

$$\frac{dx^i}{\xi^i(x^i,u^\alpha)}=\frac{du^\alpha}{\eta^\alpha(x^i,u^\alpha)}. \tag{2.24}$$

Symmetry groups.

Definition 2.15. We define the vector field X (2.9) as a Lie point symmetry of (2.2) if the determining equations

$$X^{[\pi]}\Delta_\alpha\Big|_{\Delta_\alpha=0} = 0, \quad \alpha = 1, \ldots, m, \quad \pi \geq 1, \tag{2.25}$$

are satisfied for the π-th prolongation of X, namely $X^{[\pi]}$.

Definition 2.16. The Lie group $\mathcal{G}$ is a symmetry group of (2.2) if (2.2) is form-invariant, that is

$$\Delta_\alpha\left(\bar{x}^i, \bar{u}^\alpha, \bar{u}_{(1)}, \ldots, \bar{u}_{(\pi)}\right) = 0. \tag{2.26}$$

Theorem 2.17. *The Lie group $\mathcal{G}$ (2.1) can be constructed from the infinitesimal transformations in (2.5) by integrating the Lie equations*

$$\frac{d\bar{x}^i}{d\epsilon} = \xi^i(\bar{x}^i, \bar{u}^\alpha), \quad \bar{x}^i\Big|_{\epsilon=0} = x^i, \quad \frac{d\bar{u}^\alpha}{d\epsilon} = \eta^\alpha(\bar{x}^i, \bar{u}^\alpha), \quad \bar{u}^\alpha\Big|_{\epsilon=0} = u^\alpha. \tag{2.27}$$

Lie algebras.

Definition 2.18. A vector space $\mathcal{V}_r$ of operators [8] X (2.9) is a Lie algebra if for any $X_i, X_j \in \mathcal{V}_r$,

$$[X_i, X_j] = X_iX_j - X_jX_i, \tag{2.28}$$

is in $\mathcal{V}_r$ for all $i, j = 1, \ldots, r$.

Remark 2.19. The commutator is bilinear, skew symmetric and admits to the Jacobi identity [5].

Theorem 2.20. *The set of solutions of (2.25) forms a Lie algebra [10].*

Exact solutions. The methods of (G′/G)-expansion method [7], Extended Jacobi elliptic function expansion [9] and Kudryashov [11] are usually applied after symmetry reductions.

Conservation laws. [11]

Fundamental operators.

Definition 2.21. The Euler-Lagrange operator $\frac{\delta}{\delta u^\alpha}$ is

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{\kappa\geq 1}(-1)^\kappa D_{i_1}, \ldots, D_{i_\kappa}\frac{\partial}{\partial u^\alpha_{i_1 i_2 \ldots i_\kappa}}, \tag{2.29}$$

and the Lie- Bäcklund operator in abbreviated form [11] is

$$X = \xi^i\frac{\partial}{\partial x^i} + \eta^\alpha\frac{\partial}{\partial u^\alpha} + \ldots. \tag{2.30}$$

Remark 2.22. The Lie- Bäcklund operator (2.30) in its prolonged form is

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \sum_{\kappa \geq 1} \zeta_{i_1 \dots i_\kappa} \frac{\partial}{\partial u^\alpha_{i_1 i_2 \dots i_\kappa}}, \tag{2.31}$$

for

$$\zeta^\alpha_i = D_i(W^\alpha) + \xi^j u^\alpha_{ij}, \qquad \dots, \zeta^\alpha_{i_1 \dots i_\kappa} = D_{i_1 \dots i_\kappa}(W^\alpha) + \xi^j u^\alpha_{j i_1 \dots i_\kappa}, \quad j = 1, \dots, n. \tag{2.32}$$

and the Lie characteristic function

$$W^\alpha = \eta^\alpha - \xi^j u^\alpha_j. \tag{2.33}$$

Remark 2.23. The characteristic form of Lie- Bäcklund operator (2.31) is

$$X = \xi^i D_i + W^\alpha \frac{\partial}{\partial u^\alpha} + D_{i_1 \dots i_\kappa}(W^\alpha) \frac{\partial}{\partial u^\alpha_{i_1 i_2 \dots i_\kappa}}. \tag{2.34}$$

The method of multipliers.

Definition 2.24. A function $\Lambda^\alpha \left(x^i, u^\alpha, u_{(1)}, \dots\right) = \Lambda^\alpha$, is a multiplier of (2.2) if [7]

$$\Lambda^\alpha \Delta_\alpha = D_i T^i, \tag{2.35}$$

where $D_i T^i$ is a divergence expression.

Definition 2.25. To find the multipliers Λ^α, one solves the determining equations (2.36) [10],

$$\frac{\delta}{\delta u^\alpha}(\Lambda^\alpha \Delta_\alpha) = 0. \tag{2.36}$$

Ibragimov's conservation theorem . The technique [5] enables one to construct conserved vectors associated with each Lie point symmetry of (2.2).

Definition 2.26. The adjoint equations of (2.2) are

$$\Delta^*_\alpha \left(x^i, u^\alpha, v^\alpha, \dots, u_{(\pi)}, v_{(\pi)}\right) \equiv \frac{\delta}{\delta u^\alpha}(v^\beta \Delta_\beta) = 0, \tag{2.37}$$

for a new dependent variable v^α.

Definition 2.27. The Formal Lagrangian $\mathcal{L}$ of (2.2) and its adjoint equations (2.37) is [8]

$$\mathcal{L} = v^\alpha \Delta_\alpha(x^i, u^\alpha, u_{(1)}, \dots, u_{(\pi)}). \tag{2.38}$$

Theorem 2.28. *Every infinitesimal symmetry X of (2.2) leads to conservation laws [6]*

$$D_i T^i \Big|_{\Delta_{\alpha=0}} = 0, \tag{2.39}$$

where the conserved vector

$$\begin{aligned} T^i = \xi^i \mathcal{L} + W^\alpha \left[\frac{\partial \mathcal{L}}{\partial u^\alpha_i} - D_j \left(\frac{\partial \mathcal{L}}{\partial u^\alpha_{ij}} \right) + D_j D_k \left(\frac{\partial \mathcal{L}}{\partial u^\alpha_{ijk}} \right) - \dots \right] + \\ D_j(W^\alpha) \left[\frac{\partial \mathcal{L}}{\partial u^\alpha_{ij}} - D_k \left(\frac{\partial \mathcal{L}}{\partial u^\alpha_{ijk}} \right) + \dots \right] + D_j D_k(W^\alpha) \left[\frac{\partial \mathcal{L}}{\partial u^\alpha_{ijk}} - \dots \right]. \end{aligned} \tag{2.40}$$

3. Main results

3.1. **Lie point symmetries of equal width equation(1.1).** We start first by computing Lie point symmetries of the equal width Equation (1.1), which admits the one-parameter Lie group of transformations with infinitesimal generator

$$X = \tau(t,x,u)\frac{\partial}{\partial t} + \xi(t,x,u)\frac{\partial}{\partial x} + \eta(t,x,u)\frac{\partial}{\partial u} \tag{3.1}$$

if and only if

$$X^{[3]}\Delta\Big|_{\Delta=0} = 0. \tag{3.2}$$

where

$$X^{[3]} = X + \zeta_1\frac{\partial}{\partial u_t} + \zeta_2\frac{\partial}{\partial u_x} + \zeta_{122}\frac{\partial}{\partial u_{txx}}, \tag{3.3}$$

is the third prolongation of the Lie point symmetry X as defined in (2.20) and

$$\zeta_1 = D_t(\eta) - u_t D_t(\tau) - u_x D_t(\xi), \tag{3.4}$$

$$\zeta_{12} = D_x(\zeta_1) - u_{tt} D_x(\tau) - u_{tx} D_x(\xi), \tag{3.5}$$

$$\zeta_2 = D_x(\eta) - u_t D_x(\tau) - u_x D_x(\xi), \tag{3.6}$$

$$\zeta_{122} = D_x(\zeta_{12}) - u_{ttx} D_x(\tau) - u_{txx} D_x(\xi), \tag{3.7}$$

as defined in (2.19), and

$$D_t = \frac{\partial}{\partial t} + u_t\frac{\partial}{\partial u} + u_{tx}\frac{\partial}{\partial u_x} + u_{tt}\frac{\partial}{\partial u_t} + \cdots, \tag{3.8}$$

$$D_x = \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + u_{xx}\frac{\partial}{\partial u_x} + u_{tx}\frac{\partial}{\partial u_t} + \cdots. \tag{3.9}$$

Applying the definitions of D_t and D_x given in (3.8) and (3.9), we obtain the expanded form of the ζ_s as

$$\zeta_1 = \eta_t + u_t(\eta_u - \tau_t) + u_x(-\xi_t) + u_t u_x(-\xi_u) + u_t^2(-\tau_u),$$

$$\begin{aligned}\zeta_{12} = {} & \eta_{tx} + u_x(\eta_{tu} - \xi_{tx}) + u_{tx}(\eta_u - \tau_t - \xi_x) + u_t(\eta_{xu} - \tau_{tx}) + u_t u_x(\eta_{uu} - \xi_{xu} - \tau_{tu}) \\ & + u_t u_{tx}(-2\tau_u) + u_t^2(-\tau_{xu}) + u_t^2 u_x(-\tau_{uu}) + u_{xx}(-\xi_t) + u_x^2(-\xi_{tu}) + u_x u_{tx}(-2\xi_u) \\ & + u_t u_x^2(-\xi_{uu}) + u_t u_{xx}(-\xi_u) + u_{tt}(-\tau_x) + u_x u_{tt}(-\tau_u)\end{aligned}$$

$$\zeta_2 = \eta_x + u_x(\eta_u - \xi_x) + u_t(-\tau_x) + u_t u_x(-\tau_u) + u_x^2(-\xi_u),$$

$$\zeta_{122} = \eta_{txx} + u_x(2\eta_{txu} - \xi_{txx}) + u_{xx}(\eta_{tu} - 2\xi_{tx}) + u_x^2(\eta_{tuu} - 2\xi_{txu}) + u_{txx}(\eta_u - \tau_t - 2\xi_x)$$
$$+ u_t u_x(2\eta_{xuu} - \xi_{xxu} - 2\tau_{txu}) + u_x u_{tx}(2\eta_{uu} - 4\xi_{xu} - \tau_{tu}) + u_{tx}(2\eta_{xu} - 2\tau_{tx} - \xi_{xx})$$
$$+ u_t(\eta_{xxu} - \tau_{txx}) + u_t u_{xx}(\eta_{uu} - 2\xi_{xu} - \tau_{tu}) + u_t u_x^2(\eta_{uuu} - 2\xi_{xuu} - \tau_{tuu}) + u_{tx}^2(-2\tau_u)$$
$$u_t u_{txx}(-2\tau_u) + u_t u_{tx}(-4\tau_{xu}) + u_t^2(-\tau_{xxu}) + u_x u_t^2(-2\tau_{xuu}) + u_t u_x u_{tx}(-4\tau_{uu}),$$
$$+ u_x^2 u_t^2(-\tau_{uuu}) + u_{xxx}(-\xi_t) + u_t^2 u_{xx}(-\tau_{uu}) + u_x u_{xx}(-4\xi_{tu}) + u_x^3(-\xi_{tuu}) + u_{xx} u_{tx}(-3\xi_u)$$
$$u_x u_{txx}(-2\xi_u) + u_x^2 u_{tx}(-3\xi_{uu}) + u_x u_t u_{xx}(-3\xi_{uu}) + u_t u_x^3(-\xi_{uuu}) + u_t u_{xxx}(-\xi_u)$$
$$+ u_{ttx}(-2\tau_x) + u_{tt}(-\tau_{xx}) + u_x u_{tt}(-2\tau_{xu}) + u_x^2 u_{tt}(-\tau_{uu}) + u_{xx} u_{tt}(-\tau_u) + u_x u_{ttx}(-2\tau_u - \xi_u) \tag{3.10}$$

Now from Equation (3.2), we have

$$\zeta_1 + \alpha\eta u_x + \alpha\zeta_2 u + \beta\zeta_{122}\Big|_{u_{txx} = -\frac{u_t}{\beta} - \frac{\alpha}{\beta} u u_x} = 0, \tag{3.11}$$

If we substitute for ζ_1, ζ_2 and ζ_{122} in the determining Equation (3.11), we obtain the following;

$$\eta_t + u_t(\eta_u - \tau_t) + u_x(-\xi_t) + u_t u_x(-\xi_u) + u_t^2(-\tau_u) + \alpha\eta u_x$$
$$+ \alpha u\{\eta_x + u_x(\eta_u - \xi_x) + u_t(-\tau_x) + u_t u_x(-\tau_u) + u_x^2(-\xi_u)\}$$
$$+ \beta\Big\{\eta_{txx} + u_x(2\eta_{txu} - \xi_{txx}) + u_{xx}(\eta_{tu} - 2\xi_{tx}) + u_x^2(\eta_{tuu} - 2\xi_{txu}) + u_{txx}(\eta_u - \tau_t - 2\xi_x)$$
$$+ u_t u_x(2\eta_{xuu} - \xi_{xxu} - 2\tau_{txu}) + u_x u_{tx}(2\eta_{uu} - 4\xi_{xu} - \tau_{tu}) + u_{tx}(2\eta_{xu} - 2\tau_{tx} - \xi_{xx})$$
$$+ u_t(\eta_{xxu} - \tau_{txx}) + u_t u_{xx}(\eta_{uu} - 2\xi_{xu} - \tau_{tu}) + u_t u_x^2(\eta_{uuu} - 2\xi_{xuu} - \tau_{tuu}) + u_{tx}^2(-2\tau_u)$$
$$u_t u_{txx}(-2\tau_u) + u_t u_{tx}(-4\tau_{xu}) + u_t^2(-\tau_{xxu}) + u_x u_t^2(-2\tau_{xuu}) + u_t u_x u_{tx}(-4\tau_{uu}),$$
$$+ u_x^2 u_t^2(-\tau_{uuu}) + u_{xxx}(-\xi_t) + u_t^2 u_{xx}(-\tau_{uu}) + u_x u_{xx}(-4\xi_{tu}) + u_x^3(-\xi_{tuu}) + u_{xx} u_{tx}(-3\xi_u)$$
$$u_x u_{txx}(-2\xi_u) + u_x^2 u_{tx}(-3\xi_{uu}) + u_x u_t u_{xx}(-3\xi_{uu}) + u_t u_x^3(-\xi_{uuu}) + u_t u_{xxx}(-\xi_u)$$
$$+ u_{ttx}(-2\tau_x) + u_{tt}(-\tau_{xx}) + u_x u_{tt}(-2\tau_{xu}) + u_x^2 u_{tt}(-\tau_{uu}) + u_{xx} u_{tt}(-\tau_u) + u_x u_{ttx}(-2\tau_u - \xi_u)\Big\}$$
$$\Bigg|_{u_{txx} = -\frac{u_t}{\beta} - \frac{\alpha}{\beta} u u_x} = 0 \tag{3.12}$$

Now replacing u_{txx} by $-\frac{u_t}{\beta} - \frac{\alpha}{\beta} u u_x$ in Equation (3.12), we have

$$
\begin{aligned}
&\eta_t + u_t(\eta_u - \tau_t) + u_x(-\xi_t) + u_t u_x(-\xi_u) + u_t^2(-\tau_u) + \alpha\eta u_x \\
&+ \alpha u\{\eta_x + u_x(\eta_u - \xi_x) + u_t(-\tau_x) + u_t u_x(-\tau_u) + u_x^2(-\xi_u)\} \\
&+ \beta\Bigg\{\eta_{txx} + u_x(2\eta_{txu} - \xi_{txx}) + u_{xx}(\eta_{tu} - 2\xi_{tx}) + u_x^2(\eta_{tuu} - 2\xi_{txu}) + \\
&\left[-\frac{u_t}{\beta} - \frac{\alpha}{\beta}uu_x\right](\eta_u - \tau_t - 2\xi_x) \\
&+ u_t u_x(2\eta_{xuu} - \xi_{xxu} - 2\tau_{txu}) + u_x u_{tx}(2\eta_{uu} - 4\xi_{xu} - \tau_{tu}) + u_{tx}(2\eta_{xu} - 2\tau_{tx} - \xi_{xx}) \\
&+ u_t(\eta_{xxu} - \tau_{txx}) + u_t u_{xx}(\eta_{uu} - 2\xi_{xu} - \tau_{tu}) + u_t u_x^2(\eta_{uuu} - 2\xi_{xuu} - \tau_{tuu}) + u_{tx}^2(-2\tau_u) \\
&+ u_t\left[-\frac{u_t}{\beta} - \frac{\alpha}{\beta}uu_x\right](-2\tau_u) + u_t u_{tx}(-4\tau_{xu}) + u_t^2(-\tau_{xxu}) + u_x u_t^2(-2\tau_{xuu}) + u_t u_x u_{tx}(-4\tau_{uu}), \\
&+ u_x^2 u_t^2(-\tau_{uuu}) + u_{xxx}(-\xi_t) + u_t^2 u_{xx}(-\tau_{uu}) + u_x u_{xx}(-4\xi_{tu}) + u_x^3(-\xi_{tuu}) + u_{xx}u_{tx}(-3\xi_u) \\
&u_x\left[-\frac{u_t}{\beta} - \frac{\alpha}{\beta}uu_x\right](-2\xi_u) + u_x^2 u_{tx}(-3\xi_{uu}) + u_x u_t u_{xx}(-3\xi_{uu}) + u_t u_x^3(-\xi_{uuu}) + u_t u_{xxx}(-\xi_u) \\
&+ u_{ttx}(-2\tau_x) + u_{tt}(-\tau_{xx}) + u_x u_{tt}(-2\tau_{xu}) \\
&+ u_x^2 u_{tt}(-\tau_{uu}) + u_{xx}u_{tt}(-\tau_u) + u_x u_{ttx}(-2\tau_u - \xi_u)\Bigg\} = 0
\end{aligned}
\tag{3.13}
$$

which can be written as

$$
\begin{aligned}
&\eta_t + \alpha u\eta_x + \beta\eta_{txx} + u_t(\beta\eta_{xxu} - \beta\tau_{txx} + 2\xi_x - \alpha u\tau_x) \\
&+ u_x\big(2\beta\eta_{txu} - \beta\xi_{txx} - \xi_t + \alpha u\xi_x + \alpha u\tau_t + \alpha\eta\big) \\
&+ u_t u_x(2\xi_u + 2\beta\eta_{xuu} - \beta\xi_{xxu} - 2\beta\tau_{txu} + \alpha u\tau_u) + u_t^2(\tau_u - \beta\tau_{xxu}) + \\
&u_x^2(2\alpha u\xi_u + \beta\eta_{tuu} - 2\beta\xi_{txu}) + \beta\Bigg\{u_{xx}(\eta_{tu} - 2\xi_{tx}) \\
&+ u_x u_{tx}(2\eta_{uu} - 4\xi_{xu} - \tau_{tu}) + u_{tx}(2\eta_{xu} - 2\tau_{tx} - \xi_{xx}) \\
&+ u_t u_{xx}(\eta_{uu} - 2\xi_{xu} - \tau_{tu}) + u_t u_x^2(\eta_{uuu} - 2\xi_{xuu} - \tau_{tuu}) + u_{tx}^2(-2\tau_u) \\
&+ u_t u_{tx}(-4\tau_{xu}) + u_x u_t^2(-2\tau_{xuu}) + u_t u_x u_{tx}(-4\tau_{uu}), \\
&+ u_x^2 u_t^2(-\tau_{uuu}) + u_{xxx}(-\xi_t) + u_t^2 u_{xx}(-\tau_{uu}) + u_x u_{xx}(-3\xi_{tu}) + u_x^3(-\xi_{tuu}) + u_{xx}u_{tx}(-3\xi_u) \\
&+ u_x^2 u_{tx}(-3\xi_{uu}) + u_x u_t u_{xx}(-3\xi_{uu}) + u_t u_x^3(-\xi_{uuu}) + u_t u_{xxx}(-\xi_u) \\
&+ u_{ttx}(-2\tau_x) + u_{tt}(-\tau_{xx}) + u_x u_{tt}(-2\tau_{xu}) + u_x^2 u_{tt}(-\tau_{uu}) + u_{xx}u_{tt}(-\tau_u) + u_x u_{ttx}(-2\tau_u)\Bigg\} = 0
\end{aligned}
\tag{3.14}
$$

Since the functions τ, ξ and η depend only on t, x and u and are independent of the derivatives of u, we can then split the above equation on the derivatives of u and obtain

$$\tau_x = \tau_u = \xi_u = \xi_t = \xi_x = \eta_{uu} = \eta_{tu} = 0, \tag{3.15}$$

$$\eta + u\tau_t = 0, \tag{3.16}$$

$$\eta_t + \alpha u\eta_x + \beta\eta_{txx} = 0 \tag{3.17}$$

From Equation (3.15), we find that

$$\tau = \tau(t), \tag{3.18}$$

$$\xi = C_1, \tag{3.19}$$

$$\eta = A(x)u + B(t, x). \tag{3.20}$$

Now substituting η into Equation (3.17) yields

$$B_t(t, x) + \alpha u\big[A(x)_x u + B_x(t, x)\big] + \beta B_{txx}(t, x) = 0. \tag{3.21}$$

Separation of (3.21) on powers of u gives the following equations

$$u^2 : A(x)_x = 0, \tag{3.22}$$

$$u : B_x(t, x) = 0, \tag{3.23}$$

$$u^0 : B_t(t, x) + \beta B_{txx}(t, x) = 0. \tag{3.24}$$

Integration of Equations (3.22) and (3.23) with respect to x gives that

$$A(x) = C_2 \tag{3.25}$$

$$B(t, x) = B(t). \tag{3.26}$$

Now use Equation (3.26) in Equation (3.24) to obtain $B_{txx}(t, x) = 0$ and as a result

$$B_t(t, x) = 0. \tag{3.27}$$

Integrating Equation (3.27) with respect to t gives

$$B(t, x) = C_3. \tag{3.28}$$

If we substitute $\eta = C_2 u + C_3$ into Equation (3.16), we have

$$C_2 u + C_3 + \tau_t u = 0. \tag{3.29}$$

From Equation (3.29), if we obtain

$$\tau(t) = -C_2 t - C_3\frac{t}{u} + C_4. \tag{3.30}$$

and finally;

$$\tau = -C_2 t - C_3 \frac{t}{u} + C_4, \tag{3.31}$$

$$\xi = C_1 \tag{3.32}$$

$$\eta = C_2 u + C_3. \tag{3.33}$$

We have obtained a four-dimensional Lie algebra of symmetries spanned by

$$X_1 = \frac{\partial}{\partial x}, \tag{3.34}$$

$$X_2 = u\frac{\partial}{\partial u} - t\frac{\partial}{\partial t}, \tag{3.35}$$

$$X_3 = \frac{\partial}{\partial u} - \frac{t}{u}\frac{\partial}{\partial t}, \tag{3.36}$$

$$X_4 = \frac{\partial}{\partial t}. \tag{3.37}$$

3.2. **Commutator Table for Symmetries.** We evaluate the commutation relations for the symmetry generators. By definition of Lie bracket [9], for example, we have that

$$[X_1, X_4] = X_1X_4 - X_4X_1 = \left(\frac{\partial}{\partial x}\frac{\partial}{\partial t}\right) - \left(\frac{\partial}{\partial t}\frac{\partial}{\partial x}\right) = 0. \tag{3.38}$$

Remark 3.1. The remaining commutation relations are obtained analogously. We present all

commutation relations in table (1) below.

$[X_i, X_j]$	X_1	X_2	X_3	X_4
X_1	0	0	0	0
X_2	0	0	-X_3	X_4
X_3	0	$-X_3$	0	$\frac{1}{u}X_4$
X_4	0	-X_4	- $\frac{1}{u}X_4$	0

Table 1. A commutator table for Lie algebra of equal width equation.

3.3. **Group Transformations.** The corresponding one-parameter group of transformations can be determined by solving the Lie equations [6]. Let T_{ϵ_i} be the group of transformations for each $X_i, i = 1, 2, 3, 4$. We display how to obtain T_{ϵ_i} from X_i by finding one-parameter group for the infinitesimal generator X_1, namely,

$$X_1 = \frac{\partial}{\partial x}. \tag{3.39}$$

In particular, we have the Lie equations

$$\begin{aligned} \frac{d\bar{t}}{d\epsilon} =& 0, \quad \bar{t}\Big|_{\epsilon=0} = t, \\ \frac{d\bar{x}}{d\epsilon} =& 1, \quad \bar{x}\Big|_{\epsilon=0} = x, \\ \frac{d\bar{u}}{d\epsilon} =& 0, \quad \bar{u}\Big|_{\epsilon=0} = u. \end{aligned} \tag{3.40}$$

Solving the system (3.40) one obtains,

$$\bar{t} = t, \quad \bar{x} = x + \epsilon, \quad \bar{u} = u, \tag{3.41}$$

and hence the one-parameter group T_{ϵ_4} corresponding to the operator X_1 is

$$T_{\epsilon_1}: \quad (\bar{t}, \bar{x}, \bar{u}) = (t, x + \epsilon_1, u). \tag{3.42}$$

All the five one-parameter groups are presented below :

$$\begin{aligned} T_{\epsilon_1}: &\quad (\bar{t}, \bar{x}, \bar{u}) = (t, x + \epsilon_1, u) \\ T_{\epsilon_2}: &\quad (\bar{t}, \bar{x}, \bar{u}) = (te^{-\epsilon_2}, x, ue^{\epsilon_2}) \\ T_{\epsilon_3}: &\quad (\bar{t}, \bar{x}, \bar{u}) = (te^{-\frac{\epsilon_3}{u}}, x, u + \epsilon_3) \\ T_{\epsilon_4}: &\quad (\bar{t}, \bar{x}, \bar{u}) = (t + \epsilon_4, x, u). \end{aligned} \tag{3.43}$$

3.4. **Symmetry transformations.** We now show how the symmetries we have obtained can be used to transform special exact solutions of the equal width equation into new solutions. The Lie group analysis vouches for fundamental ways of e constructing exact solutions of PDEs, that is, group transformations of known solutions and construction of group-invariant solutions. We will illustrate these methods with examples. If $\bar{u} = g(\bar{t}, \bar{x})$ is a solution of equation (1.1)

$$\phi(t, x, u, \epsilon) = g(f_1(t, x, u, \epsilon), f_2(t, x, u, \epsilon)), \tag{3.44}$$

is also a solution. The one parameter groups dictate to the following generated solutions:

$$\begin{aligned} T_{\epsilon_1}: u =& g(t, x + \epsilon_1) \\ T_{\epsilon_2}: u =& g(te^{-\epsilon_2}, x)e^{-\epsilon_2}, \\ T_{\epsilon_3}: u =& g(te^{-\frac{\epsilon_3}{u}}, x) - \epsilon_3, \\ T_{\epsilon_4}: u =& g(t + \epsilon_4, x). \end{aligned} \tag{3.45}$$

3.5. **Construction of Group-Invariant Solutions.** Now we compute the group invariant solutions of Burger's equation.

(i) $X_1 = \frac{\partial}{\partial x}$

The associated Lagrangian equations

$$\frac{dt}{0} = \frac{dx}{1} = \frac{du}{0}, \tag{3.46}$$

yield two invariants, $J_1 = t$ and $J_2 = u$. Thus using $J_2 = \Phi(J_1)$, we have

$$u(t,x) = \Phi(t). \tag{3.47}$$

The derivatives are given by :

$$u_t = \Phi'(t),$$
$$u_x = 0,$$
$$u_{txx} = 0.$$

If we substitute these derivatives into Equation (1.1) , we obtain the first order ordinary differential equation

$$\Phi'(t) = 0,$$

whose space invariant solution is

$$\Phi(t) = C_1, \tag{3.48}$$

and the group-invariant solution associated to the X_1 is

$$u(t,x) = C_1.$$

(ii) $X_2 = u\frac{\partial}{\partial u} - t\frac{\partial}{\partial t}$ The Lagrangian equations associated to this symmetry are

$$\frac{dt}{-t} = \frac{dx}{0} = \frac{du}{u}. \tag{3.49}$$

This gives the constants $J_1 = x$ and $J_2 = tu$, giving the solution

$$u = \frac{f(x)}{t}. \tag{3.50}$$

We obtain the derivatives as follows:

$$u_t = -\frac{f(x)}{t^2}, \tag{3.51}$$

$$u_x = \frac{f'(x)}{t} \tag{3.52}$$

$$u_{txx} = -\frac{f''(x)}{t^2} \tag{3.53}$$

If we substitute the above derivatives in Equation (1.1), we obtain the second order ordinary differential equation

$$f(x) - \alpha f(x)f'(x) + \beta f''(x) = 0. \tag{3.54}$$

Hence the group invariant solution to Equation to (1.1) will be given by

$$u(t,x) = \frac{f(x)}{t}, \tag{3.55}$$

where f satisfies Equation (3.54).

(iii) $X_3 = \frac{\partial}{\partial u} - \frac{t}{u}\frac{\partial}{\partial t}$

The Lagrangian system associated with the operator X_3 is

$$\frac{\mathrm{d}t}{-\frac{t}{u}} = \frac{\mathrm{d}x}{0} = \frac{\mathrm{d}u}{1}, \tag{3.56}$$

whose invariants are $J_1 = x$ and $J_2 = tu$. So, $u = \frac{g(x)}{t}$ is the group-invariant solution.

(iv) $X_4 = \frac{\partial}{\partial t}$

Characteristic equations associated to the operator X_4 are

$$\frac{\mathrm{d}t}{1} = \frac{\mathrm{d}x}{0} = \frac{\mathrm{d}u}{0}, \tag{3.57}$$

yields$J_1 = x$ and $J_2 = u$. As a result, the group-invariant solution of (1.1) for this case is $J_2 = \phi(J_1)$, for some ϕ an arbitrary function. That is,

$$u(t,x) = \phi(x). \tag{3.58}$$

The derivatives of given function are

$$u_t = 0, \tag{3.59}$$

$$u_x = \phi'(x), \tag{3.60}$$

$$u_{txx} = 0. \tag{3.61}$$

Substitution of the value of $\phi(x)$ into Equation (1.1) yields a first order nonlinear ordinary differential equation

$$\phi(x)\phi'(x) = 0. \tag{3.62}$$

From Equation (3.62), either $\phi(x) = 0$ or $\phi'(x) = 0$. The case $\phi(x) = 0 \implies \phi'(x) = 0$, and the equation is satisfied. The case $\phi(x) \neq 0$ implies that $\phi'(x) = 0$ and by integration, $\phi(x) = C_1$, hence the group invariant solution is given by

$$u(t,x) = C_2. \tag{3.63}$$

3.6. **Soliton.** We obtain a traveling wave solution of the equal width Equation(1.1) by considering a linear combination of the symmetries X_1 and X_4, namely, [7]

$$X = cX_1 + X_4 = c\frac{\partial}{\partial x} + \frac{\partial}{\partial t}, \quad \text{for some constant } c. \tag{3.64}$$

The characteristic equations are

$$\frac{\mathrm{d}t}{1} = \frac{\mathrm{d}x}{c} = \frac{\mathrm{d}u}{0} \tag{3.65}$$

We get two invariants, $J_1 = x - ct$ and $J_2 = u$. So the group-invariant solution is

$$u(t,x) = Q(x-ct), \tag{3.66}$$

for some arbitrary function φ and c the velocity of the wave.

Substitution of u into (1.1) yields a second order ordinary differential equation

$$cQ' - \alpha QQ' + \beta cQ''' = 0, \tag{3.67}$$

which can be integrated with respect to Q to give

$$cQ - \alpha\frac{Q^2}{2} + \beta cQ' = 0, \tag{3.68}$$

where we have used 0 as a constant of integration. Equation (3.68) can be rearranged and variables separated to have

$$\frac{d\xi}{2\beta c} = \frac{dQ}{\alpha Q^2 - 2cQ}, \quad \xi = x - ct. \tag{3.69}$$

The right hand side can be resolved into partial fractions to obtain

$$\frac{\xi}{2\beta c} = \frac{1}{2c}\int\left[\frac{\alpha}{\alpha Q - 2c} - \frac{1}{Q}\right]dQ = \frac{1}{2c}\ln\left|\frac{\alpha Q - 2c}{Q}\right| + \ln|C_3|, \tag{3.70}$$

where C_3 is a constant of integration. After rewriting, we have

$$Q(x-ct) = \frac{2cC_3}{\alpha C_3 - e^{\frac{x-ct}{\beta}}}. \tag{3.71}$$

Finally, the soliton solutions are given by

$$u(t,x) = \frac{2cC_3}{\alpha C_3 - e^{\frac{x-ct}{\beta}}}. \tag{3.72}$$

4. Conservation laws of equation (1.1)

We will employ multipliers in the construction of conservation laws.

4.1. **The multipliers.** We make use of the Euler-Lagrange operator defined as defined in [6] to look for a zeroth order multiplier $\Lambda = \Lambda(t,x,u)$. The resulting determining equation for computing Λ is

$$\frac{\delta}{\delta u}\left[\Lambda\{u_t + \alpha uu_x + \beta u_{txx}\}\right] = 0. \tag{4.1}$$

where

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} - D_t\frac{\partial}{\partial u_t} - D_x\frac{\partial}{\partial u_x} - D_tD_x^2\frac{\partial}{\partial u_{txx}} + \dots \tag{4.2}$$

Expansion of Equation (4.1) yields

$$\Lambda_u(u_t + \alpha uu_x + \beta u_{txx}) + \alpha u_x\Lambda - D_t(\Lambda) - \alpha D_x(u\Lambda) - \beta D_tD_x^2(\Lambda) = 0. \tag{4.3}$$

Invoking the total derivatives

$$D_t = \frac{\partial}{\partial t} + u_t\frac{\partial}{\partial u} + u_{tx}\frac{\partial}{\partial u_x} + u_{tt}\frac{\partial}{\partial u_t} + \cdots, \tag{4.4}$$

$$D_x = \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + u_{xx}\frac{\partial}{\partial u_x} + u_{tx}\frac{\partial}{\partial u_t} + \cdots. \tag{4.5}$$

on Equation (4.3) produces

$$\begin{aligned}&\Lambda_t + \alpha u\Lambda_x + \beta\Lambda_{txx} + 2\beta(\Lambda_{txu})u_x + \beta(\Lambda_{tu})u_{xx} + \beta(\Lambda_{tuu})u_x^2 + 2\beta(\Lambda_{xu})u_{tx}\\ &+ 2\beta(\Lambda_{uu})u_xu_{tx} + \beta(\Lambda_{xxu})u_t + 2\beta(\Lambda_{xuu})u_xu_x + \beta(\Lambda_{uu})u_tu_{xx} + \beta(\Lambda_{uuu})u_tu_x^2 = 0\end{aligned} \tag{4.6}$$

Splitting Equation (4.6) on derivatives of u produces an overdetermined system of four partial differential
equations, namely,

$$\Lambda_{uu} = 0, \tag{4.7}$$

$$\Lambda_{xu} = 0, \tag{4.8}$$

$$\Lambda_{tu} = 0 \tag{4.9}$$

$$\Lambda_t + \alpha u\Lambda_x + \beta\Lambda_{txx} = 0 \tag{4.10}$$

By Equation (4.7), we have

$$\Lambda = A(t,x)u + B(t,x), \tag{4.11}$$

which if used in Equations (4.8–4.9), implies that

$$\Lambda = C_1u + B(t,x). \tag{4.12}$$

If we substitute (4.12) into Equation (4.10), we obtain

$$B_t(t,x) + \alpha uB_x(t,x) + \beta B_{txx}(t,x) = 0. \tag{4.13}$$

Separation of Equation (4.13) into powers of u gives us

$$u : B_x(t,x) = 0, \tag{4.14}$$

$$u^0 : B_t(t,x) + \beta B_{txx}(t,x) = 0. \tag{4.15}$$

Equation (4.14) insists that

$$B_{txx}(t,x) = 0 \implies B_t(t,x) = 0 = B_x(t,x), \tag{4.16}$$

and thus

$$B(t,x) = C_2. \tag{4.17}$$

As a result

$$\Lambda(t,x,u) = C_1u + C_2. \tag{4.18}$$

Essentially, we extract the two multiplies

$$\Lambda_1 = 1 \tag{4.19}$$

$$\Lambda_2 = u. \tag{4.20}$$

Remark 4.1. Recall that a multiplier Λ for Equation(1.1) has the property that for the density $T^t = T^t(t, x, u, u_x)$ and flux $T^x = T^x(t, x, u, u_x, u_{tx})$,

$$\Lambda\,(u_t + \alpha u u_x + \beta u_{txx}) = D_t T^t + D_x T^x. \tag{4.21}$$

We derive a conservation law corresponding to each of the multipliers.

(i). **Conservation law for the multiplier** $\Lambda_1 = 1$

Expansion of equation (4.21) gives

$$1\{u_t + \alpha u u_x + \beta u_{txx}\} = T^t_t + u_t T^t_u + u_{tx} T^t_{u_x} + T^x_x + u_x T^x_u + u_{xx} T^x_{u_x} + u_{txx} T^x_{u_{tx}}. \tag{4.22}$$

Splitting Equation (4.22) on the third derivative of u yields

$$u_{txx} \quad : \quad T^x_{u_{tx}} = \beta, \tag{4.23}$$

$$\text{Rest} \quad : \quad u_t + \alpha u u_x = T^t_t + u_t T^t_u + u_{tx} T^t_{u_x} + T^x_x + u_x T^x_u + u_{xx} T^x_{u_x}. \tag{4.24}$$

The integration of Equation (4.23) with respect to u_{tx} gives

$$T^x = \beta u_{tx} + A(t, x, u, u_x). \tag{4.25}$$

Substituting the expression of T^x from (4.25) into Equation (4.22) we get

$$\{u_t + \alpha u u_x\} = T^t_t + u_t T^t_u + u_{tx} T^t_{u_x} + A_x + u_x A_u + u_{xx} A_{u_x} \tag{4.26}$$

which splits on second derivatives of u, to give

$$u_{xx} \quad : \quad A_{u_x} = 0, \tag{4.27}$$

$$u_{tx} \quad : \quad T^t_{u_x} = 0, \tag{4.28}$$

$$\text{Rest} \quad : \quad \{u_t + \alpha u u_x\} = T^t_t + u_t T^t_u + A_x + u_x A_u. \tag{4.29}$$

Integrating equations (4.27) and (4.28) with respect to u_x manifests that $T^t = T^t(t, x, u)$ and $A = A(t, x, u)$. Using values of A and T^t in Equation (4.29), we have

$$\{u_t + \alpha u u_x\} = T^t_t + u_t T^t_u + A_x + u_x A_u, \tag{4.30}$$

which separates on first derivatives to give us

$$u_t \quad : \quad T^t_u = 1, \tag{4.31}$$

$$u_x \quad : \quad A_u = \alpha u, \tag{4.32}$$

$$\text{Rest} \quad : \quad T^t_t + A_x = 0. \tag{4.33}$$

Equations (4.31-4.32), can be integrated with respect u to obtain

$$T^t_u = u + B(t,x), \tag{4.34}$$

$$A = \alpha\frac{u^2}{2} + C(t,x), \tag{4.35}$$

If we use the obtained values in (4.33), we have

$$B_t(t,x) + C_x(t,x) = 0. \tag{4.36}$$

Since $B(t,x)$ and $C(t,x)$ contribute to the trivial part of the conservation law, we take $B(t,x) = C(t,x) = 0$ and obtain the conserved quantities

$$T^t = u, \tag{4.37}$$

$$T^x = \alpha\frac{u^2}{2} + \beta u_{tx} \tag{4.38}$$

from which the conservation law corresponding to the multiplier $\Lambda_1 = 1$ is given by

$$D_t(u) + D_x\left(\alpha\frac{u^2}{2} + \beta u_{tx}\right) = 0. \tag{4.39}$$

(ii). **Conservation law for the multiplier $\Lambda_2 = u$**

$$u\{u_t + \alpha u u_x + \beta u_{txx}\} = T^t_t + u_t T^t_u + u_{tx}T^t_{u_x} + T^x_x + u_x T^x_u + u_{xx}T^x_{u_x} + u_{txx}T^x_{u_{tx}}. \tag{4.40}$$

Splitting Equation (4.40) on the third derivative of u yields

$$u_{txx} \quad : \quad T^x_{u_{tx}} = \beta u, \tag{4.41}$$

$$\text{Rest} \quad : \quad u_t + \alpha u u_x = T^t_t + u_t T^t_u + u_{tx}T^t_{u_x} + T^x_x + u_x T^x_u + u_{xx}T^x_{u_x}. \tag{4.42}$$

The integration of Equation (4.41) with respect to u_{tx} gives

$$T^x = \beta u u_{tx} + A(t,x,u,u_x). \tag{4.43}$$

Substituting the expression of T^x from (4.43) into Equation (4.40) we get

$$u\{u_t + \alpha u u_x\} = T^t_t + u_t T^t_u + u_{tx}T^t_{u_x} + A_x + u_x A_u + u_x\beta u_{tx} + u_{xx}A_{u_x}. \tag{4.44}$$

which splits on second derivatives of u, to give

$$u_{xx} \quad : \quad A_{u_x} = 0, \tag{4.45}$$

$$u_{tx} \quad : \quad T^t_{u_x} = -\beta u_x, \tag{4.46}$$

$$\text{Rest} \quad : \quad u\{u_t + \alpha u u_x\} = T^t_t + u_t T^t_u + A_x + u_x A_u. \tag{4.47}$$

Integrating equations (4.45) and (4.46) with respect to u_x manifests that $T^t = -\frac{\beta u_x^2}{2} + B(t,x,u)$ and $A = A(t,x,u)$. Using values of A and T^t in Equation (4.47), we have

$$u\{u_t + \alpha u u_x\} = T^t_t + u_t T^t_u + A_x + u_x A_u, \tag{4.48}$$

which separates on first derivatives to give us

$$u_t \quad : \quad B_(t,x,u)u = u, \tag{4.49}$$

$$u_x \quad : \quad A_u = \alpha u^2, \tag{4.50}$$

$$\text{Rest} \quad : \quad B_t + A_x = 0. \tag{4.51}$$

Equations (4.49-4.50), can be integrated with respect u to obtain

$$B = \frac{u^2}{2} + C(t,x), \tag{4.52}$$

$$A = \alpha\frac{u^3}{3} + D(t,x), \tag{4.53}$$

If we use the obtained values in (4.51), we have

$$C_t(t,x) + D_x(t,x) = 0. \tag{4.54}$$

Since $C(t,x)$ and $D(t,x)$ contribute to the trivial part of the conservation law, we take $C(t,x) = D(t,x) = 0$ and obtain the conserved quantities

$$T^t = -\beta\frac{u_x^2}{2} + \frac{u^2}{2}, \tag{4.55}$$

$$T^x = \beta u u_{tx} + \alpha\frac{u^3}{3} \tag{4.56}$$

from which the conservation law corresponding to the multiplier $\Lambda_2 = u$ is given by

$$D_t\left(-\beta\frac{u_x^2}{2} + \frac{u^2}{2}\right) + D_x\left(\beta u u_{tx} + \alpha\frac{u^3}{3}\right) = 0. \tag{4.57}$$

Remark 4.2. It can be shown that the two sets of conserved quantities are conservation laws. Given that $\Lambda_1 = 1$, the verification reaffirms that the equal width equation is itself a conversation law.

5. Conclusion

In this manuscript, an infinite dimensional Lie algebra of Lie point symmetries has been applied to study a third-order equal width equation. A commutator table has been constructed for the obtained Lie algebra. We have also used symmetry reductions to compute exact group-invariant solutions, including a soliton. Conservation laws have also been derived for the model with the use of zeroth order multipliers.

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Author's contribution

The author wrote the article as a scholarly duty and passion to disseminate mathematical research and hereby declares that there is no conflict of interest.

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