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MATHEMATICAL MODELLING OF THE INRUN PROFILE OF A SKI JUMPING HILL WITH THE CONTROLLED TRACK REACTION FORCE

Rafał Palej
Renata Filipowska

Cracow University of Technology, Institute of Computing Science, Cracow, Poland
e-mail: palej@mech.pk.edu.pl; renata.filipowska@op.pl

The paper deals with the problem of modelling the inrun profile of a ski jumping hill with a variable radius of curvature. It presents the possibility of reducing the inertia force acting on a ski jumper in the immediate neighbourhood of the take-off track while maintaining the inclination angles of the starting segment of the track and the take-off track which are applied in the present day constructions. The problem of a further decrease of the inertia force resulting from the increase of the inclination angle of the starting segment has also been discussed. Finally, the profiles of the inrun which do not contain the inflexion point have been determined.

Key words: inrun profile, ski jumping hill, nonlinear boundary value problem, shooting method

1. Introduction

The inrun profiles of ski jumping hills designed and constructed nowadays (Neufert, 2002) consist of two segments (AC and DE) and a circular arc CD that is tangent to them (Fig. 1).

The segment AC constitutes a rectilinear part of the inrun and it is usually inclined at an angle $\beta = 35°$. It includes the segment AB in the length of which the starting gate is situated. The arc CD forms a curvilinear part of the inrun. Its task is to reduce the inclination angle of the tangent towards the track. The segment DE plays the role of the take-off track for a ski jumper and it is usually inclined at an angle $\alpha = 10.5°$. The coordinates of point A

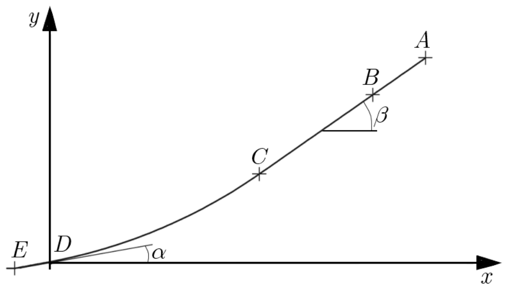

Fig. 1. A typical profile of the inrun of a ski jumping hill

and the angles α and β clearly define the length of the segment AC and the radius of curvature of the arc CD.

A normal reaction of the curvilinear part of the track is caused by the weight of the ski jumper and the normal inertia force, which is inversely proportional to the radius of curvature. This reaction is a measure of the load put on the ski jumper's legs. In the case of the ski jumping hills used nowadays, the normal reaction of the curvilinear part of the track exceeds the weight of the ski jumper at the point bordering on the take-off track by over 70%. Its value can be reduced by changing the profile of the inrun of a jumping hill. It is important to achieve the highest possible value of the radius of curvature at the point bordering on the take-off track to maximally diminish the normal inertia force. The reduction of the normal inertia force will make it easier for the ski jumper to take off and perform a good jump.

The increase of the radius of curvature at the point bordering on the take-off track can be obtained by replacing the rectilinear segment BC and the circular arc CD with one curvilinear segment BD. This exchange can be carried through in many ways. It is important that the normal reaction force should rise as the ski jumper approaches the take-off track until reaching its maximum at the point D. It turns out that it is possible to find various profiles that meet this requirement. From among the possible solutions, the profile for which the value of the normal reaction force at the point bordering on the take-off track is the smallest should be chosen.

The simplest approach, which is presented in Palej and Struk (2003b), consists in choosing a profile with a given form of the function in which the radius of curvature increases as the ski jumper approaches the take-off track. From the mathematical point of view, it is an initial value problem formulated for a nonlinear second order equation. A disadvantage of such an approach lies in the fact that the maximum of the normal reaction does not usually appear at the point bordering on the take-off track.

By applying the solutions obtained by Palej and Struk (2003b), a corrected profile of the inrun, presented in an invention project by Palej and Struk (2003a), for which the normal reaction force reaches its maximum at the point D, has been achieved.

Another approach, described by Palej and Struk (2004), consists in imposing variability of the radius of curvature of the inrun profile. The profile of the inrun results from the solution of an ordinary differential equation. From the mathematical point of view, it is a boundary value problem of a nonlinear second order equation with three boundary conditions. In this approach, it is possible to select parameters of the function which describes the radius of curvature in such a way that all the boundary conditions are fulfilled and the maximum of the normal reaction of the track occurs at the point bordering on the take-off track.

Still another approach, presented by Palej and Struk (2005), consists in assuming a constant value of the normal reaction of the track during the ski jumper's descent. The profile of the inrun is obtained there as well by means of solving a nonlinear second order differential equation with three boundary conditions. This approach brings about the smallest normal reaction at the point bordering on the take-off track.

A disadvantage of all of these approaches is a relatively big value of the inclination angle of the starting segment AB.

The paper presents, to a certain degree, a continuation and a development of the last approach.

The profile of the inrun is achieved by means of solving a nonlinear second order differential equation with four boundary conditions. Such an approach allows for the elimination of a big inclination angle of the starting segment and the obtainment of the required variability of the normal reaction force. From among various possible solutions we choose such a one for which the maximum of the reaction at the point bordering on the take-off track is the lowest and, at the same time, the ski jumper does not lose contact with the track during the descent.

2. The effect of the inrun profile on the normal reaction

The rectilinear segment BC and the circular arc CD can be replaced with one curvilinear segment BD which is tangent to the take-off track DE and the starting segment AB (Fig. 2).

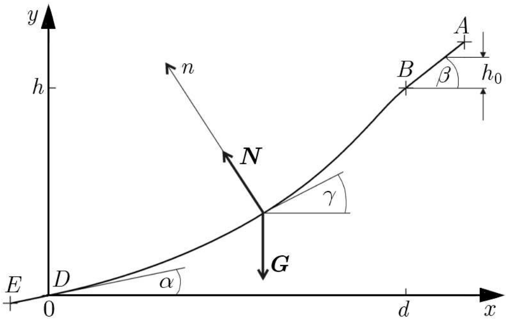

Fig. 2. A graphical illustration of the problem

The normal reaction and the radius of curvature of the curvilinear segment BD are described with the following formulas

$$N = mg\cos\gamma \pm \frac{mv^2}{r} \qquad\qquad r = \pm\frac{\sqrt{\{1+[y'(x)]^2\}^3}}{y''(x)} \tag{2.1}$$

where the sign "+" corresponds to the downward concavity of a plane curve, the sign "−" corresponds to the upward concavity, and the symbols used stand for: N – normal reaction of the track, m – mass of the ski jumper, g – gravitational acceleration, γ – inclination angle of the tangent line to the inrun, v – velocity of the ski jumper, r – radius of curvature of the track, $y(x)$ – profile of the inrun.

By applying the geometrical interpretation of the derivative, the following formula can be written down

$$\cos\gamma = \frac{1}{\sqrt{1+[y'(x)]^2}} \tag{2.2}$$

Assuming, like in the brachistochrone problem, the validity of the principle of conservation of energy, i.e. neglecting friction of the skis and the air drag, we can write

$$v^2 = 2g[h + h_0 - y(x)] \tag{2.3}$$

where h signifies the height of point B, h_0 – height of the starting gate over the point B.

By inserting Eqs. $(2.1)_2$-(2.3) into Eq. $(2.1)_1$, regardless of the type of concavity, a straightforward relation between the normal reaction N and the profile of the inrun $y(x)$ and its derivatives is achieved

$$N(x) = mg\frac{1}{\sqrt{1+[y'(x)]^2}} + 2mg\frac{h+h_0-y(x)}{\sqrt{\{1+[y'(x)]^2\}^3}}y''(x) \tag{2.4}$$

In order to facilitate the calculations, nondimensional variables have been introduced. The reference variables are: distance d (Fig. 2) – for the parameters described with length units and weight of the ski jumper G – for the normal reaction N. After the introduction of the nondimensional variables marked with letters of the Greek alphabet, Eq. (2.4) takes the following form

$$\nu(\xi) = \frac{1}{\sqrt{1+[\psi'(\xi)]^2}} + 2\frac{\eta+\eta_0-\psi(\xi)}{\sqrt{\{1+[\psi'(\xi)]^2\}^3}}\psi''(\xi) \tag{2.5}$$

where $\nu = N/(mg)$ is the nondimensional normal reaction of the track, $\eta = h/d$ – nondimensional height of the point B, $\eta_0 = h_0/d$ – nondimensional height of the starting gate over the point B, $\xi = x/d$ – nondimensional horizontal coordinate, $\psi = y/d$ – nondimensional vertical coordinate.

3. Formulation of the problem and the method of solution

By eliminating $\psi''(\xi)$ from Eq. (2.5), we obtain a differential equation with respect to $\psi(\xi)$, which contains an unknown function $\nu(\xi)$ that describes the nondimensional normal reaction

$$\psi''(\xi) = \frac{\nu(\xi)\sqrt{\{1+[\psi'(\xi)]^2\}^3} - [\psi'(\xi)]^2 - 1}{2[\eta+\eta_0-\psi(\xi)]} \tag{3.1}$$

The sought profile of the inrun has to pass points B and D and it must be tangent to the take-off track DE and the starting segment AB, which are inclined respectively at angles α and β to the horizontal. The solution to differential equation (3.1) should then fulfill the following boundary conditions

$$\begin{aligned} &\psi(0) = 0 && \psi(1) = \eta \\ &\psi'(0) = \tan\alpha && \psi'(1) = \tan\beta \end{aligned} \tag{3.2}$$

For an arbitrary function $\nu(\xi)$, the solution to Eq. (3.1) can fulfill two of four conditions (3.2) at the most. By selecting, in a particular way, the form of the function $\nu(\xi)$, a solution to Eq. (3.1) that fulfills all boundary conditions (3.2) can be found.

The function $\nu(\xi)$ should increase as the ski jumper approaches the take-off track until reaching its maximum at the point bordering on the take-off track. In addition, this function should contain two control parameters so that, through their proper selection, the solution sought fulfills all the boundary conditions.

For example, the following even polynomial functions possess the demanded properties of the normal reaction

$$\nu_i(\xi) = -a_i\xi^{2i} + b_i \qquad\qquad i = 1, 2, \ldots \tag{3.3}$$

The functions listed above impose solely the variability of the normal reaction without numerical description of its value at any point. The solution to equation (3.1), which fulfills all boundary conditions (3.2), was achieved by means of the shooting method. By giving various values of the control parameters a_i and b_i, the solution to equation (3.1) was found, regarding the problem as an initial-value problem described with the first two conditions given by (3.2). By applying the shooting method (Rao, 2002), the values of control parameters a_i and b_i were determined for which the solution to equation (3.1) fulfilled the remaining boundary conditions (3.2).

4. Numerical results

Four subsequent polynomial functions $\nu_i(\xi)$, described with formula (3.3), were considered in the calculations. For each of them, the values of control parameters a_i and b_i which guarantee the fulfillment of all boundary conditions (3.2) by the solution to equation (3.1) were determined. Table 1 shows values of the determined control parameters a_i and b_i for the following data: $\eta = 0.517677569826$ and $\eta_0 = 0.114298857$.

Table 1. A list of the control parameters a_i and b_i for which the solution to equation (3.1) fulfills all boundary conditions (3.2)

i	a_i	b_i
1	1.09211603	1.66119517
2	1.19886121	1.54432451
3	1.36668431	1.50428111
4	1.54431350	1.48370157

Figure 3 presents the obtained profiles of the inrun with the variable radius of curvature together with the profile of a typical ski jumping hill $(tsjh)$.

It is possible to observe in Fig. 3 the points of inflexion which appear in the neighbourhood of the point B. At these points, the radius of curvature tends to infinity. Figure 4 shows graphs of the radii of curvature of the solutions

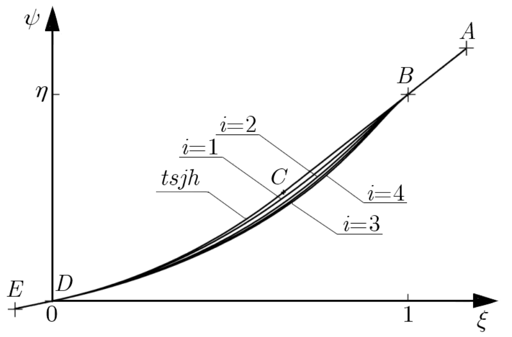

Fig. 3. Inrun profiles determined for four subsequent exponents i against the background of the profile of a typical ski jumping hill $(tsjh)$

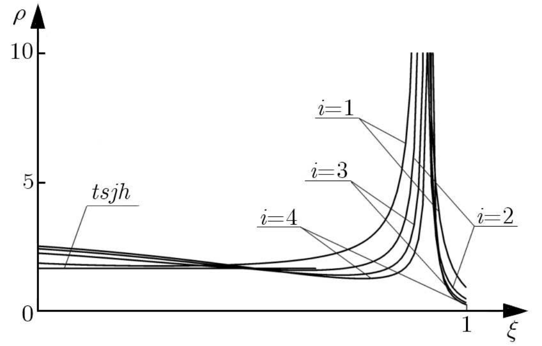

Fig. 4. Radii of curvature of the inrun profiles determined for four subsequent exponents i against the background of the radius of curvature of a typical ski jumping hill $(tsjh)$

obtained together with the graph of the radius of curvature of the profile of a typical ski jumping hill.

Together with the increase of the exponent i, the value of the radius of curvature at the point bordering on the take-off track increases as well, and the inflexion point approaches the point B. The fact that the upper part of the curvilinear segment BD is concave upward brings about the danger of losing contact with the track by the ski jumper. The graphs of the normal reaction of the track shown in Fig. 5 illustrate and explain the problem of the ski jumper's contact with the track.

For $i = 1, 2, 3$, the normal reaction of the track is positive at each point of the inrun, while for $i = 4$, the ski jumper loses contact with the track immediately after crossing the point B because the reaction is negative. A continuing growth of the exponent i increases the negative values of the normal reaction of the track in the immediate neighbourhood of the point B.

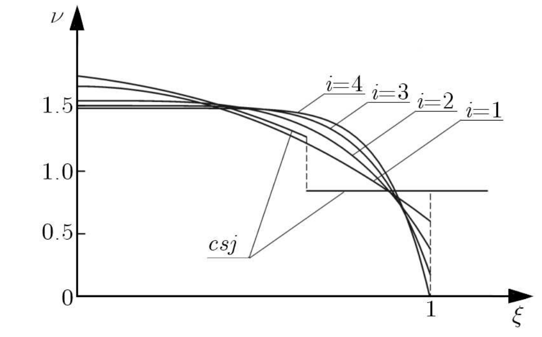

Fig. 5. Normal reactions of the inrun tracks determined for four subsequent exponents i against the background of the reaction of the inrun track of a typical ski jumping hill ($tsjh$)

5. Verification of the obtained results

In order to verify the mathematical model and the numerical results obtained, a reverse problem was considered – the one that consists in determination of the normal reaction of the track during the ski jumper's descent along a given profile. The verification was carried out for $i = 2$. A numerical solution to differential equation (3.1) was used to determine values of the function $\psi(\xi)$ in 101 equally spaced data points in the section $[0, 1]$. An interpolation polynomial in form of third-order splines was made to pass through every data point. The accuracy of the interpolation was shown in Fig. 6, which presents the difference between the numerical solution to equation (3.1) and cubic splines.

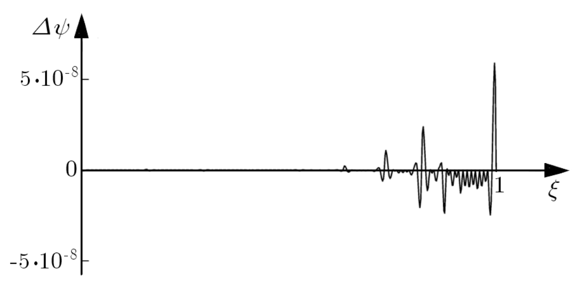

Fig. 6. The difference between the solution to equation (3.1) and cubic splines

The ski jumper's descent along the plane curve $y(x)$ (Fig. 2) is described with the following equations

$$m\ddot{x} = -N\sin\gamma \qquad\qquad m\ddot{y} = N\cos\gamma - mg \tag{5.1}$$

By eliminating the reaction N from the set of equations (5.1) and considering formulas

$$y' = \tan\gamma \qquad\qquad \ddot{y} = y''\dot{x}^2 + y'\ddot{x} \tag{5.2}$$

an equation describing ski jumper's motion along the horizontal axis x is obtained

$$\ddot{x} = -\frac{y'(x)[g + \dot{x}^2 y''(x)]}{1 + [y'(x)]^2} \tag{5.3}$$

After the introduction of nondimensional variables it takes the following form

$$\ddot{\xi} = -\frac{\psi'(\xi)[1 + \dot{\xi}^2\psi''(\xi)]}{1 + [\psi'(\xi)]^2} \tag{5.4}$$

In equation (5.4), the dot denotes differentiation with respect to nondimensional time τ, while the prime denotes differentiation with respect to nondimensional coordinate ξ. The solution to equation (5.4) should fulfill the following initial conditions

$$\xi(0) = 1 \qquad\qquad \dot{\xi}(0) = -\sqrt{2\eta_0}\cos\beta \tag{5.5}$$

Knowing the solution to the problem formulated in that way, the nondimensional normal reaction $\nu(\xi)$ can be determined as follows

$$\nu(\tau) = \frac{1 + \dot{\xi}^2\psi''(\xi)}{\sqrt{1 + [\psi'(\xi)]^2}} \tag{5.6}$$

A graph of the normal reaction versus nondimensional coordinate ξ, is shown in Fig. 7.

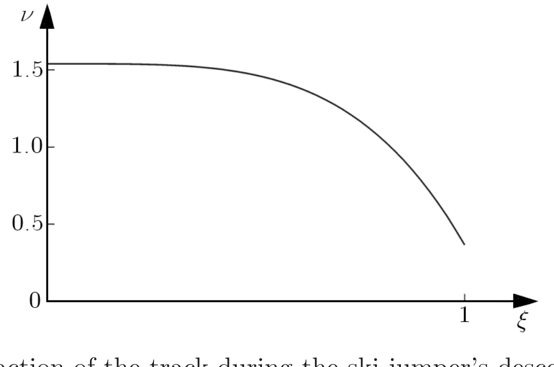

Fig. 7. Normal reaction of the track during the ski jumper's descent along a profile described with cubic splines

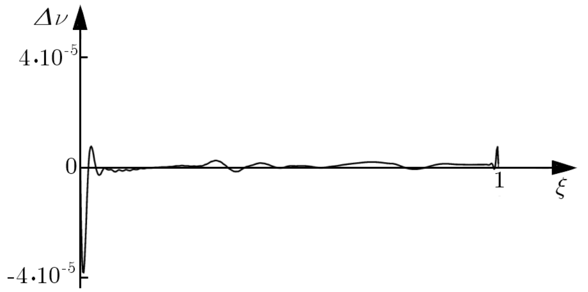

Fig. 8. The difference between the normal reaction force given with Eq. (3.3) and the one derived from Eq. (5.6)

The normal reaction force derived from Eq. (5.6) has a very similar graph to the one described with Eq. (3.3) for $i = 2$ (Fig. 5). The difference between them is shown in Fig. 8.

The coincidence between the normal reaction force given with Eq. (3.3) and the one derived from Eq. (5.6) confirms the correctness of the assumed mathematical model and the computational method applied.

6. The influence of the inclination angle of the truck starting segment on characteristics of the inrun profiles with given dynamic properties of the track

The inclination angle of the starting segment is usually determined by the degree of inclination of the slope. This angle can be increased, especially in the case of an artificially constructed inrun. The following figures present the influence of the angle β, with a determined exponent i, on: the profile of the inrun, radius of curvature and normal reaction of the track.

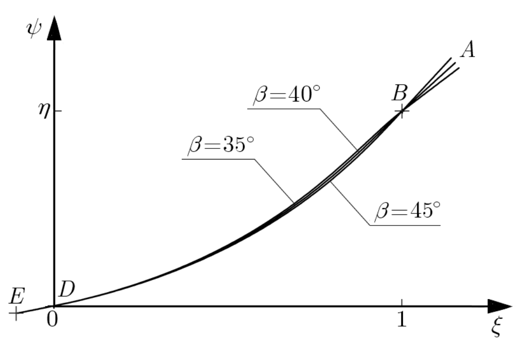

Fig. 9. Profiles of the inrun determined for three different angles β with $i = 2$

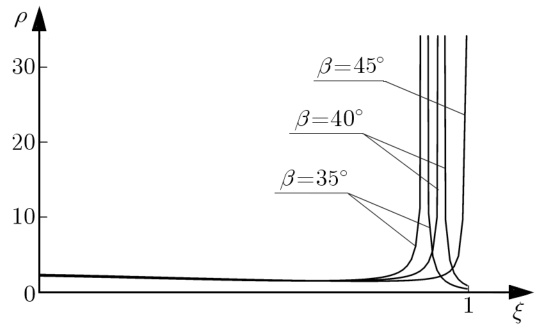

Fig. 10. Radii of curvature of the inrun profiles determined for three different angles β with $i = 2$

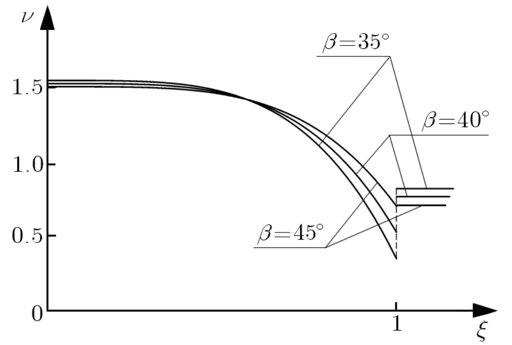

Fig. 11. Normal reactions of the inrun tracks determined for three different angles β with $i = 2$

An increase of the angle β has little effect on the profile of the inrun, especially on its lower part (Fig. 9). Together with the increase of the angle β, the inflexion point moves towards the point B (Fig. 10). The increase of the inclination angle of the starting segment slightly lowers the value of the normal track reaction at the point bordering on the take-off track (Fig. 11).

7. Inrun profiles without inflexion points

The inflexion point which can be seen in the majority of inrun profiles with a variable radius of curvature results from a relatively small inclination angle β of the starting segment. This point can be eliminated by means of a proper increase of that angle. Table 2 lists values of the angle β for which the curvilinear part of the inrun profile BD (Fig. 2) is only concave downward as the inflexion point coincides with the point B.

Table 2. A specification of the angles β for which the inflexion point coincides with the point B

i	β
1	$41°80'$
2	$45°37'$
3	$47°87'$
4	$49°68'$

The following figures show: profiles of the inrun, radii of curvature and normal reactions of the track for the solutions in which the inflexion point coincides with the point B.

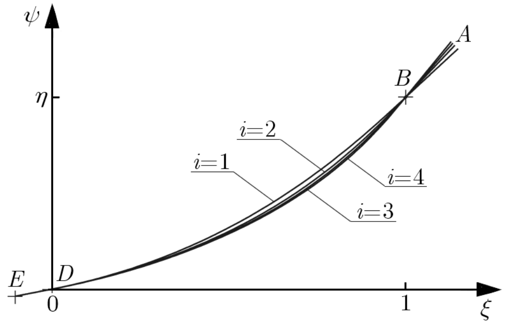

Fig. 12. Inrun profiles in which the inflexion point coincides with the point B

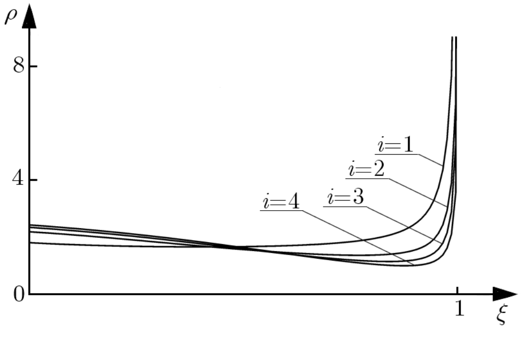

Fig. 13. Radii of curvature of the inrun profiles in which the inflexion point coincides with the point B

An increase of the exponent i causes an increase of the angle β (Table 2, Fig. 12), which guarantees the coincidence of the inflexion point with the point B. Together with the increase of the exponent i, the radius of curvature

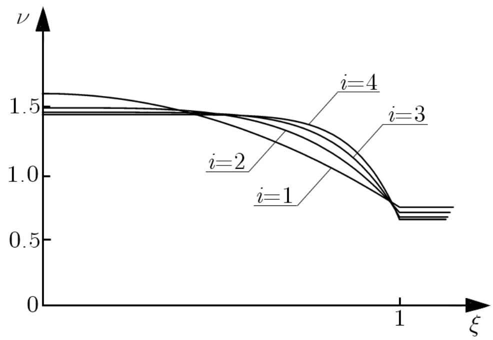

Fig. 14. Normal reactions of the inrun tracks in which the inflexion point coincides with the point B

of the track at the point bordering on the take-off track increases as well (Fig. 13), while the value of the normal reaction of the track at that point decreases (Fig. 14).

8. Conclusions

Replacing the rectilinear segment BC and the circular arc CD (Fig. 1) with one curvilinear part of the inrun profile BD (Fig. 2) allows one to increase the radius of curvature at the point D. The profiles of the inrun with a variable radius of curvature presented in the paper fulfill the requirement regarding the normal reaction of the track which should increase as the ski jumper approaches the take-off track until reaching its maximum at the point D. The nonlinear differential equation presented in the paper together with a set of boundary conditions makes it possible to design inrun profiles of ski jumping hills which are characterised by a decreased reaction of the track for various inclination angles of the starting section AB. The reduction of the inertia forces which affect the ski jumper can contribute to performing longer jumps that will, in addition, be better from the technical point of view.

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Modelowanie profilu najazdu skoczni narciarskiej o zadanych własnościach dynamicznych toru

Streszczenie

Praca dotyczy zagadnienia projektowania najazdu skoczni narciarskiej o zmiennym promieniu krzywizny. W pracy pokazano możliwość obniżenia reakcji normalnej toru w bezpośrednim sąsiedztwie z progiem, przy zachowaniu stosowanych w obecnych rozwiązaniach kątów nachylenia części startowej i progu. Zbadano również możliwość dalszego obniżenia reakcji normalnej w wyniku zwiększenia kąta nachylenia części startowej. Na koniec wyznaczono łagodne pod względem dynamicznym profile najazdu nie zawierające punktu przegięcia.

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