JOURNAL OF THEORETICAL
AND APPLIED MECHANICS
53, 4, pp. 873-884, Warsaw 2015
DOI: 10.15632/jtam-pl.53.4.873

GREEN'S FUNCTION IN FREQUENCY ANALYSIS OF CIRCULAR THIN PLATES OF VARIABLE THICKNESS

Krzysztof Kamil Żur
Bialystok University of Technology, Faculty of Management, Kleosin, Poland
e-mail: k.zur@pb.edu.pl

Free vibration analysis of homogeneous and isotropic circular thin plates with variable distribution of parameters by using Green's functions (solution to homogeneous ordinary differential equations with variable coefficients) is considered. The formula of Green's function (called the influence function) depends on the Poisson ratio and the coefficient of distribution of plate flexural rigidity, and the thickness is obtained in a closed-form. The limited independent solutions to differential Euler equations are expanded in the Neumann power series using the Volterra integral equations of the second kind. This approach allows one to obtain the analytical frequency equations as the power series rapidly convergens to exact eigenvalues for different values of the power index and different values of the Poisson ratio. The six lower natural dimensionless frequencies of axisymmetric vibration of circular plates of constant and variable thickness are calculated for different boundary conditions. The obtained results are compared with selected results presented in the literature.

Keywords: circular plates, Green's function, Neumann series

1. Introduction

The study of vibration of a thin circular plate is basic in structural mechanics because it has many applications in civil and mechanical engineering. Circular plates are the most critical structural elements in high speed rotating engineering systems such as circular saws, rotors, turbine flywheels, etc. In reality, a lot of complicating factors may come into play: non-uniform thickness, elastic constraints, anisotropic or composite materials, etc. The natural frequencies of the plates have been studied extensively for more than a century, if only because when the frequency of external load matches the natural frequency of the plate, destruction may occur.

The free vibration of circular plates of constant and variable thickness has received considerable attention in the literature. The vibration of circular plates has been discussed by many authors. The work of Leissa (1969) is an excellent source of information about methods used for free vibration analysis of plates. Free vibration analysis has been carried out by using a variety of weighting function methods (Leissa, 1969) such as the Ritz method, the Galerkin method or the finite element method. Conway (1957, 1958) analyzed the axisymmetric vibration of thin circular plates with a power function thickness variation for a particular Poisson ratio in terms of the Bessel functions. Jain *et al.* (1972) studied the axisymmetric vibration of thin circular plates with linearly varying thickness using by the Frobenius method. Yang (1993) studied the same problem using by perturbation method. Wang (1997) used the power series method for free vibration analysis of circular thin plates with power variable thickness. Wu and Liu (2001, 2002) proposed a generalized differential quadrature rule (GDQR) for free vibration analysis of circular thin plates of constant and variable thickness. Jaroszewicz and Zoryj (2006) studied free vibration of circular thin plates with variable distribution of parameters using the method of partial discretization (MPD). Taher *et al.* (2006) studied free vibration of circular and annular plates with variable thickness and different combinations of boundary conditions. Gupta *et* *al.* (2006) analyzed free vibration of nonhomogeneous circular plates with nonlinear thickness variation by using the differential quadrature method (DQM). Yalcin *et al.* (2009) studied free vibration of circular plates by using the differential transformation method (DTM). Zhou *et al.* (2011) applied the Hamiltonian approach to solution of the free vibration problem of circular and annular thin plates. Duan *et al.* (2014) proposed the DSC element method for free vibration analysis of circular thin plates with constant and stepped thickness.

In the works by Leissa (1969), Conway (1957, 1958) the solutions for free axisymmetric vibration of clamped circular plates with a power function thickness variation were presented. Those solutions were possible to obtain only for few combinations of the Poisson ratios and Bessel functions. That kind of solutions have limited practical applications. The aim of the paper is frequency analysis of circular plates with different values of the power index m of the plate parameters and different values of the Poisson ratios. The characteristic equations are obtained for two different values of the Poisson ratio and different boundary conditions such as free, clamped, simply supported, sliding and elastic supports. The limited independent solutions of differential Euler equations are expanded in the Neumann power series using the properties of integral equations. This approach allows one to obtain analytical frequency equations as the power series rapidly converges to the exact eigenvalues. The numerical results of investigation are in good agreement with selected results presented in the literature.

2. Statement of the problem

Consider an isotropic, homogeneous circular thin plate of variable thickness $h = h_R r^{m/3}$ and flexural rigidity $D = D_R r^m$ in the cylindrical coordinate system (r, θ, z) with the z-axis along the longitudinal direction. h_R and D_R are thickness and flexural rigidity of circular plates on the edge ($r = R$), respectively. The geometry and coordinate system of the considered plate are shown in Fig. 1. For free axisymmetric vibration of circular plates, the deflection is independent of θ. The partial differential equation for free vibration of thin circular plates has the following form (Timoshenko and Woinowsky-Krieger, 1959)

$$D\frac{\partial}{\partial r}\Big(\frac{\partial^2 W}{\partial r^2} + \frac{1}{r}\frac{\partial W}{\partial r}\Big) + \frac{\partial D}{\partial r}\Big(\frac{\partial^2 W}{\partial r^2} + \frac{\nu}{r}\frac{\partial W}{\partial r}\Big) + \frac{1}{r}\int\limits_0^r \rho h \frac{\partial^2 W}{\partial t^2} r\, dr = 0 \tag{2.1}$$

where ρ is mass density, r is the radial coordinate and $W(r,t)$ is the small axisymmetric deflection compared with the thickness h of the plate.

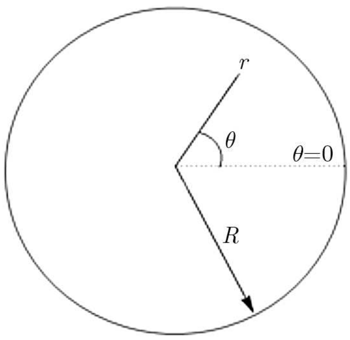

Fig. 1. Geometry and coordinate system of the circular plate

The axisymmetric deflection of a circular plate may be expressed as follows

$$W(r,t) = w(r)\mathrm{e}^{\mathrm{i}\omega t} \tag{2.2}$$

where $w(r)$ is the radial mode function and ω is natural frequency. Substituting Eq. (2.2) into Eq. (2.1) and using the dimensionless coordinate $\xi = r/R$, the governing differential equation of the circular plate becomes

$$L(w) - \lambda^2 \xi^{-2m/3} w = 0 \tag{2.3}$$

where $L(w)$ is the operator defined by

$$L(w) \equiv \frac{d^4 w}{d\xi^4} + \frac{2(m+1)}{\xi}\frac{d^3 w}{d\xi^3} + \frac{m^2 + m + \nu m - 1}{\xi^2}\frac{d^2 w}{d\xi^2} + \frac{m^2\nu - m\nu - m + 1}{\xi^3}\frac{dw}{d\xi} \tag{2.4}$$

and the dimensionless frequency λ of vibration is given by

$$\lambda = \omega R^{m/3}\sqrt{\frac{\rho h_R}{D_R}} \tag{2.5}$$

The governing differential equation for the circular plate of constant thickness has the form

$$L(w) - \lambda^2 w = 0 \tag{2.6}$$

where

$$L(w) \equiv \frac{d^4 w}{d\xi^4} + \frac{2}{\xi}\frac{d^3 w}{d\xi^3} - \frac{1}{\xi^2}\frac{d^2 w}{d\xi^2} + \frac{1}{\xi^3}\frac{dw}{d\xi} \qquad \lambda = \omega R^2\sqrt{\frac{\rho h_R}{D_R}} \tag{2.7}$$

The boundary conditions at the outer edge ($\xi = 1$) of the circular plate may be one of the following: clamped, simply supported, free, sliding supports and elastic supports. These conditions may be written in terms of the radial mode function $w(\xi)$ in the following form:

— clamped

$$w(\xi)\Big|_{\xi=1} = 0 \qquad \frac{dw}{d\xi}\Big|_{\xi=1} = 0 \tag{2.8}$$

— simply supported

$$w(\xi)\Big|_{\xi=1} = 0 \qquad M(w)\Big|_{\xi=1} = \Big(\frac{d^2 w}{d\xi^2} + \frac{\nu}{\xi}\frac{dw}{d\xi}\Big)_{\xi=1} = 0 \tag{2.9}$$

— free

$$M(w)\Big|_{\xi=1} = 0 \qquad V(w)\Big|_{\xi=1} = \Big(\frac{d^3 w}{d\xi^3} + \frac{1}{\xi}\frac{d^2 w}{d\xi^2} - \frac{1}{\xi^2}\frac{dw}{d\xi}\Big)_{\xi=1} = 0 \tag{2.10}$$

— sliding (vertical) supports

$$\frac{dw}{d\xi}\Big|_{\xi=1} = 0 \qquad V(w)\Big|_{\xi=1} = 0 \tag{2.11}$$

— elastic supports

$$\begin{aligned}
\varPhi(w)\Big|_{\xi=1} &= \Big[\Big(\frac{d^2 w}{d\xi^2} + \nu\frac{dw}{d\xi}\Big) + \phi\frac{dw}{d\xi}\Big]_{\xi=1} = 0 \\
\varPsi(w)\Big|_{\xi=1} &= \Big[\Big(\frac{d^3 w}{d\xi^3} + \frac{d^2 w}{d\xi^2} - \frac{dw}{d\xi}\Big) - \psi w\Big]_{\xi=1} = 0
\end{aligned} \tag{2.12}$$

$M(w)$ and $V(w)$ are the normalized radial bending moment and the normalized effective shear force, respectively. $\phi = K_\phi R/D_R$ and $\psi = K_\psi R^3/D_R$ are parameters of the elastic supports. K_ϕ and K_ψ are the rotational and translational spring constants (Fig. 2), respectively.

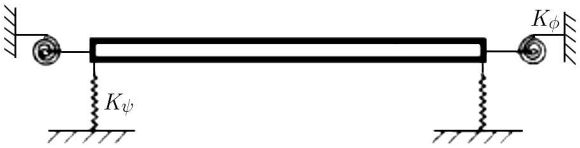

Fig. 2. Cross-section of a uniform circular plate with elastic supports

3. Finding Green's functions

The characteristic equation of a homogeneous differential Euler equation for thin circular plates with variable thickness, see Eq. (2.4)

$$L(w)=0 \tag{3.1}$$

has the following form

$$s^4+(2m-4)s^3+(m^2+m\nu-5m+4)s^2+(-m^2+m^2\nu-2m\nu+2m)s=0 \tag{3.2}$$

The roots of Eq. (3.2) are

$$s_1=0 \qquad s_2=2-m \qquad s_3=1-\frac{m}{2}-\mathcal{H} \qquad s_4=1-\frac{m}{2}+\mathcal{H} \tag{3.3}$$

where

$$\mathcal{H}=\frac{1}{2}\sqrt{m^2-4m\nu+4} \tag{3.4}$$

The general solution to Eq. (3.1) is

$$w(\xi)=C_1+C_2\xi^{2-m}+C_3\xi^{1-\frac{m}{2}-\mathcal{H}}+C_4\xi^{1-\frac{m}{2}+\mathcal{H}} \tag{3.5}$$

Green's function (solution to the homogeneous Euler equation $L(K_m(\xi,\alpha))=0$) for different values of the power index m may be received from a formula presented in the following form (Jaroszewicz and Zoryj, 2005)

$$K_m(\xi,\alpha)=\frac{A_m}{W(\alpha)_m p_0(\alpha)} \tag{3.6}$$

where $p_0(\alpha)=1$ is a coefficient placed before the highest order of the derivative of Euler differential equation (3.1) and

$$A_m=\begin{vmatrix} 1 & \alpha^{2-m} & \alpha^{1-\frac{m}{2}-\mathcal{H}} & \alpha^{1-\frac{m}{2}+\mathcal{H}} \\ 0 & \dfrac{d\alpha^{2-m}}{d\alpha} & \dfrac{d\alpha^{1-\frac{m}{2}-\mathcal{H}}}{d\alpha} & \dfrac{d\alpha^{1-\frac{m}{2}+\mathcal{H}}}{d\alpha} \\ 0 & \dfrac{d^2\alpha^{2-m}}{d\alpha^2} & \dfrac{d^2\alpha^{1-\frac{m}{2}-\mathcal{H}}}{d\alpha^2} & \dfrac{d^2\alpha^{1-\frac{m}{2}+\mathcal{H}}}{d\alpha^2} \\ 1 & \xi^{2-m} & \xi^{1-\frac{m}{2}-\mathcal{H}} & \xi^{1-\frac{m}{2}+\mathcal{H}} \end{vmatrix}$$
$$W(\alpha)_m=\begin{vmatrix} 1 & \alpha^{2-m} & \alpha^{1-\frac{m}{2}-\mathcal{H}} & \alpha^{1-\frac{m}{2}+\mathcal{H}} \\ 0 & \dfrac{d\alpha^{2-m}}{d\alpha} & \dfrac{d\alpha^{1-\frac{m}{2}-\mathcal{H}}}{d\alpha} & \dfrac{d\alpha^{1-\frac{m}{2}+\mathcal{H}}}{d\alpha} \\ 0 & \dfrac{d^2\alpha^{2-m}}{d\alpha^2} & \dfrac{d^2\alpha^{1-\frac{m}{2}-\mathcal{H}}}{d\alpha^2} & \dfrac{d^2\alpha^{1-\frac{m}{2}+\mathcal{H}}}{d\alpha^2} \\ 0 & \dfrac{d^3\alpha^{2-m}}{d\alpha^3} & \dfrac{d^3\alpha^{1-\frac{m}{2}-\mathcal{H}}}{d\alpha^3} & \dfrac{d^3\alpha^{1-\frac{m}{2}+\mathcal{H}}}{d\alpha^3} \end{vmatrix} \tag{3.7}$$

The functions 1, α^{2-m}, $\alpha^{1-\frac{m}{2}-\mathcal{H}}$, $\alpha^{1-\frac{m}{2}+\mathcal{H}}$ are linear independent solutions, then the Wronskian must satisfy the condition (Stakgold and Holst, 2011)

$$W(\alpha)_m = -\frac{\mathcal{H}}{8}(-2+m)[(-2+m)^2 - 4\mathcal{H}^2]^2\alpha^{-2(1+m)} \neq 0 \qquad \text{for } \; m \neq 0 \;\wedge\; m \neq 2 \tag{3.8}$$

After calculations, Green's function (GF) has the following form

$$\begin{aligned} K_m(\xi,\alpha) = {} & \frac{2\xi^{-m-\mathcal{H}}\alpha^{1-\mathcal{H}}}{\mathcal{H}(m-2)(4-4m+m^2-4\mathcal{H}^2)} \\ & \cdot \Big[(2-m)\xi^{1+\frac{m}{2}+2\mathcal{H}}\alpha^{1+\frac{m}{2}} + 2\xi^{m+\mathcal{H}}\mathcal{H}\alpha^{2+\mathcal{H}} - 2\xi^{2+\mathcal{H}}\mathcal{H}\alpha^{m+\mathcal{H}} + (m-2)\xi^{1+\frac{m}{2}}\alpha^{1+\frac{m}{2}+2\mathcal{H}}\Big] \end{aligned} \tag{3.9}$$

and satisfies the conditions

$$\begin{aligned} & K_m(\alpha,\alpha) = \frac{\partial K_m(\xi,\alpha)}{\partial \xi}\Big|_{\xi=\alpha} = \frac{\partial^2 K_m(\xi,\alpha)}{\partial \xi^2}\Big|_{\xi=\alpha} = 0 \\ & \frac{\partial^3 K_m(\xi,\alpha)}{\partial \xi^3}\Big|_{\xi=\alpha} = 1 \end{aligned} \tag{3.10}$$

according to the properties of influence functions (Kukla, 2009; Stakgold and Holst, 2011).

The function $K_m(\xi,\alpha)$ is indeterminate for $m = 0$ and $m = 2$. After calculation of lthe imits of the function $K_m(\xi,\alpha)$ for $m \to 0$ and $m \to 2$, the determinate Green function have the following form

$$\lim_{m\to 0} K_m(\xi,\alpha) = \frac{\alpha}{4}\Big[\alpha^2 - \xi^2 + (\xi^2+\alpha^2)\ln\frac{\xi}{\alpha}\Big] \tag{3.11}$$

when Poisson ratio $\nu = 0.25$

$$\lim_{m\to 2} K_m(\xi,\alpha) = \frac{1}{9}\xi^{-\sqrt{\frac{3}{2}}}\alpha^3\Big[\sqrt{6}\alpha^{-\sqrt{\frac{3}{2}}}(\xi^{\sqrt{6}} - \alpha^{\sqrt{6}}) - 6\xi^{\sqrt{\frac{3}{2}}}(\ln\xi + \ln\alpha)\Big] \tag{3.12}$$

and $\nu = 0.33$

$$\lim_{m\to 2} K_m(\xi,\alpha) = \frac{3\alpha^3}{16}\Big[\sqrt{3}\xi^{-\frac{2}{\sqrt{3}}}\alpha^{-\frac{2}{\sqrt{3}}}\Big(\xi^{\frac{4}{\sqrt{3}}} - \alpha^{\frac{4}{\sqrt{3}}}\Big) - 4\ln\xi + 4\ln\alpha\Big] \tag{3.13}$$

Examples of the formulas of Green's function $K_m(\xi\alpha)$ for different values of the power index m $m \in \{-3,-2,-1,0,2,3,4\}$ are presented as in the following:
— for Poisson ratio $\nu = 0.25$

$$\begin{aligned} K_{-3}(\xi,\alpha) &= \frac{1}{45\alpha^2}\Big(4\xi^5 - 5\xi^{\frac{9}{2}}\sqrt{\alpha} + 5\sqrt{\xi}\alpha^{\frac{9}{2}} - 4\alpha^5\Big) \\ K_{-2}(\xi,\alpha) &= \frac{1}{30\alpha}\Big(5\xi^4 - 5\alpha^4 - 2\sqrt{10}\xi^{2+\sqrt{\frac{5}{2}}}\alpha^{2-\sqrt{\frac{5}{2}}} + 2\sqrt{10}\xi^{2-\sqrt{\frac{5}{2}}}\alpha^{2+\sqrt{\frac{5}{2}}}\Big) \\ K_{-1}(\xi,\alpha) &= \frac{2}{9}\Big(2\xi^3 - 2\alpha^3 - \sqrt{6}\xi^{\frac{3}{2}+\sqrt{\frac{3}{2}}}\alpha^{\frac{3}{2}-\sqrt{\frac{3}{2}}} + \sqrt{6}\xi^{\frac{3}{2}-\sqrt{\frac{3}{2}}}\alpha^{\frac{3}{2}+\sqrt{\frac{3}{2}}}\Big) \\ K_0(\xi,\alpha) &= \frac{\alpha}{4}\Big[\alpha^2 - \xi^2 + (\xi^2+\alpha^2)\ln\frac{\xi}{\alpha}\Big] \\ K_2(\xi,\alpha) &= \frac{1}{9}\xi^{-\sqrt{\frac{3}{2}}}\alpha^3\Big[\sqrt{6}\alpha^{-\sqrt{\frac{3}{2}}}(\xi^{\sqrt{6}} - \alpha^{\sqrt{6}}) - 6\xi^{\sqrt{\frac{3}{2}}}(\ln\xi + \ln\alpha)\Big] \\ K_3(\xi,\alpha) &= \frac{2}{45}\Big(-10\alpha^3 + \frac{10\alpha^4}{\xi} + \sqrt{10}\xi^{-\frac{1}{2}+\sqrt{\frac{5}{2}}}\alpha^{\frac{7}{2}-\sqrt{\frac{5}{2}}} - \sqrt{10}\xi^{-\frac{1}{2}-\sqrt{\frac{5}{2}}}\alpha^{\frac{7}{2}+\sqrt{\frac{5}{2}}}\Big) \\ K_4(\xi,\alpha) &= \frac{(\xi-\alpha)^3\alpha^2(\xi+\alpha)}{12\xi^3} \end{aligned} \tag{3.14}$$

— for Poisson ratio $\nu = 0.33$

$$
\begin{aligned}
K_{-3}(\xi,\alpha) &= \frac{1}{170}\Big(\frac{17r^5}{\alpha^2} - 17\alpha^3 - 5\sqrt{17}\xi^{\frac{1}{2}(5+\sqrt{17})}\alpha^{\frac{1}{2}-\sqrt{\frac{17}{2}}} + 5\sqrt{17}\xi^{\frac{5}{2}-\sqrt{\frac{17}{2}}}\alpha^{\frac{1}{2}(1+\sqrt{17})}\\
K_{-2}(\xi,\alpha) &= \frac{3}{32\alpha}\Big(2\xi^4 - 2\alpha^4 - \sqrt{6}\xi^{2+2\sqrt{\frac{2}{3}}}\alpha^{2-2\sqrt{\frac{2}{3}}} + \sqrt{6}\xi^{2-2\sqrt{\frac{2}{3}}}\alpha^{2+2\sqrt{\frac{2}{3}}}\Big)\\
K_{-1}(\xi,\alpha) &= \frac{1}{38}\Big(19\xi^3 - 19\alpha^3 - 3\sqrt{57}\xi^{\frac{1}{6}(9+\sqrt{57})}\alpha^{\frac{1}{6}(9-\sqrt{57})} + 3\sqrt{57}\xi^{\frac{1}{6}(9-\sqrt{57})}\alpha^{\frac{1}{6}(9+\sqrt{57})}\Big)\\
K_0(\xi,\alpha) &= \frac{\alpha}{4}\Big[\alpha^2 - \xi^2 + (\xi^2+\alpha^2)\ln\frac{\xi}{\alpha}\Big]\\
K_2(\xi,\alpha) &= \frac{3\alpha^3}{16}\Big[\sqrt{3}\xi^{-\frac{2}{\sqrt{3}}}\alpha^{-\frac{2}{\sqrt{3}}}\Big(\xi^{\frac{4}{\sqrt{3}}} - \alpha^{\frac{4}{\sqrt{3}}}\Big) - 4\ln\xi + 4\ln\alpha\Big]\\
K_3(\xi,\alpha) &= \frac{(\xi-\alpha)^3\alpha^2}{6\xi^2}\\
K_4(\xi,\alpha) &= \frac{3\alpha^3}{176}\Big(\frac{11\alpha^2}{\xi^2} + \sqrt{33}\xi^{-1+\sqrt{\frac{11}{3}}}\alpha^{1-\sqrt{\frac{11}{3}}} - \sqrt{33}\xi^{-1-\sqrt{\frac{11}{3}}}\alpha^{1+\sqrt{\frac{11}{3}}} - 11\Big)
\end{aligned}
\tag{3.15}
$$

4. Solution of the problem

The ordinary differential equations with constant or variable coefficients can be transformed to the Volterra or Fredholm integral equations by using e.g. Fubini's method (Pogorzelski, 1958). The solutions to these equations are solutions to the transformed ordinary differential equation. If Green's function (kernel of integral equation) is well known (or determined), the linear independent solutions can be expanded in the Neumann (called Liouville-Neumann) power series rapidly convergent to the eigenvalues (spectrum of integral kernel) based onthe method of successive approximations (Tricomi, 1957; Shestopalov and Smirnov, 2002).

The limited (for $\xi = 0$) independent solutions of Eq. (3.1) are $w_1(\xi) = 1$ and $w_2(\xi) = \xi^{2-m}$ (or $w_2(\xi) = \xi^{1-\frac{m}{2}+\mathcal{H}}$ for $m \geqslant 2$). These solutions are expanded in the Neumann power series in the following form

$$
\begin{aligned}
K_m(\xi,\lambda)_u &= K_0(\xi)_u + \sum_{i=1}^{\eta} K_i(\xi)_u\lambda^{2i} \qquad \lambda \in \mathcal{R}^+\\
K_m(\xi,\lambda)_v &= K_0(\xi)_v + \sum_{i=1}^{\eta} K_i(\xi)_v\lambda^{2i}
\end{aligned}
\tag{4.1}
$$

where $K_i(\xi)_u$ and $K_i(\xi)_v$ are integral iterated kernels given by

$$
\begin{aligned}
K_i(\xi)_u &= \int_0^{\xi} K_m(\xi,\alpha)\alpha^{-\frac{2}{3}m}K_{i-1}(\alpha)_u\,d\alpha \qquad K_0(\alpha)_u = \chi_u\\
K_i(\xi)_v &= \int_0^{\xi} K_m(\xi,\alpha)\alpha^{-\frac{2}{3}m}K_{i-1}(\alpha)_v\,d\alpha \qquad K_0(\alpha)_v = \chi_v
\end{aligned}
\tag{4.2}
$$

and η is the degree of approximations. χ_u and χ_v are limited independent solutions to Eq. (3.1) for $\xi = 0$. $\chi_u = 1$ for all values of the parameter m. Values of χ_v depend on the power index m and the Poisson ratio ν (for $m \geqslant 2$). They are shown in Table 1.

Table 1. Values of χ_v for some considered values of the power index m

m	-3	-2	-1	0	2	3	4
χ_v	α^5	α^4	α^3	α^2	$\alpha^{2\sqrt{3}/3}$	α	$\alpha^{-1+\sqrt{11/3}}$

The characteristic equations $\Delta_m = 0$ for different boundary conditions and different values of the parameter m are obtained from well known characteristic determinants given by:
— clamped

$$\Delta_m(\lambda) \equiv \left.\begin{vmatrix} K_m(\xi,\lambda)_u & K_m(\xi,\lambda)_v \\ \dfrac{\partial K_m(\xi,\lambda)_u}{\partial \xi} & \dfrac{\partial K_m(\xi,\lambda)_v}{\partial \xi} \end{vmatrix}\right|_{\xi=1} \tag{4.3}$$

— simply supported

$$\Delta_m(\lambda) \equiv \left.\begin{vmatrix} K_m(\xi,\lambda)_u & K_m(\xi,\lambda)_v \\ M[K_m(\xi,\lambda)_u] & M[K_m(\xi,\lambda)_v] \end{vmatrix}\right|_{\xi=1} \tag{4.4}$$

— free

$$\Delta_m(\lambda) \equiv \left.\begin{vmatrix} M[K_m(\xi,\lambda)_u] & M[K_m(\xi,\lambda)_v] \\ V[K_m(\xi,\lambda)_u] & V[K_m(\xi,\lambda)_v] \end{vmatrix}\right|_{\xi=1} \tag{4.5}$$

— sliding supports

$$\Delta_m(\lambda) \equiv \left.\begin{vmatrix} \dfrac{\partial K_m(\xi,\lambda)_u}{\partial \xi} & \dfrac{\partial K_m(\xi,\lambda)_v}{\partial \xi} \\ V[K_m(\xi,\lambda)_u] & V[K_m(\xi,\lambda)_v] \end{vmatrix}\right|_{\xi=1} \tag{4.6}$$

— elastic supports

$$\Delta_m(\lambda) \equiv \left.\begin{vmatrix} \Phi[K_m(\xi,\lambda)_u] & \Phi[K_m(\xi,\lambda)_v] \\ \Psi[K_m(\xi,\lambda)_u] & \Psi[K_m(\xi,\lambda)_v] \end{vmatrix}\right|_{\xi=1} \tag{4.7}$$

For all boundary conditions, the formula of Δ_m has the following form

$$\Delta_m = a_0 + \sum_{i=1}^{\eta} (-1)^i a_i \lambda^{2i} \tag{4.8}$$

where $a_0, a_1, \ldots, a_\eta$ are coefficients of characteristic equations depending on the boundary conditions and the parameter m.

5. Results and discussion

The numerical results for dimensionless frequencies of the uniform and non-uniform circular plates with different boundary conditions are presented in Tables 2-5. The Neumann power series (Eq. (4.1)) expanded only for $\eta = 15$ allows one to obtained six lower exact eigenvalues for all considered cases. The numerical dimensionless frequencies of the uniform circular plates are presented in Table 2 with comparison to the results by Duan *et al.* (2014), Leissa (1969), Wu and Liu (2002) and Yalcin *et al.* (2009). The numerical results for uniform circular plates with elastic supports are shown in Table 3 with comparison to the results by Wu and Liu (2002).

Table 2. The first six lower dimensionless frequencies $\lambda = \omega R^2 \sqrt{\rho h_R / D_R}$ of the uniform circular plates

λ		Clamped	Simply supported $\nu = 0.3$	Simply supported $\nu = 0.25$	Free $\nu = 0.3$	Free $\nu = 0.25$	Sliding supports
λ_0	GF	10.216	4.935	4.860	9.003	8.889	14.682
	Duan *et al.* (2014)	10.215	4.935	–	9.003	–	–
	Wu and Liu (2002)	10.216	4.935	–	9.003	–	14.682
	Yalcin *et al.* (2009)	10.215	4.935	–	9.003	–	–
λ_1	GF	39.771	29.72	29.66	38.443	38.335	49.218
	Duan *et al.* (2014)	39.771	29.72	–	38.443	–	–
	Wu and Liu (2002)	39.771	29.72	–	38.443	–	49.218
	Yalcin *et al.* (2009)	39.771	29.72	–	38.443	–	–
λ_2	GF	89.104	74.156	74.101	87.750	87.645	103.499
	Duan *et al.* (2014)	89.104	74.155	–	87.753	–	–
	Wu and Liu (2002)	89.104	74.156	–	87.750	–	103.499
	Yalcin *et al.* (2009)	89.104	74.156	–	87.750	–	–
λ_3	GF	158.184	138.318	138.26	156.818	156.71	177.521
	Duan *et al.* (2014)	158.184	138.317	–	156.826	–	–
	Wu and Liu (2002)	158.184	138.318	–	156.816	–	177.521
	Yalcin *et al.* (2009)	158.184	138.318	–	156.818	–	–
λ_4	GF	247.006	222.215	222.25	245.634	245.53	271.282
	Duan *et al.* (2014)	247.006	222.213	–	245.651	–	–
	Wu and Liu (2002)	247.007	222.215	–	245.634	–	271.282
	Yalcin *et al.* (2009)	247.006	222.215	–	245.633	–	–
λ_5	GF	355.569	325.849	325.79	354.6	354.08	384.782
	Leissa (1969)	355.568	–	–	–	–	–
	Wu and Liu (2002)	355.569	325.849	–	–	–	–

GF – Green's function

Table 3. The first six lower dimensionless frequencies $\lambda = \omega R^2 \sqrt{\rho h_R / D_R}$ of the uniform circular plates with elastic supports, Poisson ratio $\nu = 0.3$

λ		$\phi = 0.1$, $\Psi = 100$	$\phi = 10$, $\Psi = 100$	$\phi = 100$, $\Psi = 100$
λ_0	GF	4.854	7.790	8.809
	Wu and Liu (2002)	4.854	7.790	8.809
λ_1	GF	22.097	22.128	22.142
	Wu and Liu (2002)	22.098	22.128	22.143
λ_2	GF	44.938	49.253	51.441
	Wu and Liu (2002)	44.938	49.254	51.442
λ_3	GF	90.469	98.741	104.413
	Wu and Liu (2002)	90.469	98.741	104.413
λ_4	GF	158.359	168.599	177.926
	Wu and Liu (2002)	158.359	168.599	177.926
λ_5	GF	246.673	258.213	271.391
	Wu and Liu (2002)	246.673	258.213	271.391

Table 4. The first six lower dimensionless frequencies $\lambda = \omega R^{m/3}\sqrt{\rho h_R/D_R}$ of the non-uniform circular plates

m	λ	Boundary conditions							
		Clamped		Simply supported		Free		Sliding supports	
		$\nu = 0.33$	$\nu = 0.25$	$\nu = 0.33$	$\nu = 0.25$	$\nu = 0.33$	$\nu = 0.25$	$\nu = 0.33$	$\nu = 0.25$
-3	λ_0	16.902	17.209	10.851	10.981	25.643	25.501	36.543	36.676
	λ_1	86.044	86.188	67.382	67.403	90.847	90.722	114.63	114.75
	λ_2	197.11	197.25	167.37	167.38	210.79	201.67	236.94	237.07
	λ_3	352.64	352.78	311.75	311.76	357.14	357.01	403.59	403.71
	λ_4	552.55	552.69	500.53	500.54	556.94	556.82	614.61	614.74
	λ_5	796.86	796.99	733.71	733.72	801.18	801.05	870.03	870.16
-2	λ_0	15.147	15.331	9.280	9.314	19.555	19.398	28.537	28.625
	λ_1	68.932	69.027	53.458	53.440	71.203	71.062	90.109	90.192
	λ_2	156.66	156.75	132.43	132.41	158.85	158.71	186.69	186.77
	λ_3	279.52	279.61	246.49	246.47	281.60	281.46	318.33	318.41
	λ_4	437.46	437.54	395.64	395.61	439.47	439.33	485.04	485.13
	λ_5	630.48	630.56	579.87	579.85	632.45	632.31	686.84	686.92
-1	λ_0	12.868	12.951	7.302	7.256	14.041	13.868	21.254	21.297
	λ_1	53.504	53.551	40.917	40.860	53.762	53.604	68.307	68.349
	λ_2	120.65	120.70	101.37	101.31	120.86	120.70	142.21	142.25
	λ_3	214.70	214.74	188.69	188.63	214.85	214.69	242.97	243.01
	λ_4	335.61	335.65	302.88	302.82	335.72	335.57	370.60	370.64
	λ_5	483.38	483.42	443.93	443.87	483.48	483.32	525.09	525.13
2	λ_0	8.894	9.111	3.297	3.334	5.302	5.412	8.876	9.193
	λ_1	25.837	26.306	19.076	19.410	22.951	23.296	28.472	29.011
	λ_2	51.575	52.278	42.759	43.337	48.776	49.363	56.510	57.279
	λ_3	86.082	87.017	75.135	75.949	83.323	84.144	93.262	94.260
	λ_4	129.36	130.52	116.25	117.30	126.62	127.67	138.76	139.99
	λ_5	181.41	182.81	166.13	167.41	178.68	179.97	193.03	194.49
3	λ_0	8.719	8.965	3.002	3.073	4.686	4.843	8.787	7.170
	[1]	8.720	–	–	–	–	–	–	–
	[7]	8.708	–	–	–	–	–	–	–
	[16]	8.719	8.965	–	–	–	–	–	–
	λ_1	21.145	21.609	15.761	16.110	18.152	18.520	21.638	22.170
	[1]	21.15	–	–	–	–	–	–	–
	[16]	21.145	21.609	–	–	–	–	–	–
	λ_2	38.453	39.122	32.031	32.595	35.607	36.187	40.402	41.133
	[1]	38.45	–	–	–	–	–	–	–
	[16]	38.453	39.122	–	–	–	–	–	–
	λ_3	60.680	61.551	53.108	53.879	57.892	58.677	63.964	64.894
	λ_4	87.834	88.910	79.076	80.052	85.077	86.066	92.411	93.577
	λ_5	119.91	121.19	109.95	111.03	117.18	118.31	125.78	126.78
4	λ_0	8.458	8.705	2.877	2.965	4.395	4.569	3.644	4.273
	λ_1	16.735	17.137	12.781	13.096	14.093	14.427	15.778	16.236
	λ_2	27.094	27.643	22.827	23.299	24.645	25.131	27.042	27.639
	λ_3	39.611	40.303	34.898	35.418	37.240	37.872	40.255	41.007
	λ_4	54.305	55.139	49.123	49.866	51.975	52.751	55.925	56.484
	λ_5	70.806	71.542	65.087	66.243	68.884	69.791	70.875	74.118

[1] – Conway (1957), [7] – Jaroszewicz and Zoryj (2006), [16] – Wang (1997)

Table 5. The first six lower dimensionless frequencies $\lambda = \omega R^{m/3}\sqrt{\rho h_R/D_R}$ of the non-uniform circular plates with elastic supports

m	λ	Elastic parameters					
		$\phi = 0.1$, $\Psi = 10$		$\phi = 100$, $\Psi = 10$		$\phi = 10$, $\Psi = 10$	
		$\nu = 0.33$	$\nu = 0.25$	$\nu = 0.33$	$\nu = 0.25$	$\nu = 0.33$	$\nu = 0.25$
-3	λ_0	3.595	3.670	2.965	3.011	0.317	0.322
	λ_1	27.042	26.920	36.525	36.650	33.285	33.379
	λ_2	91.487	91.369	113.22	113.34	104.64	104.72
	λ_3	202.25	202.13	233.81	233.93	218.85	218.90
	λ_4	357.53	357.41	398.25	398.37	376.43	376.46
	λ_5	557.30	557.18	606.59	606.71	577.83	577.84
-2	λ_0	3.818	3.873	3.344	3.377	0.354	0.357
	λ_1	21.143	21.012	28.709	28.790	26.104	26.164
	λ_2	71.857	71.723	89.165	89.246	82.728	82.772
	λ_3	159.29	159.16	184.49	184.58	173.21	173.23
	λ_4	281.972	281.83	314.55	314.63	297.95	297.95
	λ_5	439.81	439.67	479.34	479.42	457.28	457.27
-1	λ_0	3.860	3.886	3.749	3.769	0.394	0.396
	λ_1	15.947	15.818	21.624	21.662	19.520	19.546
	λ_2	54.447	54.297	67.746	67.787	63.101	63.114
	λ_3	121.29	121.14	140.76	140.80	132.60	132.60
	λ_4	215.20	215.04	240.43	240.47	228.33	228.31
	λ_5	336.06	335.88	366.76	366.77	350.51	350.48
2	λ_0	3.140	3.168	7.330	7.148	0.729	0.723
	λ_1	9.061	9.195	8.855	9.317	8.250	8.546
	λ_2	23.752	24.096	28.495	29.027	26.982	27.482
	λ_3	49.197	49.784	56.260	57.022	53.862	54.581
	λ_4	83.620	84.442	92.740	93.731	89.265	90.204
	λ_5	126.86	127.918	137.94	139.16	133.27	134.43
3	λ_0	3.100	3.156	21.623	22.157	1.050	1.023
	λ_1	8.804	8.954	40.302	41.028	6.374	6.733
	λ_2	18.947	19.309	63.727	64.651	20.771	21.271
	λ_3	36.010	36.589	92.023	93.149	38.923	39.615
	λ_4	58.170	58.955	125.22	126.24	61.783	62.668
	λ_5	85.295	86.304	153.08	155.33	89.454	90.552
4	λ_0	3.166	3.239	15.354	15.859	2.642	1.999
	λ_1	8.651	8.803	26.962	27.562	15.401	15.832
	λ_2	14.881	15.204	40.184	40.919	26.395	26.966
	λ_3	25.028	25.512	55.473	56.347	39.575	40.040
	λ_4	37.810	38.129	72.903	73.921	48.253	55.200
	λ_5	45.062	52.949	92.495	93.111	59.544	72.493

The dimensionless frequencies of the non-uniform circular plates with different boundary conditions are presented in Table 4 with comparison to the results by Conway (1957), Jaroszewicz and Zoryj (2006), Wang (1997). The numerical results for the non-uniform circular plates with elastic supports are shown in Table 5.

The dimensionless frequencies of the non-uniform circular plate (Table 4) decrease when values of the power index increase. However, the absolute values of frequencies ω increase if the power index increases, which is according to physical properties of this kind of plates with variable thickness (Wu and Liu, 2002). Additionally, the dimensionless frequencies depend on functions describing the distribution of plate parameters such as thickness or rigidity. The dimensionless frequencies and absolute values ω for the uniform and the non-uniform circular plates with elastic constraints (Table 3 and 5) depend on combination of values of the elastic parameters.

6. Conclusions

In this paper, Green's functions have been employed to solve the problem of natural vibration of uniform and non-uniform circular thin plates with different boundary conditions. The universal Green function for different power indices m and different Poisson ratios is defined. The limited solutions to the Euler equation expanded in the Neumann power series allow one to obtain characteristic equations of circular plates rapidly convergent to the exact eigenvalues. The characteristic equations have been obtained for different values of the parameter m, different values of Poisson's ratio and different boundary conditions. The considered values of Poisson's ratio have not large influence on the dimensionless eigenvalues, but the numerical results of the investigation can be used to validate the accuracy of other numerical methods as benchmark values. The obtained results are in good agreement with the results obtained by other methods presented in the literature. The calculations have been carried out with the help of Mathematica v10, which is a symbolic calculation software.

References

1. Conway H.D., 1957, An analogy between the flexural vibrations of a cone and a disc of linearly varying thickness, *Journal of Applied Mathematics and Mechanics*, **37**, 9, 406-407
2. Conway H.D., 1958, Some special solutions for the flexural vibrations of discs of varying thickness, *Ingenieur-Archiv*, **26**, 6, 408-410
3. Duan G., Wang X., Jin Ch., 2014, Free vibration analysis of circular thin plates with stepped thickness by the DSC element method, *Thin-Walled Structures*, **85**, 25-33
4. Gupta U.S., Lal R., Sharma S., 2006, Vibration analysis of non-homogeneous circular plate of nonlinear thickness variation by differential quadrature method, *Journal of Sound and Vibration*, **298**, 892-906
5. Jain R.K., Prasad C., Soni S.R., 1972, Axisymmetric vibrations of circular plates of linearly varying thickness, *ZAMP*, **23**, 941-947
6. Jaroszewicz J., Zoryj L., 2005, *Methods of Free Axisymmetric Vibration Analysis of Circular Plates Using by Influence Functions*, Bialystok University of Technology, Poland
7. Jaroszewicz J., Zoryj L., 2006, The method of partial discretization in free vibration problems of circular plates with variable distribution of parameters, *International Applied Mechanics*, **42**, 364-373
8. Kukla S., 2009, *Green's Functions and their Properties*, Czestochowa University of Technology, Poland
9. Leissa A.W., 1969, *Vibration of Plates*, Washington
10. Pogorzelski W., 1958, *Integral Equations and their Applications*, Vol. 1, PWN Warszawa
11. Shestopalov Y.V, Smirnov Y.G., 2002, *Integral Equations*, Karlstad
12. Stakgold I., Holst M., 2011, *Green's Functions and Boundary Value Problem*, Edition 3rd, Wiley

13. Taher H.R., Omidi M., Zadpoor A.A., Nikooyan A.A., 2006, Free vibration of circular and annular plates with variable thickness and different combinations of boundary conditions, *Journal of Sound and Vibration*, **296**, 1084-1092

14. Timoshenko S., Woinowsky-Krieger S., 1959, *Theory of Plates and Shells*, McGraw-Hill, New York

15. Tricomi F.G., 1957, *Integral Equations*, New York

16. Wang J., 1997, General power series solution of the vibration of classical circular plates with variable thickness, *Journal of Sound and Vibration*, **202**, 593-599

17. Wu T.Y., Liu G.R., 2001, Free vibration analysis of circular plates with variable thickness by the generalized differential quadrature rule, *International Journal of Solids and Structures*, **38**, 7967-7980

18. Wu T.Y., Liu G.R., 2002, Free vibration analysis of circular plates using generalized differential quadrature rule, *Computer Methods in Applied Mechanics and Engineering*, **191**, 5365-5380

19. Yalcin H.S., Arikoglu A., Ozkol I., 2009, Free vibration analysis of circular plates by differential transformation method, *Applied Mathematics and Computation*, **212**, 2, 377-386

20. Yang J.S., 1993, The vibration of circular plate with variable thickness, *Journal of Sound and Vibration*, **165**, 178-185

21. Zhou Z.H., Wong K.W., Xu X.S., Leung A.Y.T., 2011, Natural vibration of circular and annular plates by Hamiltonian approach, *Journal of Sound and Vibration*, **330**, 5, 1005-1017

Manuscript received October 2, 2014; accepted for print May 4, 2015