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ON THE GENERALIZED EULER EQUATIONS

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In this paper the Euler equations applied to the motion of bodies placed in the frames of reference subjected to acceleration are presented. Positions of bodies are defined with respect to these frames by n generalized coordinates. Therefore, it is possible to establish an equivalence between the generalized Euler equations, the generalized Kane's equations and the generalized Lagrange's equations valid for accelerated frame of reference. General formulae for the generalized inertial forces together with the way of their application to general cases are given. Application to a special case of a rigid body is added. This problem was considered by Lure [2] in a completely different way; however the physical interpretation given in this work allows us to formulate general and complete expressions for inertial and gyroscopic for ces and to establish the differences between them.

1. Introduction

The generalized Euler equations, representing the motion of a body or a system of bodies, relative the frame of reference S_1, which it is also subjected to an arbitrary movement defined by the velocity of it origin and by the rotation vector $\boldsymbol{\omega}_{10}$ with respect to the inertial frame of reference, have already been presented in the previous paper (cf Passos Morgado (1993)). These equations are

$$
\begin{aligned}
\Big(\frac{d\boldsymbol{P}_r}{dt}\Big)_{S_1} &= \boldsymbol{F} - M\ddot{\boldsymbol{R}}_0 - \Big[M\boldsymbol{\omega}_{10}\times\Big(\frac{d\boldsymbol{r}_c}{dt}\Big)_{S_1} + \boldsymbol{\omega}_{10}\times\boldsymbol{P}_r\Big] + \\
&- \Big(\frac{d\boldsymbol{\omega}_{10}}{dt}\Big)_{S_1}\times M\boldsymbol{r}_c - \boldsymbol{\omega}_{10}\times(\boldsymbol{\omega}_{10}\times M\boldsymbol{r}_c)
\end{aligned}
\tag{1.1}
$$

$$\begin{aligned}\left(\frac{d\boldsymbol{H}_r}{dt}\right)_{S_1} &= \boldsymbol{M}_0 - M\boldsymbol{r}_c \times \ddot{\boldsymbol{R}}_0 - \left[\left(\frac{d\boldsymbol{I}_0}{dt}\right)_{S_1}\boldsymbol{\omega}_{10} + bm\omega_{10} \times \boldsymbol{H}_r\right] + \\ &- \boldsymbol{I}_0\left(\frac{d\boldsymbol{\omega}_{10}}{dt}\right)_{S_1} - \boldsymbol{\omega}_{10} \times \boldsymbol{I}_0\boldsymbol{\omega}_{10}\end{aligned} \tag{1.2}$$

Let us assume the motion of a rigid body in S_1. In the most general case, its motion depends on n generalized coordinates and time. Therefore, the linear momentum will be defined by $\boldsymbol{P}_r = \boldsymbol{P}_r(t, q_1, ..., q_n, \dot{q}_1, ..., \dot{q}_n)$, the kinetic moment by $\boldsymbol{H}_r = \boldsymbol{H}_r(t, q_1, ..., q_n, \dot{q}_1, ..., \dot{q}_n)$, the vector of mass center positions in S_1 by $\boldsymbol{r}_c = \boldsymbol{r}_c(t, q_1, ..., q_n)$, and the inertial tensor with respect to the origin of the system S_1 by $\boldsymbol{I}_0 = \boldsymbol{I}_0(t, q_1, ..., q_n)$ origin.

The rotation vector $\boldsymbol{\omega}_{10}$ is independent of the generalized coordinates and so the velocity of the S_1 system origin.

The meaning of each term in the above expressions has already been carefully explained by Passos Morgado (1993b).

2. Equivalence of generalized Euler, Kane's and Lagrange's equations

However, what we believe deserves further explanation is that it is possible to pass from Eqs (1.1) and (1.2) to the generalized Kane's equations for accelerated systems (cf Passos Morgado (1993a)) and to the generalized Lagrange's equations directly, enabling the Author to show their equivalency. Let us suppose a body being a member of the holonomic system S_1 with n degrees of freedom. It is possible to define an instantaneous rotation vector representing the body motion as a function of the n generalized coordinates and the n generalized velocities (cf Kane (1968))

$$\boldsymbol{\omega}_{21} = \boldsymbol{\omega}_{q_1}\dot{q}_1 + ... + \boldsymbol{\omega}_{q_n}\dot{q}_n + \boldsymbol{\omega}_t \tag{2.1}$$

where the vectors $\boldsymbol{\omega}_{q_r}$ and $\boldsymbol{\omega}_t$ are functions of generalized coordinates and time.

Let $\boldsymbol{r}$ be the position vector of a point P of the body in the coordinate system S_1 and let $\boldsymbol{r}_{0'}$ be the position vector of the reference frame S_2 origin attached to the body.

The velocity of point P, as seen by an observer belonging to S_1 is

$$\boldsymbol{v}_P = \left(\frac{d\boldsymbol{r}}{dt}\right)_{S_1} = \left(\frac{d\boldsymbol{r}}{dt}\right)_{S_2} + \boldsymbol{\omega}_{21} \times \boldsymbol{r} \tag{2.2}$$

where, by differentiation by parts with respect $\dot{q}$, we obtain

$$\frac{\partial}{\partial \dot{q}}\left(\frac{d\boldsymbol{r}}{dt}\right)_{S_1} = \frac{\partial}{\partial \dot{q}}\left(\frac{d\boldsymbol{r}}{dt}\right)_{S_2} + \frac{\partial \boldsymbol{\omega}_{21}}{\partial \dot{q}} \times \boldsymbol{r} \tag{2.3}$$

or, having in mind that $\boldsymbol{r} = \boldsymbol{r}_{0'} + \boldsymbol{r}'$

$$\left(\frac{d\boldsymbol{r}}{dt}\right)_{S_2} = \left(\frac{d\boldsymbol{r}_{0'}}{dt}\right)_{S_2} \tag{2.4}$$

we have

$$\frac{\partial}{\partial \dot{q}}\left(\frac{d\boldsymbol{r}}{dt}\right)_{S_1} = \frac{\partial}{\partial \dot{q}}\left(\frac{d\boldsymbol{r}_{0'}}{dt}\right)_{S_2} + \frac{\partial \boldsymbol{\omega}_{21}}{\partial \dot{q}} \times \boldsymbol{r} \tag{2.5}$$

To pass directly from Eqs (1.1) and (1.2) to the generalized Kane's equations let us multiply Eq (1.1) by $\frac{\partial}{\partial \dot{q}}\left(\frac{d\boldsymbol{r}_{0'}}{dt}\right)_{S_2}$ and multiply Eq (1.2) by $\frac{\partial \boldsymbol{\omega}_{21}}{\partial \dot{q}}$ given by Eq (2.5).

Therefore we obtain

$$\begin{aligned}\left(\frac{d\boldsymbol{P}_r}{dt}\right)_{S_1}\frac{\partial}{\partial \dot{q}}\left(\frac{d\boldsymbol{r}_{0'}}{dt}\right)_{S_2} &= \boldsymbol{F}\frac{\partial}{\partial \dot{q}}\left(\frac{d\boldsymbol{r}_{0'}}{dt}\right)_{S_2} - M\ddot{\boldsymbol{R}}_0\frac{\partial}{\partial \dot{q}}\left(\frac{d\boldsymbol{r}_{0'}}{dt}\right)_{S_2} + \\ &-\left[M\boldsymbol{\omega}_{10} \times \left(\frac{d\boldsymbol{r}_c}{dt}\right)_{S_1} + \boldsymbol{\omega}_{10} \times \boldsymbol{P}_r\right]\frac{\partial}{\partial \dot{q}}\left(\frac{d\boldsymbol{r}_{0'}}{dt}\right)_{S_2} + \\ &-\left(\frac{d\boldsymbol{\omega}_{10}}{dt}\right)_{S_1} \times M\boldsymbol{r}_c\frac{\partial}{\partial \dot{q}}\left(\frac{d\boldsymbol{r}_{0'}}{dt}\right)_{S_2} - \boldsymbol{\omega}_{10} \times (\boldsymbol{\omega}_{10} \times M\boldsymbol{r}_c)\frac{\partial}{\partial \dot{q}}\left(\frac{d\boldsymbol{r}_{0'}}{dt}\right)_{S_2}\end{aligned} \tag{2.6}$$

for the first one, and

$$\begin{aligned}\left(\frac{d\boldsymbol{H}_r}{dt}\right)_{S_1}\frac{\partial \boldsymbol{\omega}_{21}}{\partial \dot{q}} &= \boldsymbol{M}_0\frac{\partial \boldsymbol{\omega}_{21}}{\partial \dot{q}} - M\boldsymbol{r}_c \times \ddot{\boldsymbol{R}}_0\frac{\partial \boldsymbol{\omega}_{21}}{\partial \dot{q}} + \\ &-\left[\left(\frac{dI_0}{dt}\right)_{S_1}\boldsymbol{\omega}_{10} + \boldsymbol{\omega}_{10} \times \boldsymbol{H}_r\right]\frac{\partial \boldsymbol{\omega}_{21}}{\partial \dot{q}} - I_0\left(\frac{d\boldsymbol{\omega}_{10}}{dt}\right)_{S_1}\frac{\partial \boldsymbol{\omega}_{21}}{\partial \dot{q}} - \boldsymbol{\omega}_{10} \times I_0\boldsymbol{\omega}_{10}\frac{\partial \boldsymbol{\omega}_{21}}{\partial \dot{q}}\end{aligned} \tag{2.7}$$

for the second equation.

We define, the generalized force corresponding to the coordinates q, by

$$\boldsymbol{F}\frac{\partial}{\partial \dot{q}}\left(\frac{d\boldsymbol{r}_{0'}}{dt}\right)_{S_2} + \boldsymbol{M}_0\frac{\partial \boldsymbol{\omega}_{21}}{\partial \dot{q}} = Q \tag{2.8}$$

the generalized inertial force of translation by

$$M\ddot{\boldsymbol{R}}_0\frac{\partial}{\partial \dot{q}}\left(\frac{d\boldsymbol{r}_{0'}}{dt}\right)_{S_2} + M\boldsymbol{r}_c \times \ddot{\boldsymbol{R}}_0\frac{\partial \boldsymbol{\omega}_{21}}{\partial \dot{q}} = M\ddot{\boldsymbol{R}}_0\frac{\partial \boldsymbol{r}_c}{\partial q} \tag{2.9}$$

the generalized Coriolis inertial force by

$$
\begin{aligned}
&\Big[M\boldsymbol{\omega}_{10}\times\Big(\frac{d\boldsymbol{r}_c}{dt}\Big)_{S_1}+\boldsymbol{\omega}_{10}\times\boldsymbol{P}_r\Big]\frac{\partial}{\partial\dot{q}}\Big(\frac{d\boldsymbol{r}_{0'}}{dt}\Big)_{S_2}+\\
&+\Big[\Big(\frac{dI_0}{dt}\Big)_{S_1}\boldsymbol{\omega}_{10}+\boldsymbol{\omega}_{10}\times\boldsymbol{H}_r\Big]\frac{\partial\boldsymbol{\omega}_{21}}{\partial\dot{q}}=\boldsymbol{\omega}_{10}\Big[\frac{d}{dt}\Big(\frac{\partial\boldsymbol{H}_r}{\partial\dot{q}}\Big)_{S_1}-\frac{\partial\boldsymbol{H}_r}{\partial q}\Big]
\end{aligned}
\tag{2.10}
$$

the generalized inertial force of rotation by

$$
\Big[\Big(\frac{d\boldsymbol{\omega}_{10}}{dt}\Big)_{S_1}\times M\boldsymbol{r}_c\Big]\frac{\partial}{\partial\dot{q}}\Big(\frac{d\boldsymbol{r}_{0'}}{dt}\Big)_{S_2}+I_0\dot{\boldsymbol{\omega}}_{10}\frac{\partial\boldsymbol{\omega}_{21}}{\partial\dot{q}}=\dot{\boldsymbol{\omega}}_{10}\frac{\partial\boldsymbol{H}_r}{\partial\dot{q}}
\tag{2.11}
$$

the generalized centrifugal inertial force by

$$
\boldsymbol{\omega}_{10}\times(\boldsymbol{\omega}_{10}\times M\boldsymbol{r}_c)\frac{\partial}{\partial\dot{q}}\Big(\frac{d\boldsymbol{r}_{0'}}{dt}\Big)_{S_2}+\boldsymbol{\omega}_{10}\times I_0\boldsymbol{\omega}_{10}\frac{\partial\boldsymbol{\omega}_{21}}{\partial\dot{q}}=\frac{1}{2}\boldsymbol{\omega}_{10}\frac{\partial I_0}{\partial q}\boldsymbol{\omega}_{10}
\tag{2.12}
$$

Adding Eqs (2.6) and (2.7) and substituting for the values of the generalized inertial forces the formulae given above we finally obtained

$$
\begin{aligned}
&\Big(\frac{d\boldsymbol{P}_r}{dt}\Big)_{S_1}\frac{\partial}{\partial\dot{q}}\Big(\frac{d\boldsymbol{r}_{0'}}{dt}\Big)_{S_2}+\Big(\frac{d\boldsymbol{H}_r}{dt}\Big)_{S_1}\frac{\partial\boldsymbol{\omega}_{21}}{\partial\dot{q}}=Q-M\Big[\Big(\frac{d\boldsymbol{v}_0}{dt}\Big)_{S_1}+\boldsymbol{\omega}_{10}\times\boldsymbol{v}_0\Big]\frac{\partial\boldsymbol{r}_c}{\partial q}+\\
&+\frac{1}{2}\boldsymbol{\omega}_{10}\frac{\partial I_0}{\partial q}\boldsymbol{\omega}_{10}-\dot{\boldsymbol{\omega}}_{10}\frac{\partial\boldsymbol{H}_r}{\partial\dot{q}}-\boldsymbol{\omega}_{10}\Big[\frac{d}{dt}\Big(\frac{\partial\boldsymbol{H}_r}{\partial\dot{q}}\Big)_{S_1}-\frac{\partial\boldsymbol{H}_r}{\partial q}\Big]
\end{aligned}
\tag{2.13}
$$

The equations given above are fully compatible with the generalized Euler equations for accelerated systems. It is worthy to note that Eq (2.13) is written in generalized coordinates and so it must be equivalent to both, the generalized Kane's equations and to the generalized Lagrange's equations.

To prove the aforementioned thesis, let uss note that $\boldsymbol{P}_r$ and $\boldsymbol{H}_r$ are defined by

$$
\boldsymbol{P}_r=M\Big(\frac{d\boldsymbol{r}_{0'}}{dt}\Big)_{S_1}+\boldsymbol{\omega}_{21}\times M\boldsymbol{r}'_c
\tag{2.14}
$$

$$
\boldsymbol{H}_r=\boldsymbol{r}_{0'}\times\boldsymbol{P}_r+M\boldsymbol{r}'_c\times\Big(\frac{d\boldsymbol{r}_{0'}}{dt}\Big)_{S_1}+I_{0'}\boldsymbol{\omega}_{21}
\tag{2.15}
$$

and, since

$$
\Big(\frac{d\boldsymbol{r}_{0'}}{dt}\Big)_{S_1}=\Big(\frac{d\boldsymbol{r}_{0'}}{dt}\Big)_{S_2}+\boldsymbol{\omega}_{21}\times\boldsymbol{r}_{0'}
\tag{2.16}
$$

we obtain the relationship

$$
\frac{\partial}{\partial\dot{q}}\Big(\frac{d\boldsymbol{r}_{0'}}{dt}\Big)_{S_1}=\frac{\partial}{\partial\dot{q}}\Big(\frac{d\boldsymbol{r}_{0'}}{dt}\Big)_{S_2}+\frac{\partial\boldsymbol{\omega}_{21}}{\partial\dot{q}}\times\boldsymbol{r}_{0'}
\tag{2.17}
$$

Eq (2.17) can be substituted into Eq (2.13) giving

$$
\begin{aligned}
&\Big(\frac{d\boldsymbol{P}_r}{dt}\Big)_{S_1}\Big[\frac{\partial}{\partial\dot q}\Big(\frac{d\boldsymbol{r}_{0'}}{dt}\Big)_{S_2}-\frac{\partial\boldsymbol{\omega}_{21}}{\partial\dot q}\times\boldsymbol{r}_{0'}\Big]+\Big(\frac{d\boldsymbol{H}_r}{dt}\Big)_{S_1}\frac{\partial\boldsymbol{\omega}_{21}}{\partial\dot q}=\\
&=Q-M\Big[\Big(\frac{d\boldsymbol{v}_0}{dt}\Big)_{S_1}+\boldsymbol{\omega}_{10}\times\boldsymbol{v}_0\Big]\frac{\partial\boldsymbol{r}_c}{\partial q}+ \qquad (2.18)\\
&+\frac{1}{2}\boldsymbol{\omega}_{10}\frac{\partial I_0}{\partial q}\boldsymbol{\omega}_{10}-\dot{\boldsymbol{\omega}}_{10}\frac{\partial\boldsymbol{H}_r}{\partial\dot q}-\boldsymbol{\omega}_{10}\Big[\frac{d}{dt}\Big(\frac{\partial\boldsymbol{H}_r}{\partial\dot q}\Big)_{S_1}-\frac{\partial\boldsymbol{H}_r}{\partial q}\Big]
\end{aligned}
$$

By simplifying this mathematical expression and making the substitution $\frac{\partial\boldsymbol{v}_{0'}}{\partial q}=\frac{\partial}{\partial\dot q}\Big(\frac{d\boldsymbol{r}_{0'}}{dt}\Big)_{S_1}$, we finally obtain

$$
\begin{aligned}
&\Big[\frac{d}{dt}(\boldsymbol{\omega}_{21}\times M\boldsymbol{r}'_c)_{S_1}+M\boldsymbol{a}_{0'}\Big]\frac{\partial\boldsymbol{v}_{0'}}{\partial\dot q}+\Big[\frac{d}{dt}(I_{0'}\boldsymbol{\omega}_{21})_{S_1}+M\boldsymbol{r}'_c\times\boldsymbol{a}_{0'}\Big]\frac{\partial\boldsymbol{\omega}_{21}}{\partial\dot q}= \qquad (2.19)\\
&=Q-M\ddot{\boldsymbol{R}}_0\frac{\partial\boldsymbol{r}_c}{\partial q}-\dot{\boldsymbol{\omega}}_{10}\frac{\partial\boldsymbol{H}_r}{\partial\dot q}+\frac{1}{2}\boldsymbol{\omega}_{10}\frac{\partial I_0}{\partial q}\boldsymbol{\omega}_{10}-\boldsymbol{\omega}_{10}\Big[\frac{d}{dt}\Big(\frac{\partial\boldsymbol{H}_r}{\partial\dot q}\Big)_{S_1}-\frac{\partial\boldsymbol{H}_r}{\partial q}\Big]
\end{aligned}
$$

Eq (2.19) are precisely the Kane's equations for accelerated systems, which proves the equivalency between Euler equations for mechanical systems and the generalized Kane's equations. The equivalency between the generalized Euler equations and the generalized Lagrange's equations is now straightforward (cf Passos Morgado (1993a)). To complete the discussion we present the general formulae for the forces of inertia and the moments of the refered forces in the Euler equations and the formulae for the generalized forces employed in the Lagrange's and Kane's equations.

To this end we must determined, in a general case, the values of

$$
\frac{\partial\boldsymbol{H}_r}{\partial q}\qquad\frac{\partial\boldsymbol{H}_r}{\partial\dot q}\qquad\frac{\partial I_0}{\partial q}
$$

Noting that

$$
\boldsymbol{H}_r=M\boldsymbol{r}'_c\times\Big(\frac{d\boldsymbol{r}_{0'}}{dt}\Big)_{S_1}+\boldsymbol{r}_{0'}\times(\boldsymbol{\omega}_{21}\times M\boldsymbol{r}'_c)+I_{0'}\boldsymbol{\omega}_{21} \qquad (2.20)
$$

we have immediately

$$
\frac{\partial\boldsymbol{H}_r}{\partial\dot q}=M\boldsymbol{r}'_c\times\frac{\partial}{\partial\dot q}\Big(\frac{d\boldsymbol{r}_{0'}}{dt}\Big)_{S_1}+\boldsymbol{r}_{0'}\times\Big(\frac{\partial\boldsymbol{\omega}_{21}}{\partial\dot q}\times M\boldsymbol{r}'_c\Big)+I_{0'}\frac{\partial\boldsymbol{\omega}_{21}}{\partial\dot q} \qquad (2.21)
$$

and

$$
\begin{aligned}
\frac{\partial\boldsymbol{H}_r}{\partial q}&=M\frac{\partial\boldsymbol{r}'_c}{\partial q}\times\Big(\frac{d\boldsymbol{r}_{0'}}{dt}\Big)_{S_1}+M\boldsymbol{r}'_c\times\frac{\partial}{\partial q}\Big(\frac{d\boldsymbol{r}_{0'}}{dt}\Big)_{S_1}+\frac{d\boldsymbol{r}_{0'}}{dq}\times(\boldsymbol{\omega}_{21}\times M\boldsymbol{r}'_c)+ \qquad (2.22)\\
&+\boldsymbol{r}_{0'}\times\Big(\frac{\partial\boldsymbol{\omega}_{21}}{\partial q}\times M\boldsymbol{r}'_c\Big)+\boldsymbol{r}_{0'}\times\Big(\boldsymbol{\omega}_{21}\times M\frac{\partial\boldsymbol{r}'_c}{\partial q}\Big)+\frac{\partial I_{0'}}{\partial q}\boldsymbol{\omega}_{21}+I_{0'}\frac{\partial\boldsymbol{\omega}_{21}}{\partial q}
\end{aligned}
$$

The partial derivative of the moment of inertia about the point 0, can be related to the moment of inertia about point $0'$, attached to the body and origin of the frame of reference S_2, by the following equation

$$I_0\boldsymbol{\omega}_{10} = M\boldsymbol{r}_{0'}\times(\omega_{10}\times\boldsymbol{r}_{0'})+\boldsymbol{r}_{0'}\times(\omega_{10}\times M\boldsymbol{r}_c')+M\boldsymbol{r}_c'\times(\omega_{10}\times\boldsymbol{r}_{0'})+I_0\omega_{10} \quad (2.23)$$

for the generalized centrifugal inertial forces, we have

$$\begin{aligned}\frac{1}{2}\omega_{10}\frac{\partial I_0}{\partial q}\omega_{10} &= M\Big(\omega_{10}\times\frac{\partial \boldsymbol{r}_{0'}}{\partial q}\Big)(\omega_{10}\times\boldsymbol{r}_{0'}) + M\Big(\omega_{10}\times\frac{\partial \boldsymbol{r}_{0'}}{\partial q}\Big)(\omega_{10}\times\boldsymbol{r}_c') + \\ &+M\Big(\omega_{10}\times\frac{\partial \boldsymbol{r}_c'}{\partial q}\Big)(\omega_{10}\times\boldsymbol{r}_{0'}) + \frac{1}{2}\omega_{10}\frac{\partial I_{0'}}{\partial q}\omega_{10}\end{aligned} \quad (2.24)$$

of the vectors $\boldsymbol{\omega}_{q_r} = \boldsymbol{\omega}_{q_r}(q_1,...,q_nt)$ with respect to time have to be calculated. If $\boldsymbol{c}$ is any vector fixed to the body, then

$$\Big(\frac{d\boldsymbol{c}}{dt}\Big)_{S_1} = \Big(\frac{d\boldsymbol{c}}{dt}\Big)_{S_2} + \boldsymbol{\omega}_{21}\times\boldsymbol{c} = \boldsymbol{\omega}_{21}\times\boldsymbol{c} \quad (2.25)$$

On the other hand, when $\boldsymbol{c} = \boldsymbol{c}(q_1,...,q_nt)$ is the vector attached to the body dependent on the generalized coordinates and on time, we have

$$\Big(\frac{d\boldsymbol{c}}{dt}\Big)_{S_1} = \frac{\partial\boldsymbol{c}}{\partial q_1}\dot{q}_1 + \frac{\partial\boldsymbol{c}}{\partial q_2}\dot{q}_2 + ... + \frac{\partial\boldsymbol{c}}{\partial q_n}\dot{q}_n + \frac{\partial\boldsymbol{c}}{\partial t} \quad (2.26)$$

Upon substitution of Eq (2.1) into (2.25) we have

$$\Big(\frac{d\boldsymbol{c}}{dt}\Big)_{S_1} = (\boldsymbol{\omega}_{q_1}\times\boldsymbol{c})\dot{q}_1 + (\boldsymbol{\omega}_{q_2}\times\boldsymbol{c})\dot{q}_2 + ... + (\boldsymbol{\omega}_{q_n}\times\boldsymbol{c})\dot{q}_n + \boldsymbol{\omega}_t\times\boldsymbol{c} \quad (2.27)$$

Now by comparing Eqs (2.26) and (2.27) we can conclude that

$$\frac{\partial\boldsymbol{c}}{\partial q_r} = \boldsymbol{\omega}_{q_r}\times\boldsymbol{c} \qquad \frac{\partial\boldsymbol{c}}{\partial t} = \boldsymbol{\omega}_t\times\boldsymbol{c} \quad (2.28)$$

The vectors $\boldsymbol{\omega}_{q_r}$ can be treated as operators which produce the partial derivatives when multiplied vectorially by a vector fixed to the body.

Noting

$$\frac{\partial}{\partial q_r}\Big(\frac{d\boldsymbol{c}}{dt}\Big)_{S_1} = \Big[\frac{d}{dt}\Big(\frac{\partial\boldsymbol{c}}{\partial q_r}\Big)\Big]_{S_1} \quad (2.29)$$

we can conclude, basing on the Eqs (2.25) and (2.28), that

$$\frac{\partial}{\partial q_r}(\boldsymbol{\omega}_{21}\times\boldsymbol{c}) = \Big[\frac{d}{dt}(\boldsymbol{\omega}_{q_r}\times\boldsymbol{c})\Big]_{S_1} \quad (2.30)$$

and

$$\frac{\partial \boldsymbol{\omega}_{21}}{\partial q_r} \times \boldsymbol{c} + \boldsymbol{\omega}_{21} \times \frac{\partial \boldsymbol{c}}{\partial q_r} = \left(\frac{d\omega_{q_r}}{dt}\right)_{S_1} \times \boldsymbol{c} + \boldsymbol{\omega}_{q_r} \times \left(\frac{d\boldsymbol{c}}{dt}\right)_{S_1} \tag{2.31}$$

Having in mind Eqs (2.25) and (2.28) we can rewrite Eq (2.31) as follows

$$\frac{\partial \boldsymbol{\omega}_{21}}{\partial q_r} \times \boldsymbol{c} + \boldsymbol{\omega}_{21} \times (\boldsymbol{\omega}_{q_r} \times \boldsymbol{c}) = \left(\frac{d\omega_{q_r}}{dt}\right)_{S_1} \times \boldsymbol{c} + \boldsymbol{\omega}_{q_r} \times (\boldsymbol{\omega}_{21} \times \boldsymbol{c}) \tag{2.32}$$

and introducing

$$\boldsymbol{\omega}_{q_r} \times (\boldsymbol{\omega}_{21} \times \boldsymbol{c}) - \boldsymbol{\omega}_{21} \times (\boldsymbol{\omega}_{q_r} \times \boldsymbol{c}) = \boldsymbol{c} \times (\boldsymbol{\omega}_{21} \times \boldsymbol{\omega}_{q_r}) \tag{2.33}$$

we conclude that

$$\left(\frac{d\omega_{q_r}}{dt}\right)_{S_1} = \frac{\partial \boldsymbol{\omega}_{21}}{\partial q_r} + \boldsymbol{\omega}_{21} \times \boldsymbol{\omega}_{q_r} \tag{2.34}$$

which allows us to determine the derivative of vectors $\boldsymbol{\omega}_{q_r}$ with respect to time. Also it is worthy to note the relation

$$\frac{\partial \omega_{21}}{\partial q_r} = \frac{\partial \omega_{q_1}}{\partial q_r}\dot{q}_1 + \frac{\partial \omega_{q_2}}{\partial q_r}\dot{q}_2 + \ldots + \frac{\partial \omega_{q_n}}{\partial q_r}\dot{q}_n + \frac{\partial \omega_t}{\partial q_r} \tag{2.35}$$

Let us now suppose that the origin of the system S_2 coincides with the origin of system S_1. In this case $\boldsymbol{r}_{0'} = 0$, (since $0'$ and 0 coincide), and the Eqs (2.21) and (2.22) are reduced to the forms

$$\frac{\partial \boldsymbol{H}_r}{\partial \dot{q}} = I_{0'}\frac{\partial \omega_{21}}{\partial \dot{q}} \tag{2.36}$$

$$\frac{\partial \boldsymbol{H}_r}{\partial q} = \frac{\partial I_{0'}}{\partial q} + I_{0'}\frac{\partial \omega_{21}}{\partial q} \tag{2.37}$$

Let us now to find simpler formulae for the generalized inertial forces. For the generalized rotational inertial force we have

$$\dot{\boldsymbol{\omega}}_{10}\frac{\partial \boldsymbol{H}_r}{\partial \dot{q}} = \dot{\omega}_{10}I_{0'}\frac{\partial \omega_{21}}{\partial \dot{q}} = \dot{\omega}_{10}I_{0'}\omega_q = \omega_q I_{0'}\dot{\omega}_{10} \tag{2.38}$$

For the generalized centrifugal inertial force we get

$$\frac{1}{2}\boldsymbol{\omega}_{10}\frac{\partial I_0}{\partial q}\omega_{10} = \frac{1}{2}\omega_{10}\frac{\partial I_{0'}}{\partial q}\omega_{10} = \frac{1}{2}\omega_{10}\left(\frac{\partial \boldsymbol{\omega}_{21}}{\partial \dot{q}} \times I_{0'} - I_{0'} \times \frac{\partial \boldsymbol{\omega}_{21}}{\partial \dot{q}}\right)\omega_{10} \tag{2.39}$$

and as the dyad is a symmetrical one, we can write

$$\begin{aligned}
&\frac{1}{2}\omega_{10}\frac{\partial I_0}{\partial q}\omega_{10} = \frac{1}{2}\omega_{10}\Big(\frac{\partial \omega_{21}}{\partial \dot{q}} \times I_{0'}\Big)\omega_{10} - \frac{1}{2}\omega_{10}\Big(I_{0'} \times \frac{\partial \omega_{21}}{\partial \dot{q}}\Big)\omega_{10} = \\
&= \frac{1}{2}\Big[\Big(\omega_{10} \times \frac{\partial \omega_{21}}{\partial \dot{q}}\Big)I_{0'}\omega_{10} - \omega_{10}I_{0'}\Big(\frac{\partial \omega_{21}}{\partial \dot{q}} \times \omega_{10}\Big)\Big] = \\
&= \frac{1}{2}\Big[\Big(\omega_{10} \times \frac{\partial \omega_{21}}{\partial \dot{q}}\Big)I_{0'}\omega_{10} + \Big(\omega_{10} \times \frac{\partial \omega_{21}}{\partial \dot{q}}\Big)I_{0'}\omega_{10}\Big] = \\
&= \Big(\omega_{10} \times \frac{\partial \omega_{21}}{\partial \dot{q}}\Big)I_{0'}\omega_{10} = \frac{\partial \omega_{21}}{\partial \dot{q}}(\omega_{10} \times I_{0'}\omega_{10})
\end{aligned} \tag{2.40}$$

For the Coriolis inertial force of we have

$$\begin{aligned}
&\omega_{10}\Big[\frac{d}{dt}\Big(\frac{\partial \boldsymbol{H}_r}{\partial \dot{q}}\Big)_{S_1} - \frac{\partial \boldsymbol{H}_r}{\partial q}\Big] = \omega_{10}\Big[\frac{d}{dt}\Big(I_{0'}\frac{\partial \omega_{21}}{\partial \dot{q}}\Big)_{S_1} - \Big(\frac{\partial I_{0'}}{\partial q}\omega_{21} + I_{0'}\frac{\partial \omega_{21}}{\partial q}\Big)\Big] = \\
&= \omega_{10}\Big[(\omega_{21} \times I_{0'} - I_{0'} \times \omega_{21})\frac{\partial \omega_{21}}{\partial \dot{q}} + I_{0'}\Big(\frac{\partial \omega_{21}}{\partial q} + \omega_{21} \times \frac{\partial \omega_{21}}{\partial \dot{q}}\Big) + \\
&-\Big(\frac{\partial I_{0'}}{\partial q}\omega_{21} + I_{0'}\frac{\partial \omega_{21}}{\partial q}\Big)\Big] = \omega_{10}\Big[(\omega_{21} \times I_{0'} - I_{0'} \times \omega_{21})\frac{\partial \omega_{21}}{\partial \dot{q}} + \\
&+I_{0'}\Big(\omega_{21} \times \frac{\partial \omega_{21}}{\partial \dot{q}}\Big) - \Big(\frac{\partial \omega_{21}}{\partial \dot{q}} \times I_{0'} - I_{0'} \times \frac{\partial \omega_{21}}{\partial \dot{q}}\Big)\omega_{21}\Big] = \\
&= \omega_{10}\Big[(\omega_{21} \times I_{0'})\frac{\partial \omega_{21}}{\partial \dot{q}} - (I_{0'} \times \omega_{21})\frac{\partial \omega_{21}}{\partial \dot{q}} - \Big(\frac{\partial \omega_{21}}{\partial \dot{q}} \times I_{0'}\Big)\omega_{21}\Big]
\end{aligned} \tag{2.41}$$

since

$$\Big(I_{0'} \times \frac{\partial \omega_{21}}{\partial \dot{q}}\Big)\omega_{21} = I_{0'}\Big(\frac{\partial \omega_{21}}{\partial \dot{q}} \times \omega_{21}\Big) = -I_{0'}\Big(\omega_{21} \times \frac{\partial \omega_{21}}{\partial \dot{q}}\Big)$$

To simplify Eqs (2.41) let us introduce the relation

$$(\omega_{21} \times I_{0'})\frac{\partial \omega_{21}}{\partial \dot{q}} = \omega_{21}) \times \Big(I_{0'}\frac{\partial \omega_{21}}{\partial \dot{q}}\Big)$$

and the left-side product by ω_{10} yields

$$\begin{aligned}
&-\omega_{10}(\omega_{21} \times I_{0'})\frac{\partial \omega_{21}}{\partial \dot{q}} = \omega_{10}\omega_{21} \times \Big(I_{0'}\frac{\partial \omega_{21}}{\partial \dot{q}}\Big) = \\
&= (\omega_{10} \times \omega_{21})I_{0'})\frac{\partial \omega_{21}}{\partial \dot{q}} = \frac{\partial \omega_{21}}{\partial \dot{q}}I_{0'}(\omega_{10} \times \omega_{21})
\end{aligned} \tag{2.42}$$

Introducing now the relation

$$-(I_{0'} \times \omega_{21})\frac{\partial \omega_{21}}{\partial \dot{q}} = -I_{0'}\Big(\omega_{21}\frac{\partial \omega_{21}}{\partial \dot{q}}\Big)$$

and the left-side product by $\boldsymbol{\omega}_{10}$ yields

$$
\begin{aligned}
&-\boldsymbol{\omega}_{10}(\boldsymbol{I}_{0'} \times \boldsymbol{\omega}_{21})\frac{\partial \boldsymbol{\omega}_{21}}{\partial \dot{q}} = -\boldsymbol{\omega}_{10}\boldsymbol{I}_{0'}\Big(\boldsymbol{\omega}_{21} \times \frac{\partial \boldsymbol{\omega}_{21}}{\partial \dot{q}}\Big) = \\
&= -\boldsymbol{\omega}_{10}\boldsymbol{I}_{0'}\Big(\frac{\partial \boldsymbol{\omega}_{21}}{\partial \dot{q}} \times \boldsymbol{\omega}_{21}\Big) = \Big(\frac{\partial \boldsymbol{\omega}_{21}}{\partial \dot{q}} \times \boldsymbol{\omega}_{21}\Big)\boldsymbol{I}_{0'} \times \boldsymbol{\omega}_{10} = \frac{\partial \boldsymbol{\omega}_{21}}{\partial \dot{q}}\boldsymbol{\omega}_{21} \times \boldsymbol{I}_{0'}\boldsymbol{\omega}_{10}
\end{aligned}
\tag{2.43}
$$

Another relation to be considered is the following one

$$
-\Big(\frac{\partial \boldsymbol{\omega}_{21}}{\partial \dot{q}} \times \boldsymbol{I}_{0'}\Big)\boldsymbol{\omega}_{21} = -\frac{\partial \boldsymbol{\omega}_{21}}{\partial \dot{q}}(\boldsymbol{I}_{0'}\boldsymbol{\omega}_{21})
$$

in wich, again, multiplying it by $\boldsymbol{\omega}_{10}$, we have

$$
\begin{aligned}
&-\boldsymbol{\omega}_{10}\Big(\frac{\partial \boldsymbol{\omega}_{21}}{\partial \dot{q}} \times \boldsymbol{I}_{0'}\Big)\boldsymbol{\omega}_{21} = -\boldsymbol{\omega}_{10}\frac{\partial \boldsymbol{\omega}_{21}}{\partial \dot{q}} \times (\boldsymbol{I}_{0'}\boldsymbol{\omega}_{21}) = \\
&= -\boldsymbol{\omega}_{10}\frac{\partial \boldsymbol{\omega}_{21}}{\partial \dot{q}} \times \boldsymbol{I}_{0'}\boldsymbol{\omega}_{21} = \frac{\partial \boldsymbol{\omega}_{21}}{\partial \dot{q}}\boldsymbol{\omega}_{10} \times \boldsymbol{I}_{0'}\boldsymbol{\omega}_{21}
\end{aligned}
\tag{2.44}
$$

Substitution Eqs (2.42) $\div$ (2.44) into Eq (2.41), we have for the generalized Coriolis inertial forces

$$
\begin{aligned}
&\boldsymbol{\omega}_{10}\Big[\frac{d}{dt}\Big(\frac{\partial \boldsymbol{H}_r}{\partial \dot{q}}\Big)_{S_1} - \frac{\partial \boldsymbol{H}_r}{\partial q}\Big] = \\
&= \frac{\partial \boldsymbol{\omega}_{21}}{\partial \dot{q}}\boldsymbol{I}_{0'}(\boldsymbol{\omega}_{10} \times \boldsymbol{\omega}_{21}) + \frac{\partial \boldsymbol{\omega}_{21}}{\partial \dot{q}}(\boldsymbol{\omega}_{21} \times \boldsymbol{I}_{0'}\boldsymbol{\omega}_{10} + \boldsymbol{\omega}_{10} \times \boldsymbol{I}_{0'}\boldsymbol{\omega}_{21})
\end{aligned}
\tag{2.45}
$$

which give us the Coriolis generalized force of inertia for the proposed case.

We must note that the quantity which appears in the first term of the second member of Eq (2.45), being related with the Resal acceleration, is not a gyroscopic force (cf Skalmierski (1979)) but in the reality is a term of the Coriolis generalized force.

3. Conclusions

In this paper we have generalized the Euler equations and demonstrated their equivalence to the generalized Kane's and Lagrange's equations for accelerated systems. Also we presented the general formulae for the generalized *inertial forces and the way of deriving them in the general case.* In another paper it will be presented the distinction between inertia forces and gyroscopic ones, and it will be shown that, in the most general case, there are four, and only four, different types of gyroscopic forces.

References

1. KANE T.R., 1968, *Dynamics,* Holt, Rinehart and Winston
2. LURE, *Analitical Mechanics* (in Russian), pp 430, 431
3. PASSOS MORGADO A.H., 1993a, The Equation of Motion for Accelerated Referential Frames Using Geometric Nature of Lagrange Equations, *Scientific Transaction Mechanika,* **10**, 263-270, Technical University of Bialystok
4. PASSOS MORGADO A.H., 1993b, A New Point of View on the Motion in an Accelerated Referential Frame, *Scientific Transaction Mechanika,* **10**, 279-285, TecHnical University of Bialystok
5. SKALMIERSKI B., 1979, *Mechanics and Strength of Materials,* Elsevier Scientific Publishing Company, Amsterdam-London-New York

Uogólnione równania Eulera

Streszczenie

W pracy podano równania Eulera opisujące ruch ciala rozpatrywany względem ruchomego ukladu odniesienia. Położenie cial określa się przy pomocy n współrzędnych uogólnionych i dlatego możliwe jest stwierdzenie równoważności uogólnionych równań odpowiednio Eulera, Kane'a i Lagrange'a, które również określone są względem ruchomego ukladu odniesienia. Podano ogólne wzory na uogólnione sily bezwladności oraz sposób ich stosowania. Jako przykład podano szczególny przypadek ciala sztywnego. Lure [2] rozpatrywal ten problem w zupelnie inny sposób, jednak interpretacja fizyczna podana w pracy pozwala na sformułowanie ogólnych i zupelnych wzorów na sily bezwladności i sily giroskopowe oraz na określenie różnic między nimi.

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