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**Research Paper**

# Counting the sum of cubes for Lucas and Gibonacci Numbers

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**Abstract**

Lucas and Gibonacci numbers are two sequences of numbers derived from a welknown numbers, Fibonacci numbers. The difference between Lucas and Fibonacci numbers only lies on the first and second elements. The first element in Lucas numbers is 2 and the second is 1, and nth element, n ≥ 3 determined by similar pattern as in the Fibonacci numbers, i.e : Ln = Ln-1 + Ln-2. Gibonacci numbers G_0,G_1,G_2,G_3… ; G_n=G_(n-1)+G_(n-2) are generalized of Fibonacci numbers, and those numbers are nonnegative integers. If G_0=1 and G_1=1, then the numbers are the wellknown Fibonacci numbers, and if G_0=2 and G_1=1, the numbers are Lucas numbers. Thus, the difference of those three sequences of numbers only lies on the first and second of the elements in the sequences. For Fibonacci numbers there are quite a lot identities already explored, including the sum of cubes, but there have no discussions yet about the sum of cubes for Lucas and Gibonacci numbers. In this study the sum of cubes of Lucas and Gibonacci numbers will be discussed and showed that the sum of cubes for Lucas numbers is $\sum_{i=0}^{n} (Li)^3 = \frac{L_n(L_{n+1})^2+5(-1)^n L_{n-1} + 19}{2}$ and for Gibonacci numbers is $\sum_{i=0}^{n} (Gi)^3 = \frac{[G_n(G_{n+1})^2+(G_1-G_0)^3-3{G_0}^2G_1+4{G_0}^3-(-1)^n({G_1}^2-G_1G_0-{G_0}^2)G_{n-1}]}{2}$

**Keywords**

Fibonacci numbers, Lucas numbers, Gibonacci numbers, identities, sum of cubes

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## 1. INTRODUCTION

Fibonacci number was originally stated by Leonardo Pisano or Fibonacci in his book, Liber Abaci in 1202. In his book he stated a problem about the growth of pairs rabbits. A young pair of rabbits (male and female) is placed on an island. A pair of rabbits does not breed until they are two months old. After they are two month old, each pair of rabbits produces another pair each month. Assuming that there are no rabbits ever die then the total number of pair of rabbits form a sequence 1,1,2,3,5,8,13,...and this sequence of numbers is known as Fibonacci numbers. Lucas number was introduced by François Edouard Anatole Lucas in 1877 by setting the first number is 2 and the second is 1, and the other numbers are following the pattern as on Fibonacci numbers so that it form a sequence of numbers 2, 1, 3, 4, 7, 11, 18, 29, 47,… Gibonacci numbers $G_0, G_1, G_2, G_3$,...;$G_n = G_{n-1} + G_{n-2}$ are generalized of Fibonacci numbers, and those numbers are nonnegative integers. If $G_0 = 1$ and $G_1 = 1$,then the numbers are the welknown Fibonacci numbers, and if $G_0 = 2$ and $G_1 = 1$, the numbers are Lucas numbers. Thus, the difference of those three sequence of numbers only lies on the first and second of the elements in the sequences.

There are many identities already developed that related among Gibonacci, Fibonacci, and Lucas numbers. Posamentier et al. (2007) showed that Fibonacci, Lucas, and Gibonacci numbers related to Pascal triangle, art and golden ratio. There are some researchers already investigated identities for Fibonacci Lucas and Gibonacci numbers such as Carlitz et al. (1972a,b,c,d), and Benjamin et al. (1999, 2000, 2009, 2003). Krishna (1980), investigated generalized Fibonacci sequence and William (1972) discussed how to get Fibonacci numbers from Pascal's triangle. Dunlap (2003) investigated about the golden ratio and Generalized Fibonacci numbers, including the application in biology and crystallography. Benjamin et al. (2003) proved more than one hundred Fibonacci identities by combinatorial arguments and Benjamin et al. (2009) proposed the identity for finding the sum of cubes for Fibonacci numbers. Frontczak (2018) discussed about the relationship of sum of powers for Fibonacci and Lucas numbers for sum of first, second, third, and fourth powers; and Arangala et al. (2016) investigated the relationship of the sum of power for Fibonacci and Lucas numbers for the power of square, fourth, sixth, and eighth. However, until recently there have no discussion about the sum of cubes for Lucas numbers where in the formula just contains the Lucas numbers itself, or the sum of cubes for Gibonacci numbers where in the formula just contains the Gibonacci numbers itself. Therefore, in this study we will discuss about the sum of cubes for Gibonacci numbers as an extension identity of Koshy (2001), and Benjamin et al. (1999). In Section 2 and 3 we will present the results from Koshy (2001), Benjamin et al. (1999), and Dunlap (2003), and in Section 4 we will present the results.

## 2. Some Identities for Lucas and Gibonacci Numbers

### 2.1 Some Identities for Lucas Number

The Identity 1, 2, and 3 are proposed by Koshy (2001), and the Identity 4 is proposed by Benjamin et al. (1999).

**Identity 1**

$$L_0 + L_1 + L_2 + ... + L_n = L_{n+2} - L_1$$

*Proof.* Using the definition of Lucas number we get
$L_{n-2} = L_n - L_{n-1}$ then,
$L_0 = L_2 - L_1$
$L_1 = L_3 - L_2$
$L_2 = L_4 - L_3$
$L_3 = L_5 - L_4$
.
.
.
$L_{n-1} = L_{n+1} - L_n$
Take the sum on both sides simultaneously we get :
$L_0 + L_1 + L_2 + L_3 + ... + L_n = L_{n+2} - L_1$ □

**Identity 2**

$$L_0 + L_1 + L_2 + L_3 + ... + L_{2n} = L_{2n+1} - L_1$$

*Proof.* By definition, $L_{n-1} = L_n - L_{n-2}$ then,
$L_0 = L_0$
$L_2 = L_3 - L_1$
$L_4 = L_5 - L_3$
$L_6 = L_7 - L_5$
.
.
.
$L_{2n-2} = L_{2n-1} - L_{2n-3}$
$L_{2n} = L_{2n+1} - L_{2n-1}$
Take the sum on both sides simultaneously we get :

$$\begin{aligned} L_0 + L_1 + L_2 + L_3 + ... + L_{2n} &= L_{2n+1} - L_1 + L_0 \\ &= L_{2n+1} - 1 + 2 \\ &= L_{2n+1} + 1 \\ &= L_{2n+1} L_1 \end{aligned}$$

□

**Identity 3**

$L_1 + L_3 + L_5 + ... + L_{2n-1} = L_{2n} - L_0$

*Proof.* By definition, $L_{n-1} = L_n - L_{n-2}$ then,
$L_1 = L_2 - L_0$
$L_3 = L_4 - L_2$
$L_5 = L_6 - L_4$
.
.
.
$L_{2n-3} = L_{2n-2} - L_{2n-4}$
$L_{2n-1} = L_{2n} - L_{2n-2}$
Take the sum on both sides simultaneously we get :
$L_1 + L_3 + L_5 + ... + L_{2n-1} = L_{2n} - L_0$ □

**Identity 4**

$L_{n+1}L_{n-1} = L_n^2 - (-1)^n \cdot 5$

*Proof.* Using mathematical induction, and we leaven the basic induction and just do the induction step. Asumme that the statement is true for n = k , $L_{k+1}L_{k-1} = L_k^2 - (-1)^k \cdot 5$
For n = k + 1, then :
$L_{k+1}^2 = L_{k+1} \cdot L_{k+1}$
$L_{k+1}^2 = (L_{k+2} - L_k)L_{k+1}$
$L_{k+1}^2 = L_{k+2} \cdot L_{k+1} - L_k \cdot L_{k+1}$
$L_{k+1}^2 = L_{k+2} - (L_k + L_{k-1}) \text{-} L_k \cdot L_{k+1}$
$L_{k+1}^2 = L_{k+2}L_k + L_{k-1} + L_{k+2} \cdot L_{k-1} - L_k \cdot L_{k+1}$
$L_{k+1}^2 = L_{k+2}L_k + (L_{k-1} + L_k)L_{k-1} \cdot L_k L_{k+1}$
$L_{k+1}^2 = L_{k+2}L_k + L_{k+1}L_{k-1} + L_k \cdot L_{k-1} - L_k \cdot L_{k+1}$
$L_{k+1}^2 = L_{k+2}L_k + L_{k+1} \cdot L_{k-1} + L_k(L_{k+1} - L_{k-1})$
$L_{k+1}^2 = L_{k+2}L_k + L_{k+1}L_{k-1} + L_k(-L_k)$
$L_{k+1}^2 = L_{k+2}L_k + L_{k+1}L_{k-1} - L_k^2$
Since, $L_{k-1}L_{k+2} = -L_k^2 - (-1)^k \cdot 5$, then
$L_{k+1}^2 = L_{k+2} \cdot L_k - (-1)^k \cdot 5$
$L_{k+1}^2 = L_{k+2} \cdot L_k + (-1)^{k+1} \cdot 5$
$L_{k+2} \cdot L_k = L_{k+1}^2 + (-1)^{k+1} \cdot 5$ □

### 2.2 Some Identities for Gibonacci Number

The Identity 5,6, and 7 are proposed by Dunlap (2003), Identity 8 is proposed by Benjamin et al. (2009), while Identity 9 derived from Definition 8. Those Identity will be used in the process to determine the sum of cubes for Gibonacci numbers.

**Identity 5**

$G_0 + G_1 + G_2 + ... + G_n = G_{n+2} - G_1$

*Proof.* Using the definition of Gibonacci numbers, we get
$G_{n-2} = G_n - G_{n-1}$ then,
$G_0 = G_2 - G_1$
$G_1 = G_3 - G_2$
$G_2 = G_4 - G_3$
$G_3 = G_5 - G_4$
.
.
.
$G_{n-1} = G_{n+1} - G_n$
$G_n = G_{n+2} - G_{n+1}$
Take the sum of both sides simultaneously we get:
$G_0 + G_1 + G_2 + ... + G_n = G_{n+2} - G_1$ □

**Identity 6**

$G_0 + G_1 + G_2 + G_3 + ... + G_{2n} = G_{2n+1} + G_0 - G_1$

*Proof.* Again, using the definition of Gibonacci numbers, we get
$G_{n-1} = G_n - G_{n-2}$,
$G_0 = G_0$
$G_2 = G_3 - G_1$
$G_4 = G_5 - G_3$
$G_6 = G_7 - G_5$
.
.
.
$G_{2n-2} = G_{2n-1} - G_{2n-3}$
$G_{2n} = G_{2n+1} - G_{2n-1}$
Take the sum of both sides simultaneously we get :
$G_0 + G_1 + G_2 + G_3 + ... + G_{2n} = G_{2n+1} + G_0 - G_1$ □

**Identity 7**
$G_1 + G_3 + G_5 + ... + G_{2n-1} = G_{2n} - G_0$

*Proof.* $G_{n-1} = G_n - G_{n-2}$ then,
$G_1 = G_2 - G_0$
$G_3 = G_4 - G_2$
$G_5 = G_6 - G_4$
.
.
.
$G_{2n-3} = G_{2n-2} - G_{2n-4}$
$G_{2n-1} = G_{2n} - G_{2n-2}$
Take the sum on both sides simultaneously we get :
$G_1 + G_3 + G_5 + ... + G_{2n-1} = G_{2n} - G_0$ □

**Identity 8**
$G_{n+1}G_{n-1} = G_n^2 - (-1)^n \cdot \{G_1^2 - G_1G_0 - G_0^2\}$

*Proof.* By using mathematical induction.
For n = k + 1, then :
$G_2 \cdot G_0 = G_n^2 - (-1)^n \cdot \{G_1^2 - G_1G_0 - G_0^2\}$
$(G_1 + G_0)G_0 = G_1^2 - G_1^2 - G_1G_0 - G_0^2$
$G_1G_0 + G_0^2 = G_1G_0G_0^2$
Assume that the statement is true for n = k, then
$G_{k+1}^2 = G_{k+1} \cdot G_{k+1}$
$G_{k+1}^2 = (G_{k+2} - G_k)G_{k+1}$
$G_{k+1}^2 = G_{k+2} \cdot G_{k+1} - G_k \cdot G_{k+1}$
$G_{k+1}^2 = G_{k+2} - (G_k + G_{k-1}) - G_k \cdot G_{k+1}$
$G_{k+1}^2 = G_{k+2}G_k + G_{k-1} + L_{k+2} \cdot G_{k-1} - G_k \cdot G_{k+1}$
$G_{k+1}^2 = G_{k+2}G_k + (G_{k-1} + G_k)G_{k-1} \cdot G_kG_{k+1}$
$G_{k+1}^2 = G_{k+2}G_k + G_{k+1}G_{k-1} + G_k \cdot G_{k-1} - G_k \cdot G_{k+1}$
$G_{k+1}^2 = G_{k+2}G_k + G_{k+1} \cdot G_{k-1} + G_k(G_{k+1} - G_{k-1})$
$G_{k+1}^2 = G_{k+2}G_k + G_{k+1}G_{k-1} + G_k(-G_k)$
$G_{k+1}^2 = G_{k+2}G_k + G_{k+1}G_{k-1} - G_k^2$
Because, $G_{n+1}G_{n-1} = G_n^2 - (-1)^n \cdot \{G_1^2 - G_1G_0 - G_0^2\}$ then
$G_{k+1}^2 = G_{k+2} \cdot G_k - (-1)^k \cdot \{G_1^2 - G_1G_0 - G_0^2\}$
$G_{k+1}^2 = G_{k+2} \cdot G_k + (-1)^{k+1} \cdot \{G_1^2 - G_1G_0 - G_0^2\}$
We get $G_{k+2} \cdot G_k = G_{k+1}^2 + (-1)^{k+1} \cdot \{G_1^2 - G_1G_0 - G_0^2\}$ □

The following identity was derived from Identity 8

**Identity 9**
$2G_n^2 = G_{n+1}^2 - G_nG_{n-1} - (-1)^n(G_1^2 - G_1G_0 - G_0^2)$

*Proof.* $G_n^2 = G_n \cdot G_n$
$G_n^2 = G_n \cdot (G_{n+1} - G_{n-1})$
$G_n^2 = G_n \cdot G_{n+1} - G_n \cdot G_{n-1}$
$G_n^2 = (G_{n+1} - G_{n-1}) \cdot G_{n+1} - G_n \cdot G_{n-1}$
$G_n^2 = G_{n+1}^2 - G_{n+1} \cdot G_{n-1} - G_n \cdot G_{n-1}$
Using Identity 4
$G_{n+1}G_{n-1} = G_n^2 - (-1)^n \cdot \{G_1^2 - G_1G_0 - G_0^2\}$
Then:
$G_n^2 = G_{n+1}^2\{G_n^2 - (-1)^n \cdot (G_1^2 - G_1G_0 - G_0^2)\} \cdot G_{n-1} - G_n \cdot G_{n-1}$
$2G_n^2 = G_{n+1}^2 - G_nG_{n-1} - (-1)^n(G_1^2 - G_1G_0 - G_0^2)$ □

### 3. Sum of Cubic for Lucas and Gibonacci Numbers

Based on the Definition above, we can derive an Definition for finding the sum of cubes for Lucas and Gibonacci number as follow:

**Identity 10**
The sum of cubic of Lucas numbers is
$\sum_{i=0}^{n} (Li)^3 = \frac{L_n(L_{n+1})^2 + 5(-1)^n L_{n-1} + 19}{2}$

*Proof.* Using Identity 5: $2L_n^2 = L_{n+1}^2 - L_nL_{n-1} - (-1)^n \cdot 5$, then
$2L_0^3 = 2L_0^3$
$2L_1^3 = 2L_1L_1^2$
$2L_1^3 = L_1(L_2^2L_1L_0 + (-1)^1 5)$
$2L_1^3 = L_1L_2^2 - L_1^2 \cdot -5L_1$
$2L_2^3 = 2L_2L_2^2$
$2L_2^3 = L_2(L_3^2L_2L_1 + (-1)^1 5)$
$2L_2^3 = L_2L_1^2 - L_2^2 \cdot -5L_1$
$2L_3^3 = 2L_3L_3^2$
$2L_3^3 = L_3(L_4^2L_3L_2 + (-1)^1 5)$
$2L_3^3 = L_3L_4^2 - L_3^2 \cdot -5L_1$
$2L_4^3 = 2L_4L_4^2$
$2L_4^3 = L_4(L_5^2L_4L_3 + (-1)^1 5)$
$2L_4^3 = L_4L_3^2 - L_4^2 \cdot -5L_1$
$2L_5^3 = 2L_5L_5^2$
$2L_5^3 = L_5(L_6^2L_5L_4 + (-1)^1 5)$
$2L_5^3 = L_5L_4^2 - L_5^2 \cdot -5L_1$
$2L_6^3 = 2L_6L_6^2$
$2L_6^3 = L_6(L_7^2L_6L_5 + (-1)^1 5)$
$2L_6^3 = L_6L_5^2 - L_6^2 \cdot -5L_1$
.
.
.
$2L_n^3 = 2L_nL_n^2$
$2L_n^3 = L_n(L_{n+1}^2L_nL_{n-1} + (-1)^1 5)$
$2L_n^3 = L_nL_{n-1}^2 - L_n^2 \cdot -5L_1$

Take the sum on both sides simultaneously we get :

$$2(L_0^3 + L_1^3 + L_2^3 + L_3^3 + ... + L_n^3)$$
$$2(L_0^3 - L_1^2L_0 - 5(L_1 - L_2 + L_3 - L_4 + L_5 - L_6 + ... - (-1)^n L_n)L_nL_{n+1}^2$$
$$L_nL_{n+1}^2 - 5(L_1 - L_2 + L_3 - L_4 + L_5 - L_6 + ... - (-1)^n L_n 2(L_0^3 - L_1^2L_0$$

Next, we need to find the sum of $L_1 - L_2 + L_3 - L_4 + L_5 - L_6 + ... - (-1)^n L_n$ By using Identity 2 and 3, we get:

$$-L_2 - L_4 - L_6 + ... L_{2n} = -(L_{2n+1} + L_1 - L_0) \quad (1)$$

$$-L_1 - L_3 - L_5 + ... L_{2n-1} = -(L_{2n} - L_0) \quad (2)$$

From equation (3), adding both sides with $L_{2n} - L_0$, equation (3) becomes:

$$-L_2 - L_4 - L_6 + ... L_{2n} + L_{2n} - L_0 = -(L_{2n+1} + L_1 - L_0) - (L_{2n} - L_0) \quad (3)$$

By using equation (4), equation (5) becomes :

$$\begin{aligned} -L_2 - L_4 - L_6 + ... L_{2n} + (-L_1 - L_3 - L_5 + ... L_{2n-1}) \\ = -(L_{2n+1} + L_1 - L_0) - L_{2n} - L_0 \\ L_1 - L_2 + L_3 - L_4 + L_5 - L_6 + ... + L_{2n-1} - L_{2n} \\ = -L_{2n+1} + L_1 - L_0 - L_{2n} - L_0 \\ = -L_{2n+1} + L_1 - L_{2n} \\ = -L_{2n-1} - L_{2n} \end{aligned} \quad (4)$$

Adding $L_{2n+1}$ to both sides of equation (5), we get :

$$\begin{aligned} L_1 - L_2 + L_3 - L_4 + L_5 - L_6 + ... L_{2n-1} - L_{2n} + L_{2n+1} \\ = -L_{2n-1} - L_1 + L_{2n+1} \\ L_1 - L_2 + L_3 - L_4 + L_5 - L_6 + ... L_{2n-1} - L_{2n} + L_{2n+1} \\ = L_{2n+1} - L_{2n-1} - L_1 \\ Thus, \\ L_1 - L_2 + L_3 - L_4 + L_5 - L_6 + ... L_{2n-1} - L_{2n} + L_{2n+1} \\ = L_{2n} - L_1 \end{aligned} \quad (5)$$

By equation (6) and (7), we can conclude that :

$$L_1 - L_2 + L_3 - L_4 + L_5 - L_6 + ... - (-1)^n L_n = -(-1)^n L_{n-1} - L_1 \quad (6)$$

Therefore, by using equation (8), equation (3) becomes:

$$\begin{aligned} 2({L_0}^3 + {L_1}^3 + {L_2}^3 + {L_3}^3 + ... + {L_n}^3) \\ = L_n L_{n+1}^2 - 5[-(-1)^n L_{n-1} - L_1] 2({L_0}^3 - {L_1}^2 L_0 \\ = L_{n+1}^2 + 5(-1)^n L_{n-1} + 5L_1 + 2{L_0}^3 - {L_1}^2 L_0 \end{aligned} \quad (7)$$

By using definition that $L_0 = 2$ and $L_1 = 1$, then:

$$\begin{aligned} 2({L_0}^3 + {L_1}^3 + {L_2}^3 + {L_3}^3 + ... + {L_n}^3) = \\ L_n L_{n+1}^2 - 5[-(-1)^n L_{n-1} + 5 + 16 - 2 \\ = L_n L_{n+1}^2 - 5[-(-1)^n L_{n-1} + 19 \end{aligned}$$

$$Therefore: \sum_{i=0}^{n} (Li)^3 = \frac{L_n (L_{n+1})^2 + 5(-1)^n L_{n-1} + 19}{2}$$

□

Example : Given Lucas sequence :

2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, . . . , then

$$\begin{aligned} 2({L_0}^3 + {L_1}^3 + {L_2}^3 + {L_3}^3 + ... + {L_{10}}^3) = \\ \frac{L_{10}(L_{11})^2 + 5(-1)^{10} L_9 + 19}{2} = \\ \frac{123(199)^2 + 5(-1)^{10} 76 + 19}{2} = \\ \frac{123(39601) + 5(-1)^{10} 76 + 19}{2} = \\ \frac{4870923 + 380 + 19}{2} = \\ 2435661 \end{aligned}$$

By using direct summation we get :

$$\begin{aligned} 2({L_0}^3 + {L_1}^3 + {L_2}^3 + {L_3}^3 + ... + {L_{10}}^3) = \\ 2^3 + 1^3 + 3^3 + 4^3 + 7^3 + 11^3 + 18^3 + 29^3 + 47^3 + 76^3 + 123^3 = \\ 8 + 1 + 27 + 64 + 343 + 1331 + 5832 + 24389 + 103823 + \\ 438976 + 1860867 = \\ 2435661 \end{aligned}$$

**Identity 11**

$$\sum_{i=0}^{n} (Gi)^3 = \frac{G_n(G_{n+1})^2 + G_1 - {G_0}^3 - 3{G_0}^2 G_1 4{G_0}^3 - (-1)^n({G_1}^2 - G_1 G_0 - {G_0}^2) G_{n-1}}{2}$$

*Proof.* By using Identity 9:

$2{G_n}^2 = {G_{n+1}}^2 - G_n G_{n-1} - (-1)^n({G_1}^2 - G_1 G_0 - {G_0}^2)$
$2{G_0}^3 = 2{G_0}^3$
$2{G_1}^3 = 2G_1{G_1}^2$
$2{G_1}^3 = G_1(G_2 - G_1 G_0 - (-1)^n({G_1}^2 - G_1 G_0 - {G_0}^2))$
$2{G_1}^3 = G_1{G_2}^2 - {G_2}^2 \cdot G_0 + ({G_1}^2 - G_1 G_0 - {G_0}^2)G_1$
$2{G_2}^3 = 2G_2{G_2}^2$
$2{G_2}^3 = G_2(G_3 - G_2 G_1 - (-1)^n({G_1}^2 - G_1 G_0 - {G_0}^2))$
$2{G_2}^3 = G_2{G_3}^2 - {G_3}^2 \cdot G_0 + ({G_1}^2 - G_1 G_0 - {G_0}^2)G_2$
$2{G_3}^3 = 2G_3{G_3}^2$
$2{G_3}^3 = G_3(G_4 - G_3 G_2 - (-1)^n({G_1}^2 - G_1 G_0 - {G_0}^2)$
$2{G_3}^3 = G_3{G_4}^2 - {G_4}^2 \cdot G_0 + ({G_1}^2 - G_1 G_0 - {G_0}^2)G_3$
$2{G_4}^3 = 2G_4{G_4}^2$
$2{G_4}^3 = G_4(G_5 - G_3 G_3 - (-1)^n({G_1}^2 - G_1 G_0 - {G_0}^2))$
$2{G_4}^3 = G_4{G_5}^2 - {G_5}^2 \cdot G_0 + ({G_1}^2 - G_1 G_0 - {G_0}^2)G_4$
$2{G_5}^3 = 2G_5{G_5}^2$
$2{G_5}^3 = G_5(G_6 - G_4 G_2 - (-1)^n({G_1}^2 - G_1 G_0 - {G_0}^2))$
$2{G_5}^3 = G_5{G_6}^2 - {G_5}^2 \cdot G_0 + ({G_1}^2 - G_1 G_0 - {G_0}^2)G_5$
$2{G_n}^3 = 2G_n{G_n}^2$
$2{G_n}^3 = G_n(G_{n+1} - G_n G_{n-1} - (-1)^n({G_1}^2 - G_1 G_0 - {G_0}^2))$
$2{G_n}^3 = G_n{G_{n+1}}^2 - {G_n}^2 \cdot G_{n-1}(-1)^n({G_1}^2 - G_1 G_0 - {G_0}^2)G_n$

By taking the sum of both sides simultaneously we get:

$$2({G_0}^3 + {G_1}^3 + {G_2}^3 + {G_3}^3 + ... + {G_n}^3) = 2{G_0}^3 - {G_1}^2.G_0 + ({G_1}^2 - G_1G_0 - {G_0}^2)(G_1 - G_2 + G_3 - G_4 + G_5 - G_6 + ... - (-1)^n G_n) + G_n {G_{n+1}}^2 \quad (8)$$

Next, we need to find $(G_1 - G_2 + G_3 - G_4 + G_5 - G_6 + ... - (-1)^n G_n) + G_n {G_{n+1}}^2$ By using Identity 2, we get :

$$-G_2 - G_4 - G_6 ... - G_{2n} = -(G_{2n+1} - G_1) \quad (9)$$

and Identity 7:

$$G_1 + G_3 + G_5 ... + G_{2n-1} = G_{2n} - G_0 \quad (10)$$

From equation (11), add both sides of the equation with $G_{2n} - G_0$ then equation (12) becomes :

$$-G_2 - G_4 - G_6 ... - G_{2n} + G_{2n} - G_0 = (G_{2n-1} - G_1) + G_{2n} - G_0 \quad (11)$$

By equation (12), equation(13) will be :

$$\begin{aligned} -G_2 - G_4 - G_6 ... - G_{2n} + G_{2n} + G_1 + G_3 + G_5 ... + G_{2n-1} \\ = (G_{2n-1} - G_1) + G_{2n} - G_0 \\ G_1 - G_2 + G_3 - G_4 + G_5 + G_6 + ... + G_{2n-1} - G_{2n} \\ = G_{2n-1} + G_{2n} + G_1 - G_0 \end{aligned} \quad (12)$$

From equation (13), add both sides with $G_{2n+1}$, we get:
$G_1 - G_2 + G_3 - G_4 + G_5 + G_6 + ... + G_{2n-1} - G_{2n} + G_{2n+1}$
$= G_{2n-1} + G_1 - G_0 + G_{2n+1}$
$= G_{2n+1} - G_{2n-1} + G_1 - G_0$
By using the definition we get
$G_{2n} = G_{2n+1} - G_{2n-1}$ thus,

$$\begin{aligned} G_1 - G_2 + G_3 - G_4 + G_5 + G_6 + ... + G_{2n-1} - G_{2n} + G_{2n+1} \\ = G_{2n} + G_1 - G_0 \end{aligned} \quad (13)$$

From equations (14) and (15), we can conclude that :

$$\begin{aligned} G_1 - G_2 + G_3 - G_4 + G_5 - G_6 + ... - (-1)^n G_n \\ = -(-1)^n G_{n-1+} + G_1 - G_0 \end{aligned} \quad (14)$$

By using equation (16), equation (17) become:

$$\begin{aligned} 2G_0^3 + G_1^3 + G_2^3 + G_3^3 + ... + G_n^3 \\ = 2G_0^3 - G_1^2.G_0 + (G_1^2 - G_1G_0 - G_0^2) \\ [-(-1)^n G_{n-1} + G_1 - G_0] + G_n G_{n+1}^2 \\ = G_n G_{n+1}^2 + 2G_0^3 - G_1^2 G_0 - \\ (-1)^n + G_1^2 - G_1G_0 - G_0^2 G_{n-1} + G_1 - G_0 \\ = G_n G_{n+1}^2 + 2G_0^3 - G_1^2.G_0 + (G_1^2 - G_1G_0 - G_0^2) \\ [-(-1)^n G_{n-1} + (G_1^2 - G_1G_0 - G_0^2)(+G_1 - G_0) \\ = G_n G_{n+1}^2 + 2G_0^3 - G_1^2 G_0 + (G_1^2 - G_1G_0 - G_0^2) \\ (+G_1 - G_0) - (-1)^n (G_1^2 - G_1G_0 - G_0^2) G_{n-1} \\ = G_n G_{n+1}^2 + 3G_0^3 - G_1^3 3G_1^2 G_0 - \\ (-1)^n (G_1^2 - G_1G_0 - G_0^2) G_{n-1} \\ = G_n G_{n+1}^2 + (G_1 - G_0)^3 + 4G_0^3 - 3G_0^2 G_1 - \\ (-1)^n (G_1^2 - G_1G_0 - G_0^2) G_{n-1} \end{aligned} \quad (15)$$

Therefore:
${G_0}^3 + {G_1}^3 + {G_2}^3 + ... + {G_n}^3 = \frac{[G_n(G_{n+1})^2 + (G_1 - G_0)^3 - 3{G_0}^2 G_1 + 4{G_0}^3 - (-1)^n({G_1}^2 - G_1G_0 - {G_0}^2)G_{n-1}]}{2}$ □

Example : For $G_0 = 3$ and $G_1 = 2$ , the Gibonacci sequence is 3, 2, 5, 7, 12, 19, 31, 50, 81, 131, 212,. . .
${G_0}^3 + {G_1}^3 + {G_2}^3 + ... + {G_9}^3 = \frac{[G_9(G_{10})^2 + (G_1 - G_0)^3 - 3{G_0}^2 G_1 + 4{G_0}^3 - (-1)^9({G_1}^2 - G_1G_0 - {G_0}^2)G_8]}{2}$
${G_0}^3 + {G_1}^3 + {G_2}^3 + ... + {G_9}^3 = \frac{[131(212)^2 + (2-3)^3 - 33^2 2 + 44^3 - (-1)^9 (2^2 - 23 - 3)^2 81]}{2}$
${G_0}^3 + {G_1}^3 + {G_2}^3 + ... + {G_9}^3 = \frac{[(131)(44944) - 1 - 54 - 108 + (-11)81]}{2}$
${G_0}^3 + {G_1}^3 + {G_2}^3 + ... + {G_9}^3 = \frac{[5887664 - 1\lambda 54 + 108 - 891]}{2} = 2943413$

By using direct summation we get :
${G_0}^3 + {G_1}^3 + {G_2}^3 + ... + {G_9}^3 =$
$3^3 + 2^3 + 5^3 + 7^3 + 12^3 + 19^3 + 31^3 + 50^3 + 81^3 + 131^3$
$= 9 + 8 + 125 + 343 + 1728 + 6859 +$
$29791 + 125000 + 531441 + 2248091 = 2943413$

## 4. CONCLUSIONS

From the discussion above we can conclude that the sum of cubic for Lucas number is ${L_0}^3 + {L_1}^3 + {L_2}^3 + {L_3}^3 + ... + {L_n}^3 = \sum_{i=0}^{n} (Li)^3 = \frac{L_n(L_{n+1})^2 + 5(-1)^n L_{n-1} + 19}{2}$ and for Gibonacci numbers is ${G_0}^3 + {G_1}^3 + {G_2}^3 + {G_3}^3 + ... + {G_n}^3 = \sum_{i=0}^{n} (Gi)^3 = \frac{[G_n(G_{n+1})^2 + (G_1 - G_0)^3 - 3{G_0}^2 G_1 + 4{G_0}^3 - (-1)^n({G_1}^2 - G_1G_0 - {G_0}^2)G_{n-1}]}{2}$

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