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The Use of Differential Forms to Linearize a Class of Geodesic Equations

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Abstract

Lie was the first to consider linearization of differential equations many years ago. Since then, a great deal of research has been done on linearization of differential equations using various methodologies. Surprisingly, there has not been much progress in linearizing geodesic differential equations. In particular, the use of differential forms to linearize a class of geodesic equations is not documented in the literature. Differential forms are used to linearize a class of geodesic differential equations in this research. Geodesics on a plane, geodesics on a cone, and geodesics on a sphere are examples. The solutions to these equations were discovered during the linearization process, as the findings of this study are distinctive, innovative, and original.

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1. Introduction

A geodesic is a curve that minimizes length locally. It is, in other words, a path that a particle that is not accelerating would take. Geodesics are straight lines in the plane. Geodesics are great circles on the sphere (like the equator). The term geodesic refers to the curve that would be formed if you continued on a straight path. You could travel a great circle on the surface of a sphere (think of the earth) if you kept traveling straight without turning left or right. Geodesics are the shortest distance curves on a surface. They are useful in the transportation of products and persons at low cost of time and energy, in addition to their intrinsic interest. They are also critical as emergency exit routes during flights. The methods of differential geometry can be used to locate geodesics. The equator and the other great circles on a sphere are common examples. In a curved space, a geodesic is the straightest path conceivable. A straight line is what we call space that is flat.

When you try to move straight in a curved space, you get geodesics in general. For example, while you are driving on the highway (and the road appears to be quite straight to you), you are not driving on a straight line; instead, you are driving on the straightest line conceivable on the Earth's curved surface. Ask what curve within the surface connects two neighboring places on the surface and has the smallest length. A geodesic curve is a curve that meets the condition. In a curved space, though, it is just the opposite of straight. It's a mathematical extremum,

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which means that any slight divergence from it will make it longer. The geodesic between two places on the Earth's surface is thus a great-circle path; nevertheless, the Earth's surface is curved locally as well as globally, that is, it has mountains and valleys. As a result, more than one geodesic can exist between two places. Although they are not always the same length (though they can be), they are both extrema.

Analysis of the geodesics of the Einstein-Maxwell-dilaton theory's exact solution, the four-dimensional linear dilaton black hole (LDBH) spacetime was examined in [1]. The conventional Lagrangian approach is used to investigate the test particles' geodesic movements. They demonstrated that exact analytical solutions to the radial and angular geodesic equations may be achieved after obtaining the Euler-Lagrange equations. In particular, it is demonstrated that one of the radial trajectory solutions may be expressed in terms of the Weierstrass P-function, an elliptic-type special function. In another development as can be seen in [2], some partial differential equations which were derived from the well-known two logistic distribution parameters were solved using two alternative techniques. The first approach was the conventional one, which required the solution of three partial differential equations. The well-known Darboux Theory was the second strategy. It was discovered that, the geodesic equations are two minimum or isotropic curves. Both approaches produced the same outcome, as was to be expected.

In the article [3], the differential pursuit game problem was examined, in which a finite number of pursuers chase a finite number of evaders. The problem is expressed in a Hilbert space l_2 with the motions of the pursuer and evader being defined by n^{th} and m^{th} order differential equations, respectively. Integral and geometric restrictions are placed on the control mechanisms used by the evader and pursuers, respectively. The Lie subalgebras of Noether symmetries associated with systems of geodesic equations are found to have one-dimensional optimum systems [4]. They also discovered invariants for every component of the deduced optimal system. It was demonstrated that the resulting invariants can transform systems of geodesic equations—nonlinear systems of quadratically semi-linear second-order ordinary differential equations into nonlinear systems of first-order ODEs.

The article [5] demonstrate a theorem that connects the metric collineations and the Lie symmetries of the geodesic equations in a Riemannian space. To Einstein spaces and spaces with constant curvature, the findings were applied. The utilization of the results is then demonstrated using examples. In [6], group hydrodynamical systems with right invariant L^2 or H^1 metrics that may be expressed as geodesic equations on diffeomorphism groups or on extensions of diffeomorphism groups was considered. The numerical solution of Geodesic equations was investigated in [7] where the initial value problem on an oblate spheroid, the direct geodesic problem was solved numerically using both geodesic and Cartesian coordinates. In that study, the differential geometry theory is used to formulate the geodesic equations. The initial value problem in question is reduced to a system of first-order ordinary differential equations that is solved numerically.

Not so much is done in the area of linearizing Geodesics equations. The relationship between isometries and symmetries of the system of geodesic equations was used in [8] to construct criteria for second order quadratically and cubically semi-linear equations and systems of equations. The geodesic deviation Jacobi equation that addresses finite size effects caused by gravitational tidal forces was considered in [9]. It was shown how the Jacobi problem in any spacetime that admits entirely geodesics that can be integrated can be solved. Invariant Wronskians for the Jacobi system that are linear in the 'deviation momenta' were derived by linearizing the geodesic equation and its conserved charges, resulting in a set of integrated first-order differential equations. The continuous hybrid numerical approach is taken into consideration in the work of [10] to solve second order initial value problems of ordinary differential equations in general. Utilizing the power series as the basis function, the method of collocation of the differential system resulting from the rough solution to the problem was used.

A differential form, which includes differentials, is a quantity that may be integrated. $f(x)dx$ is the differential form of the integral $\int_a^b f(x)\,dx$. This is because it is integrated over a one-dimensional region or path, this differential form has degree one. One-form refers to the differential form of degree one. The differential forms was previously used to linearize some important differential equations [11-13]. In this research, our attention is focused on the linearization of a class of Geodesics equations.

The class of Geodesics equations is obtained from a unified equation

$$y'' - 2\frac{f'(y)}{f(y)}y'^2 - f'(y)\,f(y) = 0, \tag{1}$$

as presented in [14]. Given that $f(y) = y$, equation (1) becomes

$$y'' - \frac{2}{y}y'^2 - y = 0, \tag{2}$$

which describes the geodesics on a cone. For $f(y) = d + y$, equation (1) is now

$$y'' - \frac{2}{d+y}y'^2 - d - y = 0, \tag{3}$$

which describes the geodesics on a plane, where d is a constant.

Again, for $f(y) = \sin y$, equation (1) becomes

$$y'' - 2\cot y\, y'^2 - \sin y \cos y = 0. \tag{4}$$

Equation (4) describes the geodesics on sphere.

As previously mentioned, [14] addressed the class of Geodesic equations (2) through (4). A characterization of Sundman linearizable equations in terms of one auxiliary function and ODE coefficients was the method adopted. A direct alternative method for creating the first integrals and Sundman transformations was provided by using this criterion to explicitly acquire the general solutions for the first integral. Equation (4) was also resolved in [15] using symmetry and integration techniques for differential equations. Our results utilizing the differential forms method are consistent with those obtained by other methods in every circumstance. The approach we took in this case is simpler and gets us to the findings more quickly.

2. Approach of Differential Forms

Linearization through differential forms entails that, the invertible change of independent and dependent variables
$X = F(x, y)$ and $Y = G(x, y)$,
that will map the general second order nonlinear ordinary differential equation

$$y'' = f(x, y, y'), \tag{5}$$

into a linear equation, should necessarily be in the form

$$y'' + f_0 + f_1 y' + f_2 y'^2 + f_3 y'^3 = 0, \tag{6}$$

and the coefficients f_0, f_1, f_2, and f_3 must satisfy the conditions

$$f_{0yy} + f_0\left(f_{2y} - 2f_{3x}\right) + f_2 f_{0y} - f_3 f_{0x} + \frac{1}{3}\left(f_{2xx} - 2f_{1xy} + f_1 f_{2x} - 2f_1 f_{1y}\right) = 0, \tag{7}$$

and

$$f_{3xx} + f_3\left(2f_{0y} - f_{1x}\right) + f_0 f_{3y} - f_1 f_{3x} + \frac{1}{3}\left(f_{1yy} - 2f_{2xy} + 2f_2 f_{2x} - f_2 f_{1y}\right) = 0. \tag{8}$$

Once the conditions in equations (7) and (8) are satisfied, we proceed to construct a 3×3 matrix

$$M = Pdx + Qdy, \tag{9}$$

where $P = \left(\frac{1}{3}\right)\begin{bmatrix} -2f_1 & -3f_0 & 3f_{0y} + 3f_0 f_2 \\ 0 & f_1 & 2f_{2x} - f_{1y} - 3f_0 f_3 \\ -3 & 0 & f_1 \end{bmatrix}$,

$$Q = \left(\frac{1}{3}\right)\begin{bmatrix} -f_2 & 0 & 2f_{1y} - f_{2x} + 3f_0 f_3 \\ 3f_3 & 2f_2 & 3f_{3x} - 3f_1 f_3 \\ 0 & 3 & -f_2 \end{bmatrix},$$

and solve the equation

$$dr = Mr, \tag{10}$$

where $r = \begin{bmatrix} U \\ V \\ W \end{bmatrix}$, a special solution is usually sufficient for the three components of r. We can also construct

$$K = U/W, \quad L = V/W. \tag{11}$$

Next, we construct the 2×2 matrix

$$Z = \begin{bmatrix} (2K - f_1)\,dx - Ldy & f_0 dx + Kdy \\ -Ldx - f_3 dy & Kdx + (f_2 - 2L)\,dy \end{bmatrix},$$

and solve for R from the equation

$$dR = ZR, \tag{12}$$

where $R = \begin{bmatrix} F_x \\ F_y \end{bmatrix} = \begin{bmatrix} b \\ c \end{bmatrix}$. Finally, we solve

$$dF = \begin{bmatrix} dx & dy \end{bmatrix} R; \tag{13}$$

the two independent solutions will be taken as F and G. What is given here is the summary of the method. For more detail please see [12], [11] or [13].

3. Linearization of a Class of Geodesics Equations

The unified class of geodesic equations is represented by equation (1). At this point, we want to linearize the three equations that make up the geodesic equations class.

3.1. Geodesics on a Cone

Equation (2) which describes the geodesics on a cone has the coefficients $f_0 = -y,\ f_1 = 0,\ f_2 = -\frac{2}{y},\ f_3 = 0$, that satisfy the linearizability conditions (7) and (8).

With $Pdx = \begin{bmatrix} 0 & ydx & dx \\ 0 & 0 & 0 \\ -dx & 0 & 0 \end{bmatrix}$, and

$$Qdy = \begin{bmatrix} \frac{2}{3y}dy & 0 & 0 \\ 0 & \frac{-4}{3y}dy & 0 \\ 0 & dy & \frac{2}{3y}dy \end{bmatrix},$$

$M = Pdx + Qdy$ becomes $M = \begin{bmatrix} \frac{2}{3y}dy & ydx & dx \\ 0 & \frac{-4}{3y}dy & 0 \\ -dx & dy & \frac{2}{3y}dy \end{bmatrix}$.

With this situation, $dr = \begin{bmatrix} \frac{2U}{3y}dy + Vydx + Wdx \\ -\frac{4V}{3y}dy \\ -Udx + Vdy + \frac{2W}{3y}dy \end{bmatrix}$, where $dr = Mr$ and $r = \begin{bmatrix} U \\ V \\ W \end{bmatrix}$.

If $V = 0$, then $dU = \frac{2U}{3y}dy + Wdx,\ \ dV = 0$ and $dW = -Udx + \frac{2W}{3y}dy$. Again, $U_x = W,\ \ U_y = \frac{2U}{3y},\ \ W_x = -U$ and $W_y = \frac{2W}{3y}$. $V = 0,\ \ U = y^{2/3}\sin x$ and $W = y^{2/3}\cos x$ are given as a special solution. Therefore, equation (11) is now

$$K = \frac{y^{2/3}\sin x}{y^{2/3}\cos x} = \tan x\,,\ L = 0,$$

and the matrix Z becomes
$Z = \begin{bmatrix} 2\tan x\; dx & -ydx + \tan x\; dy \\ 0 & \tan x\; dx - \frac{2}{y}dy \end{bmatrix}$,

and $dR = \begin{bmatrix} 2b\tan x\; dx - cydx + c\tan x\; dy \\ c\tan x\; dx - \frac{2c}{y}dy \end{bmatrix}$.

From dR, we see that

$$db = (2b\tan x\ - cy)\,dx + c\tan x\; dy,$$

and

$$dc = c\left(\tan x\; dx - \frac{2}{y}dy\right),$$

where $b_x = 2b\tan x\ - cy,\ \ b_y = c\tan x\,,\ \ c_x = c\tan x$ and $c_y = -\frac{2c}{y}$. Integrating
$dc = c\left(\tan x\; dx - \frac{2}{y}dy\right)$, we have $\ln c\ + \ln y^2 - \ln\sec x\ = k$. That is, $\frac{cy^2}{\sec x} = e^k = k$, and then

$$c = k\frac{\sec x}{y^2},$$

where k is a constant.

We notice that $b_y = c_x$ so that $b_y = k\frac{\sec x \tan x}{y^2}$. Integrating, we have

$$b = -k\frac{\sec x \tan x}{y} + g(x). \tag{14}$$

When we differentiate equation (14) with respect to x, we get

$$b_x = -k\frac{\left(\sec x + 2\sec x \tan^2 x\right)}{y} + g'(x). \tag{15}$$

But b_x is also expressed as $b_x = 2b\tan x - cy$, that is

$$b_x = -2k\frac{\sec x \tan^2 x}{y} + 2g(x)\tan x - k\frac{\sec x}{y}.$$

Now comparing the above with equation (15) and simplifying, one sees that

$$g'(x) - 2g(x)\tan x = 0.$$

Using the method of integrating factor with $p(x) = -2g(x)\tan x$ and $q(x) = 0$, one obtains the integrating factor as $\frac{1}{\sec^2 x}$. Therefore, $\frac{g(x)}{\sec^2 x} = m$, that is $g(x) = m\sec^2 x$ or $g(x) = \frac{m}{\cos^2 x}$, where m is another constant.

Substituting $g(x) = m\sec^2 x$ into equation (14), one obtains that

$$b = -k\frac{\sec x \tan x}{y} + m\sec^2 x.$$

For $dF = \begin{bmatrix} dx & dy \end{bmatrix}\begin{bmatrix} b \\ c \end{bmatrix}$, you have that

$$dF = \left(\frac{-k\sec x \tan x}{y} + m\sec^2 x\right)dx + k\frac{\sec x}{y^2}dy.$$

On integration,

$$F = -ky^{-1}\sec x + m\tan x - ky^{-1}\sec x,$$

and finally, $F = \frac{-2k\sec x}{y} + m\tan x$.

Taking the coefficients proportional to the constants k and m to be the linearizing point transformation, we have

$$X = \tan x, \quad Y = \frac{2\sec x}{y}.$$

With the transformation $Y = c_1 X + c_2$, one sees that $\frac{2\sec x}{y} = c_1\tan x + c_2$, that is

$2\sec x = c_1 y\tan x + c_2 y$ is the solution of equation (2).

3.2. Geodesics on Plane

We now proceed to linearize the equation that describes the geodesics on plane. The equation is designated as equation (3). Equation (3) is in the form of (6) and its coefficients: $f_0 = -d - y$, $f_1 = 0$, $f_2 = \frac{-2}{d+y}$, $f_3 = 0$ satisfy the linearizability conditions (7) and (8), and therefore, it is linearizable using differential forms.

Now,

$$Pdx = \begin{bmatrix} 0 & (d+y)\,dx & dx \\ 0 & 0 & 0 \\ -dx & 0 & 0 \end{bmatrix},$$

$$Qdy = \begin{bmatrix} \frac{2dy}{3(d+y)} & 0 & 0 \\ 0 & \frac{-4dy}{3(d+y)} & 0 \\ 0 & dy & \frac{2dy}{3(d+y)} \end{bmatrix},$$

so that equation (9) becomes

$$M = \begin{bmatrix} \frac{2dy}{3(d+y)} & (d+y)\,dx & dx \\ 0 & \frac{-4dy}{3(d+y)} & 0 \\ -dx & dy & \frac{2dy}{3(d+y)} \end{bmatrix},$$

and $dr = \begin{bmatrix} \frac{2Udy}{3(d+y)} + V(d+y)\,dx + Wdx \\ \frac{-4Vdy}{3(d+y)} \\ -Udx + Vdy + \frac{2Wdy}{3(d+y)} \end{bmatrix}.$

Letting $V = 0$ gives $dU = \frac{2Udy}{3(d+y)} + Wdx$, $dV = 0$, $dW = -Udx + \frac{2Wdy}{3(d+y)}$, and $U_x = W$, $U_y = \frac{2U}{3(d+y)}$, $W_x = -U$, $W_y = \frac{2W}{3(d+y)}$.

The situation above is satisfied by a special solution $U = (d+y)^{\frac{2}{3}}\sin x$ and

$W = (d+y)^{\frac{2}{3}}\cos x$. Therefore $K = \tan x$, $L = 0$. Constructing the matrix Z, one sees that

$$Z = \begin{bmatrix} 2\tan x\,dx & -(d+y)\,dx + \tan x\,dy \\ 0 & \tan x\,dx - \frac{2dy}{d+y} \end{bmatrix},$$

and equation (12) is now

$$dR = \begin{bmatrix} 2b\tan x\,dx - (d+y)\,cdx + c\tan x\,dy \\ c\tan x\,dx - \frac{2cdy}{d+y} \end{bmatrix}.$$

This situation becomes

$$db = (2b\tan x - (d+y)\,c)\,dx + c\tan x\,dy, \tag{16}$$

and

$$dc = c\left(\tan x\,dx - \frac{2dy}{d+y}\right). \tag{17}$$

Integrating (17) above, we have

$$\ln c = \ln\sec x - 2\int\frac{dy}{d+y} + k,$$

so that

$$\frac{c(d+y)^2}{\sec x} = e^k = k.$$

Making c the subject, we have

$$c = \frac{k\sec x}{(d+y)^2}, \tag{18}$$

where k is a constant.

Again, from equation (16), $b_x = 2b\tan x - (d+y)\,c$, $b_y = c\tan x$, $c_x = c\tan x$ and $c_y = -\frac{2c}{d+y}$, where $b_y = c_x$.

Since $b_y = c_x$, we have that $b_y = k\frac{\tan x \sec x}{(d+y)^2}$, and on integration, we have

so that

$$b = \frac{ky\tan x \sec x}{d(d+y)} + g(x). \tag{19}$$

Differentiating b with respect to x, we have

$$b_x = \frac{ky\sec x}{d(d+y)}\left(2\tan^2 x + 1\right) + g'(x).$$

One notes that b_x is also expressed as $b_x = 2b\tan x - (d+y)c$, that is

$$b_x = \frac{2ky\tan^2 x \sec x}{d(d+y)} + 2\tan x\, g(x) - \frac{k\sec x}{d+y}.$$

Comparing the two expressions of b_x and simplifying, one has

$$g'(x) - 2\tan x\, g(x) = \frac{-kd\sec x - ky\sec x}{d(d+y)}.$$

This implies that

$$g'(x) - 2\tan x\, g(x) = \frac{-k\sec x}{d}. \tag{20}$$

Equation (20) has now been transformed into a first-order linear differential equation that can be solved using the integrating factor $\frac{1}{\sec^2 x}$.

Multiplication of the integrating factor and equation (20) gives

$$\cos^2 x\, g'(x) - 2\sin x \cos x\, g(x) = -\frac{k\cos x}{d}.$$

This becomes

$$\cos^2 x\, g(x) = \frac{-k}{d}\sin x + m,$$

where m is another constant.

Simplifying, the equation above becomes

$$g(x) = \frac{-k\tan x \sec x}{d} + m\sec^2 x,$$

and the value of b from equation (19) is now

$$b = \frac{ky\tan x \sec x}{d(d+y)} - \frac{k\tan x \sec x}{d} + m\sec^2 x.$$

Referring to equation (13), that is, $dF = \begin{bmatrix} dx & dy \end{bmatrix}\begin{bmatrix} b \\ c \end{bmatrix}$, one sees that

$$dF = \left(\frac{ky\tan x \sec x}{d(d+y)} - \frac{k\tan x \sec x}{d} + m\sec^2 x\right)dx + \frac{k\sec x}{(d+y)^2}dy.$$

Integrating the equation above, you have

$$F = k\left(\frac{2y\sec x}{d(d+y)} - \frac{\sec x}{d}\right) + m\tan x.$$

Therefore

$$X = \tan x, \quad Y = \frac{(y-d)\sec x}{d(d+y)}$$

is the linearizing point transformation of equation (3). The general solution can be readily expressed using the transformation $Y = c_1 X + c_2$, as

$$y = d + c_1 d(d+y)\sin x + c_2 d(d+y)\cos x.$$

3.3. Geodesics on Sphere

Next to be considered is equation (4) that describes the geodesics on a sphere. The equation has the coefficients $f_0 = -\sin y \cos y$, $f_1 = 0$, $f_2 = -2\cot y$, $f_3 = 0$ that satisfied the linearizability conditions (7) and (8).

The 3×3 matrix $M = Pdx + Qdy$ is now

$$M = \begin{bmatrix} \frac{2}{3}\cot y\, dy & \sin y \cos y\, dx & dx \\ 0 & -\frac{4}{3}\cot y\, dy & 0 \\ -dx & dy & \frac{2}{3}\cot y\, dy \end{bmatrix},$$

where $Pdx = \begin{bmatrix} 0 & \sin y \cos y\, dx & dx \\ 0 & 0 & 0 \\ -dx & 0 & 0 \end{bmatrix}$

and $Qdy = \begin{bmatrix} \frac{2}{3}\cot y\, dy & 0 & 0 \\ 0 & -\frac{4}{3}\cot y\, dy & 0 \\ 0 & dy & \frac{2}{3}\cot y\, dy \end{bmatrix}$.

Now, $dr = \begin{bmatrix} \frac{2}{3}U\cot y\, dy + V\sin y \cos y\, dx + Wdx \\ -\frac{4}{3}V\cot y\, dy \\ -Udx + Vdy + \frac{2}{3}W\cot y\, dy \end{bmatrix}$, so that, letting $V = 0$, we have that

$dU = \frac{2}{3}U\cot y\, dy + Wdx$, $dV = 0$ and $dW = -Udx + \frac{2}{3}W\cot y\, dy$. Also, $U_x = W$, $U_y = \frac{2}{3}U\cot y$, $W_x = -U$ and $W_y = \frac{2}{3}W\cot y$.

A special solution $U = \sin x(\sin y)^{2/3}$, $V = 0$, $W = \cos x(\sin y)^{2/3}$ satisfies the situation above. Construction of K and L shows that $K = \frac{\sin x (\sin y)^{2/3}}{\cos x (\sin y)^{2/3}} = \tan x$ and $L = 0$. With K and L, the 2×2 matrix Z becomes

$$Z = \begin{bmatrix} 2\tan x\, dx & -\sin y \cos y\, dx + \tan x\, dy \\ 0 & \tan x\, dx - 2\cot y\, dy \end{bmatrix},$$

and

$$dR = \begin{bmatrix} 2b\tan x\, dx - c\sin y \cos y\, dx + c\tan x\, dy \\ c\tan x\, dx - 2c\cot y\, dy \end{bmatrix},$$

so that

$$db = 2b\tan x\, dx - c\sin y \cos y\, dx + c\tan x\, dy, \tag{21}$$

and

$$dc = c\tan x\, dx - 2c\cot y\, dy. \tag{22}$$

From equations (21) and (22), one sees that

$$b_x = 2b\tan x - c\sin y \cos y, \quad b_y = c\tan x, \quad c_x = c\tan x,$$

$$c_y = -2c\cot y.$$

Integrating equation (22), we have that

$$c = \frac{k\sec x}{\sin^2 y}, \tag{23}$$

where $k = e^k$ is a constant.

One notes that $b_y = c_x$, therefore,

$$b_y = \frac{k\tan x \sec x}{\sin^2 y}.$$

Integrating the equation above, you see that

$$b = -k\tan x \sec x \cot y + g(x), \tag{24}$$

for some function $g(x)$.

We can see that when we differentiate equation (24) with respect to x, we get

$$b_x = -k\cot y \tan^2 x \sec x - k\cot y \sec^3 x + g'(x). \tag{25}$$

But b_x is also expressed as $b_x = 2b\tan x - c\sin y \cos y$ that is

$$b_x = -2k\tan^2 x \sec x \cot y + 2\tan x\, g(x) - k\sec x \cot y. \tag{26}$$

Now, comparing equations (25) and (26) and simplifying, we have

$$g'(x) - 2\tan x\, g(x) = -k\sec x \cot y \left(\tan^2 x - \sec^2 x + 1\right).$$

Using the identity $\tan^2 x + 1 = \sec^2 x$, one finally has that

$$g'(x) - 2\tan x\, g(x) = 0. \tag{27}$$

Solving equation (27) using the integrating factor $\frac{1}{\sec^2 x}$, one sees that $g(x) = m\sec^2 x$, where m is another constant.

Substituting $g(x)$ into equation (24), we now have that

$$b = -k\tan x \sec x \cot y + m\sec^2 x.$$

Now,

$$dF = \left(-k\tan x \sec x \cot y + m\sec^2 x\right)dx + \frac{k\sec x}{\sin^2 y},$$

and on integration, $F = -2k\cot y \sec x + m\tan x$. The linearizing point transformation is now

$$X = \tan x, \quad Y = 2\cot y \sec x,$$

and the solution of equation (4) is now $2\cot y = c_1\sin x + c_2\cos x$.

4. Conclusion

The linearization of geodesics on spheres, planes, and cones is accomplished using the differential forms technique. These equations have convincing solutions as obtained from the transformation $Y = c_1X + c_2$. These three geodesics equations are considered under the umbrella geodesics equation. The solution or result of equation (4) is similar to the one obtained in [15] where symmetry and integration method was used. It is shown that the equation to be linearized must have the form (6), and the coefficients must satisfy the linearizability conditions (7) and (8). The findings of this study are distinctive, innovative, and original.

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